Numerical Algorithms manuscript No.
(will be inserted by the editor)

Natural Volterra Runge-Kutta methods

D. Conte · R. D'Ambrosio · G. Izzo · Z. Jackiewicz

Received: date / Accepted: date

Abstract A very general class of Runge-Kutta methods for Volterra integral equations of the second kind is analyzed. Order and stage order conditions are derived for methods of order p and stage order $q = p$ up to the order four. We also investigate stability properties of these methods with respect to the basic and the convolution test equations. The systematic search for A- and V_0-stable methods is described and examples of highly stable methods are presented up to the order $p = 4$ and stage order $q = 4$.

Keywords Volterra integral equation · Volterra Runge-Kutta methods · order and stage order conditions · stability analysis · A-stability · V_0-stability.

Mathematics Subject Classification (2000) 65R20

Dajana Conte
University of Salerno
Fisciano - Italy
E-mail: dajconte@unisa.it

Raffaele D'Ambrosio
University of Salerno
Fisciano - Italy
E-mail: rdambrosio@unisa.it

Giuseppe Izzo
University of Naples "Federico II"
Naples - Italy
Partially supported by the School of Sciences and Technology - University of Naples "Federico II" under the project "F.A.R.O." *Control and stability of diffusive processes in the environment.*
E-mail: giuseppe.izzo@unina.it

Zdzislaw Jackiewicz
Arizona State University
Tempe - USA
and AGH University of Science and Technology
Kraków, Poland E-mail: jackiewicz@asu.edu

1 Introduction

For the numerical solution of Volterra integral equations (VIEs) of the second kind

$$y(t) = g(t) + \int_{t_0}^{t} k\big(t, s, y(s)\big) ds, \quad t \in [t_0, T], \tag{1.1}$$

where

$$g : [t_0, T] \to \mathbb{R}^m, \quad k : \varDelta \times \mathbb{R}^m \to \mathbb{R}^m, \quad \varDelta := \big\{(t, s) : \ t_0 \le s \le t \le T\big\},$$

are sufficiently smooth, we consider a very general class of so-called Volterra Runge-Kutta (VRK) methods defined by

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{\mu} \alpha_{ij} k\left(t_n + d_{ij}h, t_n + e_{ij}h, \sum_{l=1}^{\nu} \beta_{ijl} Y_l^{[n]}\right) + \widetilde{F}_n(t_n + c_i h), \\ y_{n+1} &= \sum_{j=1}^{\nu} w_j Y_j^{[n]}, \end{aligned} \tag{1.2}$$

$i = 1, 2, \ldots, \nu$; $n = 0, 1, \ldots, N-1$. Here, μ is a fixed integer, $t_n = t_0 + nh$, $n = 0, 1, \ldots, N$, $Nh = T - t_0$, is the uniform grid, and $\widetilde{F}_n(t_n + c_i h)$ is an approximation to the tail $F_n(t_n + c_i h)$ defined by $F_n(t) = g(t) + \int_{t_0}^{t_n} k\big(t, s, y(s)\big) ds$. With this notation the equation (1.1) can be rewritten as

$$y(t) = F_n(t) + \int_{t_n}^{t} k\big(t, s, y(s)\big) ds, \quad t \in [t_n, t_{n+1}], \ n = 0, 1, \ldots, N-1. \tag{1.3}$$

The tail approximation $\widetilde{F}_n(t_n + c_i h)$ to $F_n(t_n + c_i h)$ should be chosen in such a way that it preserves the order of convergence of the VRK method and that it is as efficient as possible in terms of the number of evaluations of the kernel function k appearing in (1.1), (1.2), and (1.3). Similarly as in [18] we can define the tail approximation of the form

$$\widetilde{F}_n(t) = g(t) + h \sum_{\kappa=1}^{n} \sum_{j=1}^{\nu} \gamma_j k(t, t_{\kappa-1} + c_j h, Y_j^{[\kappa-1]}), \tag{1.4}$$

with weight vector $\gamma = [\gamma_1, \ldots, \gamma_\nu]^T$. The resulting method (1.2) with the tail approximations $\widetilde{F}(t_n + c_i h)$ defined by (1.4) was referred to in [8], as an extended VRK method. A different approach to tail approximations based on natural continuous extensions of the numerical solution of degree $d \le p$ was proposed by Bellen et al. in [4]. These natural continuous extensions $u(t_n + \theta h)$ are defined by

$$u(t_n + \theta h) = \sum_{j=1}^{\nu} w_j(\theta) Y_j^{[n]}, \tag{1.5}$$

$n = 0, 1, \ldots, N-1$; $\theta \in [0, 1]$, where $w_j(\theta)$ are polynomials of degree d,

$$\lfloor p/2 \rfloor \le d \le \min\{\nu - 1, p\}.$$

Here, $\lfloor p/2 \rfloor$ stands for the integer part of $p/2$. These polynomials satisfy the linear system of equations

$$\sum_{j=1}^{\nu} w_j(\theta) c_j^k = \theta^k, \quad k = 0, 1, \dots, d. \tag{1.6}$$

See [4] for comparison. We then consider the following tail approximation

$$\widetilde{F}_n(t) = g(t) + h \sum_{\kappa=1}^{n} \sum_{j=1}^{m} v_j k\big(t, t_{\kappa-1} + \xi_j h, u(t_{\kappa-1} + \xi_j h)\big), \tag{1.7}$$

where the weight v_j and abscissas ξ_j correspond to a quadrature rule of order greater or equal to p. The resulting formulas with tail approximation defined by (1.7) will be referred to as natural VRK methods.

Natural continuous extensions of Runge-Kutta (RK) methods for ODEs were introduced by Zennaro in [26].

The coefficients α_{ij}, β_{ijl}, and w_i, and the abscissas c_i, d_{ij}, and e_{ij} will be chosen so that the VRK method has order p and stage order $q = p$, and some desirable stability properties with respect to the basic and the convolution test equations.

Since, in general, $k(t, s, y)$ is defined only for $s \le t$, we will always assume the so-called kernel condition $e_{ij} \le d_{ij}$.

The VRK methods (1.2) were first introduced in [4] and further investigated in [5]. They include as special cases Pouzet-type methods [23]

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{m} a_{ij} k(t_n + c_i h, t_n + c_j h, Y_j^{[n]}) + \widetilde{F}_n(t_n + c_i h), \\ y_{n+1} &= h \sum_{j=1}^{m} b_j k(t_n + h, t_n + c_j h, Y_j^{[n]}) + \widetilde{F}_n(t_n + h), \end{aligned} \tag{1.8}$$

$i = 1, 2, \dots, m$; $n = 0, 1, \dots, N-1$, and Bel'tyukov-type methods [6]

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{m} a_{ij} k(t_n + e_j h, t_n + c_j h, Y_j^{[n]}) + \widetilde{F}_n(t_n + c_i h), \\ y_{n+1} &= h \sum_{j=1}^{m} b_j k(t_n + e_j h, t_n + c_j h, Y_j^{[n]}) + \widetilde{F}_n(t_n + h), \end{aligned} \tag{1.9}$$

$i = 1, 2, \dots, m$; $n = 0, 1, \dots, N-1$. Pouzet-type methods (1.8) correspond to $\mu = m$, $\nu = m+1$, $c = [c_1, c_2, \dots, c_m, 1]^T$, and

$$\begin{gathered} \alpha_{ij} = a_{ij}, \quad i, j = 1, 2, \dots, m, \quad \alpha_{m+1,j} = b_j, \quad j = 1, 2, \dots, m, \\ d_{ij} = c_i, \quad i, j = 1, 2, \dots, m, \quad d_{m+1,j} = 1, \quad j = 1, 2, \dots, m, \\ e_{ij} = c_j, \quad i = 1, 2, \dots, m+1, \quad j = 1, 2, \dots, m, \\ \beta_{ijl} = \delta_{jl}, \quad i, l = 1, 2, \dots, m+1, \quad j = 1, 2, \dots, m, \end{gathered}$$

$$w_j = \delta_{m+1,j}, \quad j = 1, 2, \ldots, m+1.$$

Similarly, Bel'tyukov-type methods (1.9) correspond to $\mu = m$, $\nu = m + 1$, $c = [c_1, c_2, \ldots, c_m, 1]^T$, and

$$\alpha_{ij} = a_{ij}, \quad i, j = 1, 2, \ldots, m, \quad \alpha_{m+1,j} = b_j, \quad j = 1, 2, \ldots, m,$$

$$d_{ij} = e_j, \quad i, j = 1, 2, \ldots, m+1, \quad j = 1, 2, \ldots, m,$$

$$e_{ij} = c_j, \quad i = 1, 2, \ldots, m+1, \quad j = 1, 2, \ldots, m,$$

$$\beta_{ijl} = \delta_{jl}, \quad i, l = 1, 2, \ldots, m+1, \quad j = 1, 2, \ldots, m,$$

$$w_j = \delta_{m+1,j}, \quad j = 1, 2, \ldots, m+1.$$

Choosing $c_\nu = 1$, $w_j = 0$, $j = 1, 2, \ldots, \nu - 1$, $w_\nu = 1$, we obtain a class of methods for which the external approximation y_{n+1} is equal to the last internal stage, i.e., $y_{n+1} = Y_\nu^{[n]}$.

The numerical solution of Volterra integral and integro-differential equations, including Volterra equations with weakly singular kernels, is discussed in a monograph [8]. Two-step Runge-Kutta methods for Volterra integral equations have been introduced and analyzed in [10–12].

In the next section we derive conditions on the coefficients of (1.2) so that the resulting VRK methods have order p and stage order $q = p$. In Sections 3 and 4 we investigate stability properties of VRK methods (1.2) with respect to the basic and the convolution test equations. We are mainly interested in the derivation of A-stable and V_0-stable methods. These stability concepts are defined in Sections 3 and 4, respectively. In Section 5 we derive examples of A-stable and V_0-stable natural VRK methods with $p = q = \mu = \nu$ for $\nu = 1$, 2, 3, and 4. Numerical experiments which confirm the expected order and stage order are reported in Section 6. Finally, in Section 7 some concluding remarks are given and plans for future work are briefly outlined.

