Runge-Kutta-Nyström stability for a class of General Linear Methods for $y'' = f(x,y)$

R. D'Ambrosio and B. Paternoster

Dipartimento di Matematica e Informatica, Università degli Studi di Salerno, Italy.
E-mail: {rdambrosio,beapat}@unisa.it

Abstract. Aim of this paper is to begin an investigation on the linear stability analysis of a class of General Linear Methods for special second order Ordinary Differential Equations, taking as a starting point a class of two-step Runge-Kutta-Nyström methods, which is obtained through a transformation of the two-step Runge-Kutta methods. We construct methods with one and two stages possessing the same stability properties of the indirect collocation Gauss-Legendre Runge-Kutta-Nyström methods, which are P-stable.

Keywords: General linear methods, two-step Runge-Kutta-Nyström methods, linear stability, A-stability, L-stability, P-stability
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INTRODUCTION

General Linear Methods (GLM) for first order Ordinary Differential Equations (ODEs) were introduced in the late 60's by Butcher [1] to provide an unifying theory of the basic questions of consistency, stability and convergence of numerical methods for ODEs. Later they were used as a framework for studying accuracy questions, and many new GLMs were derived (see for instance [1, 3] and the references therein contained). For special second order ODEs $y\text{''} = f(x,y)$ no systematic investigation on GLMs has begun till now, even if many linear and nonlinear methods appeared in the literature (see, for instance, [2]). We would like to begin our investigation on GLMs for second order ODEs, starting from the linear stability analysis of a special class of two-step Runge-Kutta-Nyström (TSRKN) methods presented in [5]-[9],

$$\begin{aligned}
Y_j^{[i-1]} &= y_{i-1} + hc_j y'_{i-1} + h^2 \sum_{s=1}^m \bar{a}_{js} f(x_{i-1}+c_s h, Y_s^{[i-1]}), \qquad j=1,\dots,m\\
Y_j^{[i]} &= y_i + hc_j y'_i + h^2 \sum_{s=1}^m \bar{a}_{js} f(x_i+c_s h, Y_s^{[i]}), \qquad j=1,\dots,m,\\
y_{i+1} &= (1-\theta)y_i + \theta y_{i-1} + h\sum_{j=1}^m v_j y'_{i-1} + h\sum_{j=1}^m w_j y'_i + h^2 \sum_{j=1}^m \left(\bar{v}_j f(x_{i-1}+c_j h, Y_j^{[i-1]}) + \bar{w}_j f(x_i+c_j h, Y_j^{[i]})\right),\\
y'_{i+1} &= (1-\theta)y'_i + \theta y'_{i-1} + h\sum_{j=1}^m \left(v_j f(x_{i-1}+c_j h, Y_j^{[i-1]}) + w_j f(x_i+c_j h, Y_j^{[i]})\right),
\end{aligned} \tag{1}$$

represented by the Butcher array

$$\begin{array}{c|c}
\mathbf{c} & \bar{\mathbf{A}} \\
\hline
 & \bar{\mathbf{v}}^T \\
\theta & \bar{\mathbf{w}}^T \\
 & \mathbf{v} \\
 & \mathbf{w}
\end{array} \; . \tag{2}$$

The class (1) was derived in [5] as a family of indirect methods [11] through a transformation of two-step Runge-Kutta (TSRK) methods [4]. The analysis of convergence and consistency for TSRKN methods has already been provided in [5]. In this paper we derive new TSRKN methods, possessing the same stability properties of the best Runge-Kutta-Nyström (RKN) methods, that are the indirect Gauss-Legendre collocation-based methods [11], introducing the concept of RKN-stability, following the lines already drawn in the derivation of GLM methods for first order ODEs [1, 3].

STABILITY ANALYSIS

In this section we present the framework on which the analysis of the linear stability properties of TSRKN methods is based. The stability (or amplification) matrix of TSRKN methods is (see [5])

$$\mathbf{M}(z^2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \theta+\alpha(\overline{\mathbf{v}},\mathbf{e}) & 1-\theta+\alpha(\overline{\mathbf{w}},\mathbf{e}) & \mathbf{v}^T\mathbf{e}+\alpha(\overline{\mathbf{v}},\mathbf{c}) & \mathbf{w}^T\mathbf{e}+\alpha(\overline{\mathbf{w}},\mathbf{c}) \\ 0 & 0 & 0 & 1 \\ \alpha(\mathbf{v},\mathbf{e}) & \alpha(\mathbf{w},\mathbf{e}) & \mathbf{v}^T\mathbf{e}+\theta+\alpha(\mathbf{v},\mathbf{c}) & 1-\theta+\alpha(\mathbf{w},\mathbf{c}) \end{bmatrix}, \tag{3}$$

where $\alpha(\mathbf{x},\mathbf{y}) = -z^2\mathbf{x}^T\mathbf{N}^{-1}\mathbf{y}$, $\mathbf{N} = \mathbf{I}+z^2\mathbf{A}$, $\mathbf{I}$ is the identity matrix of order m and $e = (1,\ldots,1) \in \mathbb{R}^m$.

The characteristic polynomial of the matrix (3) is called *stability polynomial*. We next consider the following definitions (see [10, 11]).

Definition 1. $(0,\beta^2)$ *is a stability interval for the method* (1) *if,* $\forall z^2 \in (0,\beta^2)$, *the spectral radius* $\rho(\mathbf{M}(z^2))$ *of the matrix* $\mathbf{M}(z^2)$ *is such that*

$$\rho(\mathbf{M}(z^2)) < 1. \tag{4}$$

The condition (4) is equivalent to the fact that the roots of stability polynomial are in modulus less than 1, $\forall z^2 \in (0,\beta^2)$. In particular, setting $S(z^2) = trace(\mathbf{M}^2(z^2))$ and $P(z^2) = det(\mathbf{M}^2(z^2))$, for a one-step RKN method condition (4) is equivalent to

$$P(z^2) < 1, \quad |S(z^2)| < P(z^2)+1, \quad \forall z \in (0,\beta^2).$$

Definition 2. *The method* (1) *is A-stable if* $(0,\beta^2) = (0,+\infty)$.

If the eigenvalues $r_1(z^2), r_2(z^2), r_3(z^2)$, and $r_4(z^2)$ of the stability matrix (3) (or, equivalently, the roots of the stability polynomial) are on the unit circle, then the interval of stability becomes an interval of periodicity, according to the following definition.

Definition