2 Stage order and order conditions

To discuss order conditions for VRK methods (1.2), with no loss of generality (cfr. [4], [7]), we will consider a simpler form of VIE (1.1), where the kernel function $k(t, s, y(s))$ is independent of s. This can be accomplished if we define, for example,

$$\widetilde{y}(t) = \begin{bmatrix} t \\ y(t) \end{bmatrix}, \quad \widetilde{g}(t) = \begin{bmatrix} t_0 \\ g(t) \end{bmatrix}, \quad \widetilde{k}\big(t, \widetilde{y}(s)\big) = \begin{bmatrix} 1 \\ k\big(t, s, y(s)\big) \end{bmatrix}.$$

Then the equation (1.1) can be reduced to the form

$$y(t) = g(t) + \int_{t_0}^{t} k\big(t, y(s)\big) ds, \quad t \in [t_0, T], \tag{2.1}$$

where for convenience we have written y, g and k instead of $\widetilde{y}$, $\widetilde{g}$ and $\widetilde{k}$. The VRK method for (2.1) now takes the form

$$\begin{aligned} Y_i^{[n]} &= h\sum_{j=1}^{\mu}\alpha_{ij}k\left(t_n+d_{ij}h,\sum_{l=1}^{\nu}\beta_{ijl}Y_l^{[n]}\right)+\widetilde{F}_n(t_n+c_ih),\\ y_{n+1} &= \sum_{j=1}^{\nu}w_jY_j^{[n]}, \end{aligned} \tag{2.2}$$

$i=1,2,\ldots,\nu$, $n=0,1,\ldots,N-1$, where $\widetilde{F}_n(t_n+c_ih)$ is an approximation to the tail $F_n(t_n+c_ih)$ of sufficiently high order defined by

$$F_n(t)=g(t)+\int_{t_0}^{t_n}k\big(t,y(s)\big)ds,\quad t\in[t_n,t_{n+1}],\quad n=0,1,\ldots,N-1.$$

The tail approximation (1.4) now takes the form

$$\widetilde{F}_n(t)=g(t)+h\sum_{\kappa=1}^{n}\sum_{j=1}^{\nu}\gamma_jk(t,Y_j^{[\kappa-1]}), \tag{2.3}$$

and the tail approximation (1.7) based on natural continuous extensions (1.5) takes the form

$$\widetilde{F}_n(t)=g(t)+h\sum_{\kappa=1}^{n}\sum_{j=1}^{m}v_jk\big(t,u(t_{\kappa-1}+\xi_jh)\big). \tag{2.4}$$

It follows from definition of $\widetilde{k}(t,\widetilde{y}(s))$ that with abscissas e_{ij} defined by

$$e_{ij}=\sum_{l=1}^{\nu}\beta_{ijl}c_l,\quad i=1,2,\ldots,\nu,\quad j=1,2,\ldots,\mu. \tag{2.5}$$

the order conditions for (2.2) applied to (2.1) are the same as the order conditions for (1.2) applied to (1.1). As in [4] we also assume that

$$\sum_{l=1}^{\nu}\beta_{ijl}=1,\quad i=1,2,\ldots,\nu,\quad j=1,2,\ldots,\mu. \tag{2.6}$$

Order conditions for VRK methods (2.2) were derived in [4] using a generalization of the RK theory for VIEs of the second kind developed in [7]. This theory is based on formally transforming the VIE (2.1) into an infinite system of partitioned ordinary differential equations (ODEs), and then using the theory of P-series developed by Hairer [14], which for RK methods for VIEs (2.1) reduces to the so-called V-series [7]. Alternatively, order conditions for VRK methods can be also obtained using a generalization of the approach proposed by Albrecht [1,2] for RK methods for ODEs, and extended in [21] to the general class of two-step RK methods. This was illustrated recently by Garrappa [13], who derived order conditions for some classes of RK methods for Volterra integral equations with weakly singular kernels.

In this section we derive the conditions which guarantee that VRK methods (2.2) have order p and stage order $q=p$. This means that

$$hd_i = O(h^{p+1}), \quad i=1,2,\dots,\nu, \quad h\to 0, \tag{2.7}$$

and

$$h\widehat{d} = O(h^{p+1}), \quad h\to 0, \tag{2.8}$$

where hd_i, $i=1,2,\dots,\nu$, are local discretization errors of the stage values $Y_i^{[n]}$, and $h\widehat{d}$ is the local discretization error of y_{n+1}. These errors are defined as residues obtained by replacing $Y_i^{[n]}$ by $y(t_n+c_ih)$, $\widetilde{F}_n(t_n+c_ih)$ by $F_n(t_n+c_ih)$, and y_{n+1} by $y(t_{n+1})$ in (2.2), where $y(t)$ is the solution to (2.1), i.e.,

$$hd_i := y(t_n+c_ih)-h\sum_{j=1}^{\mu}\alpha_{ij}k\Big(t_n+d_{ij},\sum_{l=1}^{\nu}\beta_{ijl}y(t_n+c_lh)\Big)-F_n(t_n+c_ih), \tag{2.9}$$

$i=1,2,\dots,\nu$, and

$$h\widehat{d} := y(t_{n+1})-\sum_{j=1}^{\nu}w_jy(t_n+c_jh). \tag{2.10}$$

It follows from (2.7) and (2.8) that the stage order and order conditions can be obtained by expanding hd_i, $i=1,2,\dots,\nu$, and $h\widehat{d}$ into Taylor series around the point t_n and equating to zero the coefficients of the resulting elementary differentials up to stage order $q=p$ and order p. These elementary differentials depend on the y, k, derivatives of y, partial derivatives of k, and their combinations. In what follows we will illustrate this process to derive stage order and order conditions up to the order $p=4$ and stage order $q=4$.

The expressions hd_i and $h\widehat{d}$ have the following Taylor expansions up to the order four

$$\begin{aligned}
hd_i &= y(t_n)-F_n(t_n)+(y'(t_n)-F_n'(t_n))c_ih+(y''(t_n)-F_n''(t_n))\frac{c_i^2h^2}{2}\\
&+(y'''(t_n)-F_n'''(t_n))\frac{c_i^3h^3}{6}+(y^{(4)}(t_n)-F_n^{(4)}(t_n))\frac{c_i^4h^4}{24}\\
&-h\sum_{j=1}^{\mu}\alpha_{ij}k\Big(t_n+d_{ij}h,y(t_n)+\sum_{l=1}^{\nu}\beta_{ijl}c_ly'(t_n)h\\
&+\sum_{l=1}^{\nu}\beta_{ijl}\frac{c_l^2}{2}y''(t_n)h^2+\sum_{l=1}^{\nu}\beta_{ijl}\frac{c_l^3}{6}y'''(t_n)h^3\Big)+O(h^5),
\end{aligned}$$

$i=1,2,\dots,\nu$, where we have used (2.6), and

$$\begin{aligned}
h\widehat{d} &= \Big(1-\sum_{j=1}^{\nu}w_j\Big)y(t_n)+\Big(1-\sum_{j=1}^{\nu}w_jc_j\Big)y'(t_n)h+\Big(1-\sum_{j=1}^{\nu}w_jc_j^2\Big)y''(t_n)\frac{h^2}{2}\\
&+\Big(1-\sum_{j=1}^{\nu}w_jc_j^3\Big)y'''(t_n)\frac{h^3}{6}+\Big(1-\sum_{j=1}^{\nu}w_jc_j^4\Big)y^{(4)}(t_n)\frac{h^4}{24}+O(h^5).
\end{aligned}$$

$p=q$	differential	conditions for order p and stage order q
$p=0$	y	$\sum_{j=1}^{\nu} w_j = 1$
$p=1$	y'	$\sum_{j=1}^{\nu} w_j c_j = 1$
$q=1$	k	$\sum_{j=1}^{\mu} \alpha_{ij} = c_i$
$p=2$	y''	$\sum_{j=1}^{\mu} w_j c_j^2 = 1$
$q=2$	$\frac{\partial k}{\partial t}$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij} = c_i^2$
$q=2$	$\frac{\partial k}{\partial y} y'$	$\sum_{j=1}^{\mu} \alpha_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l = \frac{c_i^2}{2}$

Table 1 Order and stage order conditions for $p=0$, $p=q=1$, and $p=q=2$

We evaluate next $y(t)-F_n(t)$, $y'(t)-F_n'(t)$, $y''(t)-F_n''(t)$, $y'''(t)-F_n'''(t)$, and $y^{(4)}(t)-F_n^{(4)}(t)$ for $t=t_n$. After some computations it follows from (1.3) that

$$\begin{aligned}
&y - F_n = 0, \quad y' - F_n' = k, \quad y'' - F_n'' = 2\tfrac{\partial k}{\partial t} + \tfrac{\partial k}{\partial y} y', \\
&y''' - F_n''' = 3\tfrac{\partial^2 k}{\partial t^2} + 3\tfrac{\partial^2 k}{\partial t \partial y} y' + \tfrac{\partial^2 k}{\partial y^2} y'^{\,2} + \tfrac{\partial k}{\partial y} y'', \\
&y^{(4)} - F_n^{(4)} = 4\tfrac{\partial^3 k}{\partial t^3} + 6\tfrac{\partial^3 k}{\partial t^2 \partial y} y' + 4\tfrac{\partial^3 k}{\partial t \partial y^2} y'^{\,2} + 4\tfrac{\partial^2 k}{\partial t \partial y} y'' \\
&\qquad\qquad + \tfrac{\partial^3 k}{\partial y^3} y'^{\,3} + 3\tfrac{\partial^2 k}{\partial y^2} y' y'' + \tfrac{\partial k}{\partial y} y'''.
\end{aligned}$$

where we have skipped the arguments in y, F_n, and their derivatives and in k and its partial derivatives. Substituting the above expressions into hd_i, $i=1,2,\ldots,\nu$, and expanding the function k appearing in hd_i into Taylor series around the point $(t_n, y(t_n))$, and then equating to zero the coefficients of the resulting elementary differentials we obtain stage order conditions up to the stage order $q=4$. Similarly, equating to zero the powers of h in $h\widehat{d}$ up to the order four we obtain the order conditions of the form $\sum_{j=1}^{\nu} w_j c_j^k = 1$, $k=0,1,2,3,4$. We will refer to these conditions as quadrature order conditions. These order and stage order conditions are listed in Table 1 for $p=0$, $p=q=1$ and $p=q=2$, in Table 2 for $p=q=3$, and in Table 3 for $p=q=4$. Observe that multiplying the stage order conditions by w_i and then summing the resulting expressions from $i=1$ to $i=\nu$ and taking into account the quadrature order conditions we obtain stage order conditions derived in [4]. Observe also that for $p=q=3$ the last two of the stage order conditions listed in Table 1 in [4] are not necessary.

The class of methods we are interested in this paper are natural VRK methods with $p=q=\mu=\nu$, i.e., methods with the tail approximation defined by (2.4).

We conclude this section by listing in Table 4 the number of conditions (which include order and stage order conditions and relations (2.6)) and the number of free parameters c_j, w_j, α_{ij}, d_{ij}, and β_{ijl} for natural VRK methods with $p=q=\mu=\nu$, for $\nu=1$, 2, 3, and 4. Constructing such methods we will usually assume that the last component c_ν of the abscissa vector c is equal to one.

$p=q$	differential	conditions for order p and stage order q
$p=3$	y'''	$\sum_{j=1}^{\nu} w_j c_j^3 = 1$
$q=3$	$\frac{\partial^2 k}{\partial t^2}$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij}^2 = c_i^3$
$q=3$	$\frac{\partial^2 k}{\partial t \partial y} y'$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l = \frac{c_i^3}{2}$
$q=3$	$\frac{\partial^2 k}{\partial y^2} y'^{\,2}$	$\sum_{j=1}^{\mu} \alpha_{ij} \big(\sum_{l=1}^{\nu} \beta_{ijl} c_l \big)^2 = \frac{c_i^3}{3}$
$q=3$	$\frac{\partial k}{\partial y} y''$	$\sum_{j=1}^{\mu} \alpha_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l^2 = \frac{c_i^3}{3}$

Table 2 Order and stage order conditions for $p = q = 3$

$p=q$	differential	conditions for order p and stage order q
$p=4$	$y^{(4)}$	$\sum_{j=1}^{\nu} w_j c_j^4 = 1$
$q=4$	$\frac{\partial^3 k}{\partial t^3}$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij}^3 = c_i^4$
$q=4$	$\frac{\partial^3 k}{\partial t \partial y^2} y'^{\,2}$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij} \big(\sum_{l=1}^{\nu} \beta_{ijl} c_l \big)^2 = \frac{c_i^4}{3}$
$q=4$	$\frac{\partial^3 k}{\partial t^2 \partial y} y'$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij}^2 \sum_{l=1}^{\nu} \beta_{ijl} c_l = \frac{c_i^4}{2}$
$q=4$	$\frac{\partial^2 k}{\partial t \partial y} y''$	$\sum_{j=1}^{\mu} \alpha_{ij} d_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l^2 = \frac{c_i^4}{3}$
$q=4$	$\frac{\partial^2 k}{\partial y^2} y' y''$	$\sum_{j=1}^{\mu} \alpha_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l \sum_{l=1}^{\nu} \beta_{ijl} c_l^2 = \frac{c_i^4}{4}$
$q=4$	$\frac{\partial^3 k}{\partial y^3} y'^{\,3}$	$\sum_{j=1}^{\mu} \alpha_{ij} \big(\sum_{l=1}^{\nu} \beta_{ijl} c_l \big)^3 = \frac{c_i^4}{4}$
$q=4$	$\frac{\partial k}{\partial y} y'''$	$\sum_{j=1}^{\mu} \alpha_{ij} \sum_{l=1}^{\nu} \beta_{ijl} c_l^3 = \frac{c_i^4}{4}$

Table 3 Order and stage order conditions for $p = q = 4$

$p=q=\mu=\nu$	# of conditions: $q=p$	# of parameters
1	4	5
2	13	20
3	34	51
4	77	104

Table 4 Number of conditions and number of free parameters for natural VRK methods with $p = q = \mu = \nu$, for $\nu = 1$, 2, 3, and 4

3 Stability analysis with respect to the basic test equation

In this section we investigate stability properties of VRK methods (2.2) with the tail approximation defined by (2.4) with respect to the basic test equation

$$y(t) = 1 + \lambda \int_0^t y(s)ds, \quad t \geq 0, \tag{3.1}$$

where $\lambda \in \mathbb{C}$. We will follow the approach of [4]. Applying VRK method (2.2) to (3.1) we obtain

$$\begin{aligned} Y_i^{[n]} &= h\lambda \sum_{l=1}^{\nu} \sum_{j=1}^{\mu} \alpha_{ij}\beta_{ijl} Y_l^{[n]} + \widetilde{F}_n(t_n + c_i h), \\ y_{n+1} &= \sum_{j=1}^{\nu} w_j Y_j^{[n]}, \end{aligned} \tag{3.2}$$

$n = 0, 1, \ldots$, where the tail approximation (2.4) takes now the form

$$\widetilde{F}_n(t) = 1 + h\lambda \sum_{\kappa=1}^{n} \sum_{l=1}^{m} v_l u(t_{\kappa-1} + \xi_l h). \tag{3.3}$$

It follows from (1.5) that $u(t_{k-1} + \xi_l h) = \sum_{j=1}^{\nu} w_j(\xi_l) Y_j^{[k-1]}$ and the relation (3.3) can be written as

$$\widetilde{F}_n(t) = 1 + h\lambda \sum_{\kappa=1}^{n} \sum_{j=1}^{\nu} b_j Y_j^{[\kappa-1]} \tag{3.4}$$

if we define $b_j = \sum_{l=1}^{m} v_l w_j(\xi_l)$. Putting $z = h\lambda$, $e = [1, \ldots, 1]^T \in \mathbb{R}^\nu$,

$$b = [b_1, \ldots, b_\nu]^T, \quad A = [a_{il}]_{i,l=1}^{\nu}, \quad a_{il} = \sum_{j=1}^{\mu} \alpha_{ij}\beta_{ijl}, \quad Y^{[\kappa]} = \left[Y_1^{[\kappa]}, \ldots, Y_\nu^{[\kappa]}\right]^T,$$

and assuming that $I - zA$ is invertible the relation (3.2) with $\widetilde{F}_n(t)$ given by (3.4) can be written in the vector form

$$Y^{[n]} = \left(1 + z \sum_{\kappa=1}^{n} b^T Y^{[\kappa-1]}\right)(I - zA)^{-1} e. \tag{3.5}$$

Here, I is the identity matrix of dimension ν. Observe that for VRK methods of stage order $q \geq 1$ we have

$$\sum_{l=1}^{\nu} a_{il} = \sum_{l=1}^{\nu} \sum_{j=1}^{\mu} \alpha_{ij}\beta_{ijl} = \sum_{j=1}^{\mu} \alpha_{ij} \sum_{l=1}^{\nu} \beta_{ijl} = \sum_{j=1}^{\mu} \alpha_{ij} = c_i,$$

$i = 1, 2, \ldots, \nu$, where we have used (2.6) and stage order condition corresponding to $q = 1$. Let $R(z)$ be the rational function defined by

$$R(z) := 1 + z b^T (I - zA)^{-1} e. \tag{3.6}$$

Then it can be demonstrated that

$$Y^{[n]} = \big(R(z)\big)^n (I - zA)^{-1} e, \tag{3.7}$$

$n = 0, 1, \ldots$, compare [5]. It follows from the second formula in (3.2) and (3.7) that $y_{n+1} = \big(R(z)\big)^n w^T (I - zA)^{-1} e$, where $w = [w_1, \ldots, w_\nu]^T$. Hence,

$$y_{n+1} = R(z) y_n, \tag{3.8}$$

$n = 0, 1, \ldots$. As observed before in [5], and in [17] in the context of extended Pouzet-type methods, it follows from relation (3.8) that stability properties of VRK methods (2.2) with the tail approximation defined by (2.4) with respect to the test equation (3.1) are the same as stability properties of the underlying RK method for ODEs with respect to $y' = \lambda y$, $t \geq 0$. This underlying RK method for ODEs is given by the Butcher table

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,\nu} \\ \vdots & \vdots & \ddots & \vdots \\ c_\nu & a_{\nu,1} & \cdots & a_{\nu,\nu} \\ \hline & b_1 & \cdots & b_\nu \end{array} . \tag{3.9}$$

In particular, the region of absolute stability of a VRK method is given by

$$\mathcal{A} = \big\{ z \in \mathbb{C} : \ |R(z)| < 1 \big\},$$

where the stability function $R(z)$ is defined by (3.6), and the VRK method is A-stable, i.e., its region of absolute stability includes the left half of the complex plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, if the underlying RK method for ODEs is A-stable.

The stability function $R(z)$ can be written as $R(z) = P(z)/Q(z)$, $z \in \mathbb{C}$, where $P(z)$ and $Q(z)$ are polynomials of degree less than or equal to ν. Then it follows that the RK method (3.9) is A-stable if

$$|R(iy)| \leq 1, \quad y \in \mathbb{R}, \tag{3.10}$$

and $R(z)$ is analytic for $\mathrm{Re}(z) < 0$, i.e., the polynomial $Q(z)$ does not have zeros in the negative half plane (compare [15]). The condition (3.10) is equivalent to the fact that the so-called Nørsett polynomial defined by

$$E(y) := |Q(iy)|^2 - |P(iy)|^2 = Q(iy)Q(-iy) - P(iy)P(-iy)$$

satisfies the condition

$$E(y) \geq 0, \quad y \in \mathbb{R}. \tag{3.11}$$

The above observations were used in [5] to characterize VRK methods of collocation type. In Section 5 we will use these results to investigate if the natural VRK methods of order p and stage order $q = p$, for $p = 1$, 2, 3, and 4, are A-stable.

4 Stability analysis with respect to the convolution test equation

In this section we investigate stability properties of VRK methods (2.2) with tail approximation defined by (2.4) with respect to the convolution test equation

$$y(t) = 1 + \int_0^t \big(\lambda + \xi(t-s)\big)y(s)ds, \quad t \geq 0, \tag{4.1}$$

where $\lambda, \xi \in \mathbb{R}$. The solution $y(t)$ to this equation tends to zero as t tends to infinity if and only if $\lambda < 0$ and $\xi \leq 0$ (compare [9]), and we will investigate whether this property is inherited by the numerical solution $\{y_n\}_{n=0}^{\infty}$ obtained by application of the VRK method (2.2) with tail approximation given by (2.4) to the test equation (4.1). It can be verified that this numerical solution depends on the parameters $h\lambda$ and $h^2\xi$, where h is the stepsize used. The VRK method is said to be stable for given $(h\lambda, h^2\xi)$ if $y_n = y_n(h\lambda, h^2\xi) \to 0$ as $n \to \infty$. The region of stability $\mathcal{S}$ of a VRK method with respect to (4.1) is the set of all $(h\lambda, h^2\xi)$ for which the method is stable, i.e.,

$$\mathcal{S} := \big\{(h\lambda, h^2\xi) \in \mathbb{R}^2 : \; y_n(h\lambda, h^2\xi) \to 0 \quad \text{as} \quad n \to \infty\big\}. \tag{4.2}$$

The VRK method is said to be V_0-stable if its region of stability includes the set $h\lambda < 0$ and $h^2\xi < 0$, i.e., $\{(h\lambda, h^2\xi) \in \mathbb{R}^2 : h\lambda < 0,\ h^2\xi < 0\} \subset \mathcal{S}$.

The V_0-stability proved to be a very demanding property of numerical methods and only a few examples of such formulas were discovered so far in the literature on the subject. Wolkenfelt [25] has proved that no such formulas exist in the class of reducible quadrature methods for (1.1). Further negative results about nonexistence of V_0-stable methods of some types were obtained in [3,16,17]. A first example of a V_0-stable method was given in the report [9]. This is a first order RK method of Bel'tyukov type with tail approximation given by the composite right rectangular quadrature formula. A first example of a VRK method of second order was reported in [5]. V_0-stable RK methods of Bel'tyukov type of order $p = 3$ and $p = 4$ were discovered only very recently by Izzo et al. [19]. In this paper we will present new examples of V_0-stable natural VRK methods of order p and stage order $q = p$ up to the order four.

We will follow again the approach of [5]. Applying VRK method (2.2) to (4.1) we obtain

$$\begin{aligned}
Y_i^{[n]} &= h\lambda \sum_{l=1}^{\nu}\sum_{j=1}^{\mu} \alpha_{ij}\beta_{ijl}Y_l^{[n]} + h^2\xi \sum_{l=1}^{\nu}\sum_{j=1}^{\mu} \alpha_{ij}d_{ij}\beta_{ijl}Y_l^{[n]} \\
&\quad - h^2\xi \sum_{l=1}^{\nu}\sum_{j=1}^{\mu} \alpha_{ij}e_{ij}\beta_{ijl}Y_l^{[n]} + \widetilde{F}_n(t_n + c_i h), \\
y_{n+1} &= \sum_{j=1}^{\nu} w_j Y_j^{[n]},
\end{aligned} \tag{4.3}$$

$i = 1, 2, \ldots, \nu$, where the tail approximation (2.4) now takes the form

$$\widetilde{F}_n(t_n + c_i h) = 1 + h\lambda \sum_{\kappa=1}^{n} \sum_{l=1}^{m} v_l u(t_{\kappa-1} + \xi_l h)$$
$$+ h^2\xi \sum_{\kappa=1}^{n} \sum_{l=1}^{m} v_l (n - \kappa + 1) u(t_{\kappa-1} + \xi_l h) + h^2\xi \sum_{\kappa=1}^{n} \sum_{l=1}^{m} v_l (c_i - \xi_l) u(t_{\kappa-1} + \xi_l h). \tag{4.4}$$

Putting

$$b_j = \sum_{l=1}^{m} v_l w_j(\xi_l), \quad r_j = \sum_{l=1}^{m} v_l \xi_l w_j(\xi_l), \tag{4.5}$$

it follows from (1.5) that (4.4) can be rewritten in the form

$$\begin{aligned} \widetilde{F}_n(t_n + c_i h) &= 1 + h\lambda \sum_{\kappa=1}^{n} \sum_{j=1}^{\nu} b_j Y_j^{[\kappa-1]} \\ &+ h^2\xi \sum_{\kappa=1}^{n} \sum_{j=1}^{\nu} b_j (n - \kappa + 1) Y_j^{[\kappa-1]} + h^2\xi \sum_{\kappa=1}^{n} \sum_{j=1}^{\nu} (b_j c_i - r_j) Y_j^{[\kappa-1]}. \end{aligned} \tag{4.6}$$

Putting $x = h\lambda$, $y = h^2\xi$, $r = [r_1, \ldots, r_\nu]^T$, $e = [1, \ldots, 1]^T \in \mathbb{R}^\nu$,

$$A = [a_{il}]_{i,l=1}^{\nu}, \quad a_{il} = \sum_{j=1}^{\mu} \alpha_{ij} \beta_{ijl}, \quad B = [b_{il}]_{i,l=1}^{\nu}, \quad b_{il} = \sum_{j=1}^{\mu} \alpha_{ij} d_{ij} \beta_{ijl},$$

$$C = [c_{il}]_{i,l=1}^{\nu}, \quad c_{il} = \sum_{j=1}^{\mu} \alpha_{ij} e_{ij} \beta_{ijl}, \quad b = [b_1, \ldots, b_\nu]^T, \quad c = [c_1, \ldots, c_\nu]^T,$$

$$Y^{[\kappa]} = \left[Y_1^{[\kappa]}, \ldots, Y_\nu^{[\kappa]}\right]^T, \quad F^{[\kappa]} = \left[\widetilde{F}_\kappa(t_\kappa + c_1 h), \ldots, \widetilde{F}_\kappa(t_\kappa + c_\nu h)\right]^T,$$

the relation (4.3) can be written in the vector form

$$Y^{[n]} = \big(xA + y(B - C)\big) Y^{[n]} + F^{[n]}, \tag{4.7}$$

and the tail approximation (4.6) takes the form

$$F^{[n]} = e + \sum_{\kappa=1}^{n} \big(xeb^T + y(n - \kappa + 1) eb^T + y(cb^T - er^T)\big) Y^{[\kappa-1]}, \tag{4.8}$$

where $r = b.c := [b_1 c_1, \ldots, b_\nu c_\nu]^T$. As in [5] we can reduce (4.7) and (4.8) to a vector difference equation of order two. Putting

$$Q = Q(x, y) = I - xA - y(B - C) \tag{4.9}$$

the relation (4.7) takes the form $QY^{[n]} = F^{[n]}$. It can be verified that (4.8) yields

$$\begin{aligned} F^{[n+2]} - 2F^{[n+1]} + F^{[n]} &= \big((x + y) eb^T + y(cb^T - er^T)\big) Y^{[n+1]} \\ &- \big(xeb^T + y(cb^T - er^T)\big) Y^{[n]}, \end{aligned}$$

and substituting this into the relation

$$Q\big(Y^{[n+2]} - 2Y^{[n+1]} + Y^{[n]}\big) = F^{[n+2]} - 2F^{[n+1]} + F^{[n]}$$

we obtain

$$\begin{aligned} QY^{[n+2]} &= \big(2Q + (x+y)eb^T + y(cb^T - er^T)\big)Y^{[n+1]} \\ &\quad - \big(Q + xeb^T + y(cb^T - er^T)\big)Y^{[n]}, \end{aligned} \tag{4.10}$$

$n = 0, 1, \ldots$. This is the desired vector recurrence relation of order two. In what follows we assume that the matrix Q defined by (4.9) is nonsingular and we define the vectors $\widetilde{e} = Q^{-1}e$, $\widetilde{c} = Q^{-1}c$. We will look for solutions to (4.10) in the form

$$Y^{[n]} = \alpha_n \widetilde{e} + \beta_n \widetilde{c}, \tag{4.11}$$

where α_n and β_n are some scalars which have to be determined. Substituting (4.11) into (4.10) and comparing the coefficients of e and c in the resulting expressions, and taking into account that the vectors e and c are linearly independent we obtain

$$\begin{aligned} \alpha_{n+2} &= \big(2 + (x+y)b^T\widetilde{e} - yr^T\widetilde{e}\big)\alpha_{n+1} + \big((x+y)b^T\widetilde{c} - yr^T\widetilde{c}\big)\beta_{n+1} \\ &\quad - (1 + xb^T\widetilde{e} - yr^T\widetilde{e})\alpha_n - (xb^T\widetilde{c} - yr^T\widetilde{c})\beta_n, \end{aligned}$$

$$\beta_{n+2} = yb^T\widetilde{e}\alpha_{n+1} + (2 + yb^T\widetilde{c})\beta_{n+1} - yb^T\widetilde{e}\alpha_n - (1 + yb^T\widetilde{c})\beta_n,$$

$n = 0, 1, \ldots$. Next, we will reduce the above recurrence relations for α_n and β_n to a vector difference equation of the first order. Let

$$M = M(x,y) = \begin{bmatrix} 2 + (x+y)b^T\widetilde{e} - yr^T\widetilde{e} & (x+y)b^T\widetilde{c} - yr^T\widetilde{c} \\ yb^T\widetilde{e} & 2 + yb^T\widetilde{c} \end{bmatrix},$$

$$N = N(x,y) = \begin{bmatrix} -1 - xb^T\widetilde{e} + yr^T\widetilde{e} & -xb^T\widetilde{c} + yr^T\widetilde{c} \\ -yb^T\widetilde{e} & -1 - yb^T\widetilde{c} \end{bmatrix},$$

Then the system for α_{n+2} and β_{n+2} can be written in a compact form

$$v_{n+1} = S\, v_n, \tag{4.12}$$

$n = 0, 1, \ldots$, where

$$S = S(x,y) = \left[\begin{array}{c|c} M & N \\ \hline I & 0 \end{array}\right] \in \mathbb{R}^{4\times 4}, \quad v_n = \begin{bmatrix} \alpha_{n+1} & \beta_{n+1} & \alpha_n & \beta_n \end{bmatrix}^T \in \mathbb{R}^4.$$

It follows from (4.7) and (4.8) that $Y^{[0]} = Q^{-1}F^{[0]} = \widetilde{e}$,

$$Y^{[1]} = Q^{-1}F^{[1]} = \big(1 + (x+y)b^T\widetilde{e} - yr^T\widetilde{e}\big)\widetilde{e} + yb^T\widetilde{e}\,\widetilde{c},$$

which implies that $\alpha_0 = 1$, $\beta_0 = 0$, $\alpha_1 = 1 + (x+y)b^T\widetilde{e} - yr^T\widetilde{e}$, $\beta_1 = yb^T\widetilde{e}$. Hence, the initial vector v_0 of the recurrence equation (4.12) takes the form

$$v_0 = \Big[\alpha_1\ \beta_1\ \alpha_0\ \beta_0\Big]^T = \Big[1 + (x+y)b^T\widetilde{e} - yr^T\widetilde{e} \quad yb^T\widetilde{e} \quad 1 \quad 0\Big]^T.$$

The stability properties of VRK method (2.2) with the tail approximation (2.4) with respect to the convolution test equation (4.1) are determined by the characteristic polynomial $\phi(\theta)$ of the amplification matrix $S = S(x,y)$ appearing in (4.12). This polynomial takes the form

$$\phi(\theta) = \det(\theta I - S) = \det(\theta^2 I - \theta M - N).$$

It can be verified that $\phi(\theta) = (\theta-1)^2\psi(\theta)$ with

$$\begin{aligned}\psi(\theta) = \theta^2 &- \big(2 + (x+y)b^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e})\big)\theta \\ &+ 1 + xb^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e}) + y^2(b^T\widetilde{e}\,r^T\widetilde{c} - b^T\widetilde{c}\,r^T\widetilde{e}),\end{aligned} \tag{4.13}$$

where $\theta = 1$ is an eigenvalue of S of algebraic multiplicity two and geometric multiplicity one. It was demonstrated in [5] that this double eigenvalue $\theta = 1$ does not affect the stability properties of VRK methods (2.2) with respect to (4.1). As a result, the stability region of VRK method (2.2) with respect to the test equation (4.1) can be characterized as

$$\mathcal{S} = \big\{(x,y) \in \mathbb{R}^2 : \ |\theta_1(x,y)| < 1 \quad \text{and} \quad |\theta_2(x,y)| < 1\big\},$$

where $\theta_1 = \theta_1(x,y)$ and $\theta_2 = \theta_2(x,y)$ are the roots of the polynomial $\psi(\theta)$ defined by (4.13). Putting $y = 0$ in (4.13) this polynomial $\psi(\theta)$ reduces to

$$\psi(\theta) = \theta^2 - (2 + xb^T\widetilde{e})\theta + 1 + xb^T\widetilde{e},$$

with the roots $\theta = 1$ and $\theta = 1 + xb^T\widetilde{e} = 1 + xb^T(I - xA)^{-1}e$. The latter root corresponds to the stability function $R(z)$ defined by (3.6) with $z = x$, which was encountered in the stability analysis of VRK methods (2.2) with respect to the basic test equation (3.1).

We will use the Schur criterion [24,22,20] to find conditions under which the roots $\theta_1 = \theta_1(x,y)$ and $\theta_2 = \theta_2(x,y)$ of $\psi(\theta)$ are inside of the unit circle. This criterion implies that this is the case if and only if

$$\big|1 + xb^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e}) + y^2(b^T\widetilde{e}\,r^T\widetilde{c} - b^T\widetilde{c}\,r^T\widetilde{e})\big| < 1, \tag{4.14}$$

$$\begin{aligned}&\big|2 + (x+y)b^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e})\big| \\ &\quad < \ \big|2 + xb^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e}) + y^2(b^T\widetilde{e}\,r^T\widetilde{c} - b^T\widetilde{c}\,r^T\widetilde{e})\big|.\end{aligned} \tag{4.15}$$

To search for V_0-stable methods it is more convenient to reformulate (4.14) and (4.15) in the form

$$\varphi(x,y) > 0, \quad x, y \le 0, \tag{4.16}$$

$$\psi(x,y) > 0, \quad x, y \le 0, \tag{4.17}$$

where the rational functions $\varphi(x,y)$ and $\psi(x,y)$ are defined by

$$\varphi(x,y) := 1 - \big(1 + xb^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e}) + y^2(b^T\widetilde{e}\,r^T\widetilde{c} - b^T\widetilde{c}\,r^T\widetilde{e})\big)^2,$$

$$\begin{aligned}\psi(x,y) := {} & \big(2 + xb^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e}) + y^2(b^T\widetilde{e}\,r^T\widetilde{c} - b^T\widetilde{c}\,r^T\widetilde{e})\big)^2 \\ & - \big(2 + (x+y)b^T\widetilde{e} + y(b^T\widetilde{c} - r^T\widetilde{e})\big)^2.\end{aligned}$$

For VRK methods with $\mu = \nu$ these functions take the form

$$\varphi(x,y) = \frac{\displaystyle\sum_{0\le i+j\le 2\nu} (-1)^{i+j}\xi_{ij}^{(1)} x^i y^j}{\Bigg(\displaystyle\sum_{0\le i+j\le \nu} \eta_{ij}^{(1)} x^i y^j\Bigg)^2}, \quad \psi(x,y) = \frac{\displaystyle\sum_{0\le i+j\le 2\nu} (-1)^{i+j}\xi_{ij}^{(2)} x^i y^j}{\Bigg(\displaystyle\sum_{0\le i+j\le \nu} \eta_{ij}^{(2)} x^i y^j\Bigg)^2},$$

$\xi_{00}^{(1)} = \xi_{00}^{(2)} = 0$, $\eta_{00}^{(1)} = \eta_{00}^{(2)} = 1$, where the coefficients $\xi_{ij}^{(1)}$, $\eta_{ij}^{(1)}$, $\xi_{ij}^{(2)}$, and $\eta_{ij}^{(2)}$ depend on the remaining free parameters of the methods. Then the sufficient conditions for V_0-stability take the form

$$\xi_{ij}^{(k)} > 0, \quad i,j = 0,1,\dots,\nu, \quad 1 \le i+j \le 2\nu, \quad k = 1,2. \tag{4.18}$$

To enforce the conditions (4.16) and (4.17) we can also consider the polynomials

$$\gamma(x,t) = \sum_{k=0}^{2\nu}\Bigg(\sum_{l=0}^{k}\eta_l^{(1)}t^l\Bigg)(-x)^k, \quad \delta(x,t) = \sum_{k=0}^{2\nu}\Bigg(\sum_{l=0}^{k}\eta_l^{(2)}t^l\Bigg)(-x)^k,$$

obtained by substituting $y = t\,x$ in the numerators of $\varphi(x,y)$ and $\psi(x,y)$. Then the sufficient conditions for V_0-stability of VRK methods, which are less restrictive than the conditions (4.18), take the form

$$\eta_k^{(1)} > 0, \quad \eta_k^{(2)} > 0, \quad k = 0,1,\dots,2\nu, \tag{4.19}$$

and

$$t_i^{(1)} \le 0, \quad t_i^{(2)} \le 0, \tag{4.20}$$

where $t_i^{(1)}$ and $t_i^{(2)}$ are real roots of the polynomials $\sum_{l=0}^{k}\eta_l^{(1)}t^l$ and $\sum_{l=0}^{k}\eta_l^{(2)}t^l$, $k = 1,2,\dots,2\nu$, appearing in $\gamma(x,t)$ and $\delta(x,t)$.

It is easy to see that the conditions (4.18) imply the conditions (4.19) and (4.20) and that the converse is not true. For example, the V_0-stable Bel'tyukov VRK methods of order $p = 3$ and $p = 4$ constructed in [19] satisfy (4.19) and (4.20), but do not satisfy (4.18).

In the next section we present examples of natural VRK methods up to order four and stage order four which satisfy the conditions (4.18) or (4.19) and (4.20) as well as the condition (3.11) in Section 3. This will lead to methods which are both A- and V_0-stable.

5 Examples of *A*- and *V*$_0$-stable natural VRK methods

5.1 Natural VRK methods with $p = q = \mu = \nu = 1$

It follows from order and stage order conditions and relations (2.5) and (2.6) that $w_1 = 1$, $c_1 = 1$, $\alpha_{11} = 1$, $\beta_{111} = 1$, and $e_{11} = 1$. Moreover, d_{11} must satisfy the kernel condition $e_{11} \leq d_{11}$. Put $d = d_{11}$. It follows from (1.6) that $w_1(\theta) = 1$, $\theta \in (0, 1]$, and the resulting method is

$$\begin{aligned} &Y_1^{[n]} = hk(t_n + dh, Y_1^{[n]}) + \widetilde{F}_n(t_{n+1}), \\ &y_{n+1} = Y_1^{[n]}, \quad u(t_n + \theta h) = Y_1^{[n]}, \quad \theta \in (0, 1], \end{aligned} \tag{5.1}$$

$n = 0, 1, \ldots, N-1$. Observe that u is discontinuous at the grid points. Choosing $m = 1$, $v_1 = 1$, $\xi_1 = 1$, the tail approximation (2.4) takes the form

$$\widetilde{F}_n(t) = g(t) + h \sum_{\kappa=1}^{n} k\big(t, u(t_\kappa)\big) = g(t) + h \sum_{\kappa=1}^{n} k\big(t, Y_1^{[\kappa-1]}\big). \tag{5.2}$$

We have $b_1 = v_1 w_1 = 1$, $a_{11} = \alpha_{11}\beta_{111} = 1$, and the underlying RK method is the backward Euler method which is A- and L-stable.

We next investigate V_0-stability. It can be verified that for the method (5.1), with the tail approximation defined by (5.2), the stability polynomial $\psi(\theta)$ equivalent to (4.13) takes the form

$$\psi(\theta) = \big(1 - x + (1-d)y\big)\theta^2 - \big(2 - x + (3 - 2d)y\big)\theta + 1 + (1-d)y.$$

It now follows from the Schur criterion that the region of V_0-stability is the set of all points $(x, y) \in \mathbb{R}^2$ such that

$$x\big(-2 + x - 2(1-d)y\big) \geq 0 \quad \text{and} \quad y\big(-2(2-x) - (5 - 4d)y\big) \geq 0.$$

Hence, the method (5.1)-(5.2) is V_0-stable if and only if $d \geq 5/4$. This is consistent with the results of [5].

5.2 Natural VRK methods with $p = q = \mu = \nu = 2$

Natural VRK methods with $\mu = \nu = 2$ take the form

$$\begin{aligned} Y_1^{[n]} &= h\alpha_{11} k\big(t_n + d_{11}h, \beta_{111}Y_1^{[n]} + \beta_{112}Y_2^{[n]}\big) \\ &\quad + h\alpha_{12} k\big(t_n + d_{12}h, \beta_{121}Y_1^{[n]} + \beta_{122}Y_2^{[n]}\big) + \widetilde{F}_n(t_n + c_1 h), \\ Y_2^{[n]} &= h\alpha_{21} k\big(t_n + d_{21}h, \beta_{211}Y_1^{[n]} + \beta_{212}Y_2^{[n]}\big) \\ &\quad + h\alpha_{22} k\big(t_n + d_{22}h, \beta_{221}Y_1^{[n]} + \beta_{222}Y_2^{[n]}\big) + \widetilde{F}_n(t_n + c_2 h), \\ y_{n+1} &= w_1 Y_1^{[n]} + w_2 Y_2^{[n]}, \end{aligned} \tag{5.3}$$

where

$$\widetilde{F}_n(t) = h \sum_{\kappa=1}^{n} \Big(v_1 k\big(t, u(t_{\kappa-1} + \xi_1 h)\big) + v_2 k\big(t, u(t_{\kappa-1} + \xi_2 h)\big)\Big), \tag{5.4}$$

and u is the natural continuous extension defined by

$$u(t_{\kappa-1}+\theta h)=w_1(\theta)Y_1^{[\kappa-1]}+w_2(\theta)Y_2^{[\kappa-1]}, \tag{5.5}$$

$k=1,2,\ldots,n$, $\theta\in[0,1]$.

To satisfy the kernel conditions, $e_{ij}\le d_{ij}$, we assume that $d_{ij}=e_{ij}+p_{ij}$, where $p_{ij}\ge 0$, $i,j=1,2$. The order conditions are

$$w_1+w_2=1,\quad w_1c_1+w_2c_2=1,\quad w_1c_1^2+w_2c_2^2=1, \tag{5.6}$$

and the stage order conditions expressed in terms of p_{ij} and e_{ij} take the form

$$\begin{aligned}&\alpha_{11}+\alpha_{12}=c_1,\ \alpha_{11}p_{11}+\alpha_{12}p_{12}=\frac{c_1^2}{2},\ \alpha_{11}e_{11}+\alpha_{12}e_{12}=\frac{c_1^2}{2},\\ &\alpha_{21}+\alpha_{22}=c_2,\ \alpha_{21}p_{21}+\alpha_{22}p_{22}=\frac{c_2^2}{2},\ \alpha_{21}e_{21}+\alpha_{22}e_{22}=\frac{c_2^2}{2}.\end{aligned} \tag{5.7}$$

Moreover, the conditions on the weights v_i and abscissas ξ_i of the quadrature rule (5.4) are

$$v_1+v_2=1,\quad v_1\xi_1+v_2\xi_2=\frac{1}{2},\quad v_1\xi_1^2+v_2\xi_2^2=\frac{1}{3}, \tag{5.8}$$

and the conditions on continuous weights $w_i(\theta)$ of natural continuous extension (5.5) are

$$w_1(\theta)+w_2(\theta)=1,\quad w_1(\theta)c_1+w_2(\theta)c_2=\theta,\quad \theta\in[0,1]. \tag{5.9}$$

Moreover, the coefficients β_{ijl} satisfy the conditions

$$\sum_{l=1}^{2}\beta_{ijl}=1,\quad \sum_{l=1}^{2}\beta_{ijl}c_l=e_{ij},\quad i,j=1,2,$$

which can be used to express β_{ijl} in terms of e_{ij} by solving the above system of linear equations with respect to β_{ijl}.

To satisfy order conditions (5.6) we choose $w_1=0$, $w_2=1$, and $c_2=1$. We also fix $c_1=\frac{1}{6}$. Then we solve stage order conditions (5.7) with respect to α_{11}, α_{12}, α_{21}, α_{22}, e_{11} and e_{21}, the system (5.8) with respect to v_1, v_2, and ξ_1, assuming that $\xi_2=1$, and the system (5.9) with respect to $w_1(\theta)$ and $w_2(\theta)$. This leads to a six parameter family of methods (5.3) depending on p_{11}, p_{12}, p_{21}, p_{22}, e_{12}, and e_{22}, with

$$v_1=\frac{3}{4},\quad v_2=\frac{1}{4},\quad \xi_1=\frac{1}{3},\quad w_1(\theta)=\frac{6}{5}(1-\theta),\quad w_2(\theta)=\frac{1}{5}(6\theta-1).$$

It can be verified that the underlying RK formula of order 2 for ODEs of this family of VRK methods is

$$\begin{array}{c|c} c & A\\ \hline & b^T\end{array}=\begin{array}{c|cc}\frac{1}{6} & \frac{11}{60} & -\frac{1}{60}\\ 1 & \frac{3}{5} & \frac{2}{5}\\ \hline & \frac{3}{5} & \frac{2}{5}\end{array}.$$

The Nørsett polynomial of this method is $E(y) = y^4/36$ and it follows that the above family of VRK methods is A-stable for all choices of the free parameters p_{ij} and e_{ij}, $i, j = 1, 2$.

We will describe next the search for V_0-stable methods. The polynomial equivalent to (4.13), denoted by the same symbol $\psi(\theta)$, takes now the form

$$\psi(\theta) = p_2(x,y)\theta^2 - p_1(x,y)\theta + p_0(x,y),$$

where $p_2(x,y)$, $p_1(x,y)$, and $p_0(x,y)$ also depend on the free parameters of the method. It can be verified using the Schur criterion or the conditions (4.14) and (4.15) that $\psi(\theta)$ is a Schur polynomial if these parameters satisfy a complicated system of eight inequalities which are not listed here. Choosing

$$e_{12} = \frac{8 + p_{11}}{12p_{11} - 1}, \quad e_{22} = \frac{3p_{21} - 1}{3(2p_{21} - 1)},$$

so some of these inequalities become equalities, we obtain a four-parameter family of V_0-stable VRK methods depending on p_{11}, p_{12}, p_{21} and p_{22}. The coefficients of the resulting methods are

$$c = \begin{bmatrix} \frac{1}{6} & 1 \end{bmatrix}^T, \quad w = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad v = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}^T, \quad \xi = \begin{bmatrix} \frac{1}{3} & 1 \end{bmatrix}^T,$$

$$\alpha = \begin{bmatrix} \frac{1-12p_{12}}{72(p_{11}-p_{12})} & \frac{12p_{11}-1}{72(p_{11}-p_{12})} \\ \frac{1-2p_{22}}{2(p_{21}-p_{22})} & \frac{2p_{21}-1}{2(p_{21}-p_{22})} \end{bmatrix}, \quad E = \begin{bmatrix} \frac{8+p_{12}}{12p_{12}-1} & \frac{8+p_{11}}{12p_{11}-1} \\ \frac{1-3p_{22}}{3(1-2p_{22})} & \frac{1-3p_{21}}{3(1-2p_{21})} \end{bmatrix},$$

$$D = \begin{bmatrix} \frac{8-p_{11}+p_{12}+12p_{11}p_{12}}{12p_{12}-1} & \frac{8+p_{11}-p_{12}+12p_{11}p_{12}}{12p_{11}-1} \\ \frac{1+3p_{21}-3p_{22}-6p_{21}p_{22}}{3(1-2p_{22})} & \frac{1-3p_{21}+3p_{22}-6p_{21}p_{22}}{3(1-2p_{21})} \end{bmatrix},$$

$$\beta_1 = \begin{bmatrix} \beta_{111} & \beta_{121} \\ \beta_{211} & \beta_{221} \end{bmatrix} = \begin{bmatrix} \frac{6(9-11p_{12})}{5(1-12p_{12})} & \frac{6(9-11p_{11})}{5(1-12p_{11})} \\ \frac{2(2-3p_{22})}{5(1-2p_{22})} & \frac{2(2-3p_{21})}{5(1-2p_{21})} \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} \beta_{112} & \beta_{122} \\ \beta_{212} & \beta_{222} \end{bmatrix} = \begin{bmatrix} \frac{49-6p_{12}}{5(12p_{12}-1)} & \frac{49-6p_{11}}{5(12p_{11}-1)} \\ \frac{4p_{22}-1}{5(2p_{22}-1)} & \frac{4p_{21}-1}{5(2p_{21}-1)} \end{bmatrix}.$$

5.3 Natural VRK methods with $p = q = \mu = \nu = 3$

Similarly as in the previous subsection we assume that $d_{ij} = e_{ij} + p_{ij}$, where $p_{ij} \geq 0$, $i, j = 1, 2, 3$. The order conditions are

$$\sum_{i=1}^{3} w_i c_i^k = 1, \quad k = 0, 1, 2, 3, \tag{5.10}$$

and the stage order conditions corresponding to $q=1$, $q=2$, and $q=3$ are

$$\sum_{j=1}^{3}\alpha_{ij}=c_i,\quad i=1,2,3, \tag{5.11}$$

$$\sum_{j=1}^{3}\alpha_{ij}p_{ij}=\frac{c_i^2}{2},\quad \sum_{j=1}^{3}\alpha_{ij}e_{ij}=\frac{c_i^2}{2},\quad i=1,2,3, \tag{5.12}$$

$$\sum_{j=1}^{3}\alpha_{ij}p_{ij}^2=\frac{c_i^3}{3},\ \sum_{j=1}^{3}\alpha_{ij}e_{ij}p_{ij}=\frac{c_i^3}{6},\ \sum_{j=1}^{3}\alpha_{ij}e_{ij}^2=\frac{c_i^3}{3},\ \sum_{j=1}^{3}\alpha_{ij}f_{ij}=\frac{c_i^3}{3}, \tag{5.13}$$

$i=1,2,3$, where $e_{ij}=\sum_{l=1}^{3}\beta_{ijl}c_l$, $f_{ij}=\sum_{l=1}^{3}\beta_{ijl}c_l^2$, $i,j=1,2,3$. The conditions on the weights v_i and abscissa ξ_i of the quadrature rule (2.4) corresponding to $m=3$ are $\sum_{i=1}^{3}v_i\xi_i^k=1/(k+1)$, $k=0,1,2$, and the continuous weights of the natural continuous extension (1.5) corresponding to $\nu=3$ are polynomials of degree d, $1=\lfloor p/2\rfloor\le d\le\min\{\nu-1,p\}=2$. We assume $d=2$. The system of equations for $w_j(\theta)$ is $\sum_{j=1}^{3}w_j(\theta)c_j^k=\theta^k$, $k=0,1,2$. To satisfy the order conditions (5.10) we will always assume that $w_1=w_2=0$, $w_3=1$, and $c_3=1$. We can also express the coefficients β_{ijk} in terms of e_{ij} and f_{ij} by solving the system of linear equations

$$\sum_{l=1}^{3}\beta_{ijl}=1,\quad \sum_{l=1}^{3}\beta_{ijl}c_l=e_{ij},\quad \sum_{l=1}^{3}\beta_{ijl}c_l^2=f_{ij},\quad i,j=1,2,3.$$

We next compute the coefficients α_{ij} from stage order conditions (5.11) and (5.12) corresponding to $q=1$ and $q=2$, and e_{ij}, f_{11}, f_{21}, and f_{31} from stage order conditions (5.13) corresponding to $q=3$. This leads to a seventeen parameter family of natural VRK methods of order $p=3$ and stage order $q=3$ depending on c_1, c_2, f_{12}, f_{13}, f_{22}, f_{23}, f_{32}, f_{33}, and p_{ij}, $i,j=1,2,3$. It can be verified that the Nørsett polynomial $E(y)$ for this family of methods depends only on c_1 and c_2 and takes the form

$$E(y)=\frac{-1+2(c_1+c_2)-6c_1c_2}{36}\,y^4+\frac{c_1^2c_2^2}{36}\,y^6.$$

Hence, it follows that the methods are A-stable if $1-2(c_1+c_2)+6c_1c_2\le 0$, and to guarantee A-stability we compute c_2 from $1-2(c_1+c_2)+6c_1c_2=0$, i.e., $c_2=(2c_1-1)/(2(3c_1-1))$.

We next investigate V_0-stability. Performing an extensive computer search in this space of free parameters c_1, f_{12}, f_{13}, f_{22}, f_{23}, f_{32}, f_{33}, and p_{ij}, $i,j=1,2,3$, where we started with many random guesses, we were able to find methods of order $p=3$ and stage order $q=3$ which are A- and V_0-stable. An example of such a method is given in the Appendix.

To our knowledge this is the first example of a VRK method of order $p=3$ and stage order $q=3$ which is A- and V_0-stable. The Bel'tyukov VRK method of order $p=3$ with four stages constructed in [19] is also A- and V_0-stable, but its stage order is only $q=1$.

5.4 Natural VRK methods with $p = q = \mu = \nu = 4$

We assume again that $d_{ij} = e_{ij} + p_{ij}$, where $p_{ij} \geq 0$, $i, j = 1, 2, 3, 4$. The order conditions take the form

$$\sum_{i=1}^{4} w_i c_i^k = 1, \quad k = 0, 1, 2, 3, 4, \tag{5.14}$$

and the stage order conditions up to the stage order $q = 4$ expressed in terms of p_{ij}, e_{ij}, f_{ij}, and g_{ij} are

$$\sum_{j=1}^{4} \alpha_{ij} = c_i, \quad i = 1, 2, 3, 4, \tag{5.15}$$

$$\sum_{j=1}^{4} \alpha_{ij} p_{ij} = \frac{c_i^2}{2}, \quad \sum_{j=1}^{4} \alpha_{ij} e_{ij} = \frac{c_i^2}{2}, \quad i = 1, 2, 3, 4, \tag{5.16}$$

$$\sum_{j=1}^{4} \alpha_{ij} p_{ij}^2 = \frac{c_i^3}{3}, \sum_{j=1}^{4} \alpha_{ij} e_{ij} p_{ij} = \frac{c_i^3}{6}, \sum_{j=1}^{4} \alpha_{ij} e_{ij}^2 = \frac{c_i^3}{3}, \sum_{j=1}^{4} \alpha_{ij} f_{ij} = \frac{c_i^3}{3}, \tag{5.17}$$

$$\begin{aligned} &\sum_{j=1}^{4} \alpha_{ij} p_{ij}^3 = \frac{c_i^4}{4}, \sum_{j=1}^{4} \alpha_{ij} e_{ij}^2 p_{ij} = \frac{c_i^4}{12}, \sum_{j=1}^{4} \alpha_{ij} e_{ij} p_{ij}^2 = \frac{c_i^4}{12}, \\ &\sum_{j=1}^{4} \alpha_{ij} p_{ij} f_{ij} = \frac{c_i^4}{12}, \sum_{j=1}^{4} \alpha_{ij} e_{ij} f_{ij} = \frac{c_i^4}{4}, \sum_{j=1}^{4} \alpha_{ij} e_{ij}^3 = \frac{c_i^4}{4}, \sum_{j=1}^{4} \alpha_{ij} g_{ij} = \frac{c_i^4}{4}, \end{aligned} \tag{5.18}$$

$i = 1, 2, 3, 4$, where

$$e_{ij} = \sum_{l=1}^{4} \beta_{ijl} c_l, \quad f_{ij} = \sum_{l=1}^{4} \beta_{ijl} c_l^2, \quad g_{ij} = \sum_{l=1}^{4} \beta_{ijl} c_l^3, \quad i, j = 1, 2, 3, 4.$$

The conditions on the weights v_i and abscissa ξ_i of the quadrature rule (2.4) corresponding to $m = 4$ are $\sum_{i=1}^{4} v_i \xi_i^k = 1/(k+1)$, $k = 0, 1, 2, 3$, and the continuous weights of the natural continuous extension (1.5) corresponding to $\nu = 4$ are polynomials of degree d, $2 = \lfloor p/2 \rfloor \leq d \leq \min\{\nu - 1, p\} = 3$. We assume $d = 3$. The system of equations for $w_j(\theta)$ is $\sum_{j=1}^{4} w_j(\theta) c_j^k = \theta^k$, $k = 0, 1, 2, 3$. To satisfy the order conditions (5.14) we will always assume that the weights w_i are given by $w_1 = w_2 = w_3 = 0$, $w_4 = 1$, and that $c_4 = 1$. We can also express the coefficients β_{ijk} in terms of e_{ij}, f_{ij}, and g_{ij} by solving the system of linear equations

$$\sum_{l=1}^{4} \beta_{ijl} = 1, \quad \sum_{l=1}^{4} \beta_{ijl} c_l = e_{ij}, \quad \sum_{l=1}^{4} \beta_{ijl} c_l^2 = f_{ij}, \quad \sum_{l=1}^{4} \beta_{ijl} c_l^3 = g_{ij},$$

$i, j = 1, 2, 3, 4$. We next compute the coefficients α_{ij} from the first equations of the stage order conditions corresponding to $q = 1$, $q = 2$, $q = 3$, and $q = 4$, i.e., from the system of linear equations

$$\sum_{j=1}^{4} \alpha_{ij} p_{ij}^k = \frac{c_i^{k+1}}{k+1}, \quad i = 1, 2, 3, 4, \quad k = 0, 1, 2, 3.$$

We also compute f_{ij}, $i = 1,2,3,4$, $j = 1,2$, from the fourth equations of the stage order conditions corresponding to $q = 3$ and $q = 4$, and $g_{i,1}$, $i = 1,2,3,4$, from the last stage condition corresponding to $q = 4$. Then we try to compute the parameters e_{ij}, $i,j = 1,2,3,4$, from the remaining stage order conditions, which is the system of nonlinear equations of dimension 24. We have used for this purpose the function `FindMinimum` from Mathematica, with high precision options, where the objective function was defined as the sum of squares of equations corresponding to the stage order conditions, and where we were looking for solutions for which this minimum was equal to zero (or was very small). This program turned out to be successful for many choices of free parameters and leads to a thirty-nine parameter family of natural VRK methods of order $p = 4$ and stage order $q = 4$, depending on c_1, c_2, c_3, f_{ij}, $i = 1,2,3,4$, $j = 3,4$, g_{ij}, $i = 1,2,3,4$, $j = 2,3,4$, and p_{ij}, $i,j = 1,2,3,4$.

The next main challenge is then to identify the VRK methods which are A- and V_0-stable. It can be verified that the Nørsett polynomial $E(y)$ for this family of methods depends only on c_1, c_2, and c_3 and takes the form

$$E(y) = a_6(c_1,c_2,c_3)\,y^6 + a_8(c_1,c_2,c_3)\,y^8,$$

where $a_6(c_1,c_2,c_3)$ is not listed here and $a_8(c_1,c_2,c_3) = c_1^2c_2^2c_3^2/576$. Hence, the method is A-stable if and only if $a_6(c_1,c_2,c_3) \geq 0$. To identify natural VRK methods which are V_0-stable we have performed an extensive computer search based on (4.19) and (4.20) in the space of all free parameters. We have found methods of order $p = 4$ and stage order $q = 4$ which are both A- and V_0-stable. An example of such a method is given in the Appendix.

It can be verified that this method, listed in the Appendix, satisfies also the stability conditions (4.18). To our knowledge this is the first example of a VRK method of order $p = 4$ and stage order $q = 4$ which is A- and V_0-stable. The Bel'tyukov VRK method of order $p = 4$ with eight stages constructed in [19] is also A- and V_0-stable, but its stage order is only $q = 1$.

6 Numerical experiments

We implemented the methods reported in Section 5 and Appendixes 1 and 2 in a fixed stepsize code and performed numerical tests on several linear and nonlinear problems from literature ([8]). We report here the results obtained on the following nonlinear tests problems:

$$y(t) = 1 + \sin^2(t) - \int_0^t 3\sin(t-s)y^2(s)ds, \quad t \in [0,5], \tag{6.1}$$

with exact solution $y(t) = cos(t)$, and

$$y(t) = e^{-t} + \int_0^t e^{s-t}(y(s) + e^{-y(s)})ds, \quad t \in [0,20], \tag{6.2}$$

with exact solution $y(t) = \log(t+e)$.

For each test and for each numerical method we have chosen the stepsize $h = (T - t_0)/N$, with $N = 2^i$, $i = 6, 7, \ldots, 10$ and have computed the corresponding numerical solution y_N and the stages $Y^{[N-1]}$. Then we have computed the correct digits as

$$cdy_N = -log_{10}(|y(T) - y_N|), \quad cdY_N = -log_{10}(||Y - Y^{[N-1]}||)$$

and the *estimated orders* as

$$\widetilde{p} = (cdy_N - cdy_{N/2})/log_{10}(2), \quad \widetilde{q} = (cdY_N - cdY_{N/2})/log_{10}(2),$$

where $Y = [y(t_{N-1} + c_1 h), \ldots, y(t_{N-1} + c_\nu h)]^T$.

The results reported in Tables 1, 2 and 3 show as the methods converge and attain the expected order and stage order. Observe that for the method with $p = q = 3$ the estimated order and stage order are close to four. This can be partially explained by the fact the this method satisfies the first four of stage order conditions corresponding to $q = 4$. However, this method does not satisfy the last three stage order conditions corresponding to $q = 4$.

	Problem (6.1)				Problem (6.2)			
N	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$
64	0.62	–	0.67	–	1.91	–	1.88	–
128	1.39	2.54	1.41	2.47	2.80	2.95	2.76	2.95
256	2.09	2.33	2.10	2.30	3.70	2.98	3.66	2.98
512	2.75	2.19	2.75	2.17	4.61	3.02	4.57	3.02
1024	3.38	2.10	3.38	2.09	5.53	3.08	5.50	3.08

Table 1 Numerical results for VRK method of Section 5.2 with $p_{11} = 1$, $p_{12} = 0$, $p_{21} = 1$, $p_{22} = 0$, $p = q = 2$.

	Problem (6.1)				Problem (6.2)			
N	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$
64	1.11	–	1.10	–	2.05	–	2.05	–
128	2.24	3.74	2.23	3.77	3.16	3.67	3.16	3.69
256	3.41	3.90	3.41	3.91	4.31	3.83	4.32	3.84
512	4.60	3.95	4.60	3.96	5.49	3.91	5.50	3.92
1024	5.80	3.98	5.80	3.98	6.68	3.96	6.69	3.96

Table 2 Numerical results for VRK method of Appendix 1 with $p = q = 3$.

	Problem (6.1)				Problem (6.2)			
N	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$	cdY_N	$\widetilde{q}$	cdy_N	$\widetilde{p}$
64	2.05	–	1.99	–	2.89	–	2.87	–
128	3.27	4.07	3.24	4.17	4.09	3.98	4.05	3.93
256	4.49	4.05	4.48	4.10	5.33	4.12	5.30	4.14
512	5.70	4.03	5.69	4.05	6.57	4.10	6.55	4.14
1024	6.91	4.01	6.91	4.02	7.79	4.06	7.78	4.09

Table 3 Numerical results for VRK method of Appendix 2 with $p = q = 4$.

7 Concluding remarks

We described a systematic search for highly stable natural VRK methods (1.2) of order p and stage order $q = p$ for Volterra integral equations (1.1). Examples of methods which are both A- and V_0-stable are presented with $p = q = \mu = \nu$ for $\nu = 1$, 2, 3, and 4. To our knowledge the methods of order $p = 3$ and stage order $q = 3$ and of order $p = 4$ and stage order $q = 4$ are the first examples of VRK methods which are both A- and V_0-stable. The previous examples of the Bel'tyukov VRK methods of order $p = 3$ and $p = 4$ which are A- and V_0-stable, had stage order of $q = 1$ only.

The future work will address the construction of highly stable VRK methods with more "balanced" coefficients and with the components of the abscissa vector c in the unit interval $[0, 1]$, and various implementation aspects related to these methods.

References

1. P. Albrecht, A new theoretical approach to Runge-Kutta methods, SIAM J. Numer. Anal. 24(1987), 391–406.
2. P. Albrecht, Runge-Kutta theory in a nutshell, SIAM J. Numer. Anal. 33(1996), 1712–1735.
3. S. Amini, Stability analysis of methods employing reducible rules for Volterra integral equations, BIT 23(1983), 322–328.
4. A. Bellen, Z. Jackiewicz, R. Vermiglio, and M. Zennaro, Natural continuous extensions of Runge-Kutta methods for Volterra integral equations of the second kind and their applications, Math. Comput. 52(1989), 49–63.
5. A. Bellen, Z. Jackiewicz, R. Vermiglio, and M.Zennaro, Stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind, IMA J. Numer. Anal. 10(1990), 103–118.
6. B.A. Bel'tyukov, An analogue of the Runge-Kutta method for the solution of nonlinear integral equations of Volterra type, Differential Equations 1(1965), 417–433.
7. H. Brunner, E. Hairer, and S.P. Nørsett, Runge-Kutta theory for Volterra integral equations of the second kind, Math. Comput. 39(1982), 147–163.
8. H. Brunner and P.J. van der Houwen, The Numerical Solution of Volterra Equations, CWI Monographs 3, North-Holland, Amsterdam 1986.
9. H. Brunner, S.P. Nørsett, and P.H. M. Wolkenfelt, On V_0-stability of numerical methods for Volterra integral equations of the second kind, Report NW 84/80, Mathematisch Centrum, Amsterdam 1980.

10. D. Conte, Z. Jackiewicz, B. Paternoster, Two-step almost collocation methods for Volterra Integral equations, Appl. Math. Comp. 204(2008), 839–853.
11. D. Conte, B. Paternoster, Multistep collocation methods for Volterra Integral Equations, Appl. Numer. Math. 59(2009), 1721–1736.
12. D. Conte, R. D'Ambrosio, B. Paternoster, Two-step diagonally-implicit collocation based methods for Volterra Integral Equations, Appl. Numer. Math. 62(2012), 1312–1324.
13. R. Garrappa, Order conditions for Volterra Runge-Kutta methods, Appl. Numer. Math. 60(2010), 561–573.
14. E. Hairer, Order conditions for numerical methods for partitioned ordinary differential equations, Numer. Math. 36(1981), 431–445.
15. E. Hairer and G. Wanner, Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems. Springer Verlag, Berlin, Heidelberg, New York 1996.
16. P.J. van der Houwen, Convergence and stability results in Runge-Kutta methods for Volterra integral equations of the second kind, BIT 20(1980), 375–377.
17. P.J. van der Houwen, P.J. Wolkenfelt, and C.T.H. Baker, Convergence and stability analysis for modified Runge-Kutta methods in the numerical treatment of second-kind Volterra integral equations, IMA J. Numer. Anal. 1(1981), 303–328.
18. G. Izzo, Z. Jackiewicz, E. Messina, and A. Vecchio, General linear methods for Volterra integral equations, J. Comput. Appl. Math. 234(2010), 2768–2782.
19. G. Izzo, E. Russo, and C. Chiapparelli, Highly stable Runge-Kutta methods for Volterra integral equations, Appl. Numer. Math. 62(2012), 1001-1013.
20. Z. Jackiewicz, General Linear Methods for Ordinary Differential Equations, John Wiley & Sons, Hoboken, New Jersey 2009.
21. Z. Jackiewicz and S. Tracogna, A general class of two-step Runge-Kutta methods for ordinary differential equations, SIAM J. Numer. Anal. 32(1995), 1390–1427.
22. J.D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley & Sons, Chichester, New York 1973.
23. P. Pouzet, Etude en vue de leur traitement numérique des équations intégrales de type Volterra, Rev. Français Traitement Information (Chiffres), 6(1963), 79–112.
24. J. Schur, Über Potenzreihen die im Innern des Einheitskreises beschränkt sind, J. Reine Angew. Math. 147(1916), 205–232.
25. P.H.M. Wolkenfelt, On the numerical stability of reducible quadrature methods for second kind Volterra integral equations, Z. Angew. Math. Mech. 61(1981), 399-401.
26. M. Zennaro, Natural continuous extensions of Runge-Kutta methods, Math. Comput. 46(1986), 119–133.

Appendix

1. Coeffcients of A- and V_0-stable VRK method with $p = q = \mu = \nu = 3$:

$$c = \xi = \begin{bmatrix} 0.2986793639978812 & 1.936484620788317 & 1 \end{bmatrix}^T, \ w = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T,$$

$$\alpha = \begin{bmatrix} 32.56961736991217 & -41.99309472466154 & 9.722156718747259 \\ -4.161355012670768 & 4.879541694296535 & 1.218297939162551 \\ 0.6776823435367453 & -0.3281654092330291 & 0.6504830656962838 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2986793639978812 & 0.2986793639978812 & 0.2986793639978812 \\ 1.936484620788317 & 1.936484620788317 & 1.936484620788317 \\ 1 & 1 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} -1.346553231360564 & -1.430077578837826 & -1.661365425814824 \\ 1.323898168980016 & 1.033127308380405 & 1.923182148594542 \\ 0.7375804560553820 & -0.4657284124449499 & -0.2347197261846119 \end{bmatrix},$$

$$\beta_1 = \begin{bmatrix} 4.921882655709681 & 5.348697045172605 & 6.655082395586630 \\ -1.414299318231974 & -0.4574205056380096 & -2.523779005744676 \\ 0.7556717097159309 & 2.976938883652643 & 1.787408095661566 \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} 1.180227222673612 & 1.410673496755580 & 2.142033246435904 \\ -0.7132836071749915 & -0.3071819068549123 & -0.9042285693173371 \\ 0.2856946224028209 & 0.6642503731178405 & 0.02010332667742674 \end{bmatrix},$$

$$\beta_3 = \begin{bmatrix} -5.102109878383293 & -5.759370541928185 & -7.797115642022534 \\ 3.127582925406965 & 1.764602412492922 & 4.428007575062013 \\ -0.04136633211875173 & -2.641189256770483 & -0.8075114223389929 \end{bmatrix},$$

$$v = \begin{bmatrix} 0.6978557058854169 & -0.01129692071379275 & 0.3134412148283758 \end{bmatrix}^T,$$

$$w_1(\theta) = 1.685913054558716 - 2.556517982922362\,\theta + 0.8706049283636465\,\theta^2,$$
$$w_2(\theta) = 0.1947342344115391 - 0.8467184619289128\,\theta + 0.6519842275173737\,\theta^2,$$
$$w_3(\theta) = -0.8806472889702551 + 3.403236444851275\,\theta - 1.522589155881020\,\theta^2.$$

2. Coeffcients of A- and V_0-stable VRK method with $p = q = \mu = \nu = 4$:

$$c = \xi = \begin{bmatrix} 2.762397779083248 & 1.913469432180418 & 0.1536783627086710 & 1 \end{bmatrix}^T, \quad w = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$\alpha = \begin{bmatrix} 47.23599624170400 & 9.533061854029514 & -0.06153867452370376 & -53.94512164212656 \\ -45.31266297570112 & 20.42562237707649 & -2.636574632525243 & 29.43708466333030 \\ 19.47734452896614 & -99.80597747586084 & -1.972074947692187 & 82.45438625729556 \\ 651.9198600162418 & -840.3237901698746 & -0.06675199086140641 & 189.4706821444943 \end{bmatrix},$$

$$D = \begin{bmatrix} 2.762397779083248 & 2.762397779083248 & 2.762397779083248 & 2.762397779083248 \\ 1.913469432180418 & 1.913469432180418 & 1.913469432180418 & 1.913469432180418 \\ 0.1536783627086710 & 0.1536783627086710 & 0.1536783627086710 & 0.1536783627086710 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1.508806675989335 & 1.008138078375737 & -0.4893268821398639 & 1.429143517912420 \\ -0.1188587575071992 & -0.6049753222140734 & -1.426290901096432 & 0.1712584972842957 \\ -1.431301040995621 & -1.209945614977351 & -1.857257552290057 & -1.170740675514594 \\ -0.03229009987419818 & -0.04685707497725945 & -1.731189146518958 & -0.09468545513375631 \end{bmatrix},$$

$$\beta_1 = \begin{bmatrix} 0.4279842620145835 & 0.5508875619460129 & -1.066855714840414 & 0.4676116389578849 \\ -0.7274604094091049 & -1.179408759403227 & -1.785109812837321 & -0.4617559306556133 \\ -1.227019361090619 & -0.4733522004909464 & -0.5790851512353563 & -0.2970327009823673 \\ 0.3531777482403369 & 0.4257605737627339 & -0.7913769844447518 & 0.6729778609723476 \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} -1.647056352800810 & -2.363199068791738 & 3.877918109504065 & -1.883078753607592 \\ 2.698766475031944 & 4.307714056830686 & 6.819446353268455 & 1.788220855616697 \\ 4.291843816285965 & 2.043708990116466 & 2.601799888135156 & 1.522464387138038 \\ -1.341082547123978 & -1.548577463191051 & 3.392665514727985 & -2.253336506354883 \end{bmatrix},$$

$$\beta_3 = \begin{bmatrix} -1.487689477636443 & -1.413133165310170 & 3.723717147679234 & -1.565788496191245 \\ 2.720035416436076 & 4.089878959703036 & 6.510022758334166 & 1.947757085496399 \\ 4.949977400312023 & 3.831376034801120 & 4.978419136014973 & 3.589622961720761 \\ 0.5077169804395160 & 0.4521193500783359 & 5.240991311800451 & 0.2627678578843441 \end{bmatrix},$$

$$\beta_4 = \begin{bmatrix} 3.706761568422670 & 4.225444672155895 & -5.534779542342885 & 3.981255610840952 \\ -3.691341482058915 & -6.218184257130496 & -10.54435929876530 & -2.274222010457483 \\ -7.014801855507369 & -4.401732824426639 & -6.001133872914773 & -3.815054647876432 \\ 1.480187818444125 & 1.670697539349981 & -6.842279842083684 & 2.317590787498192 \end{bmatrix},$$

$$v = \begin{bmatrix} 0.02924952029775191 & -0.1395352885214494 & 0.5010956238747390 & 0.6091901443489585 \end{bmatrix}^T,$$

$$w_1(\theta) = -0.07534109200536251 + 0.6049669560288470\,\theta - 0.7858367964652556\,\theta^2 + 0.2562109324417711\,\theta^3,$$
$$w_2(\theta) = 0.3110804920142088 - 2.447923892057980\,\theta + 2.869623785788667\,\theta^2 - 0.7327803857448962\,\theta^3,$$
$$w_3(\theta) = 1.360453870544642 - 2.563932117959241\,\theta + 1.460859005472815\,\theta^2 - 0.2573807580582160\,\theta^3,$$
$$w_4(\theta) = -0.5961932705534880 + 4.406889053988373\,\theta - 3.544645994796227\,\theta^2 + 0.7339502113613411\,\theta^3.$$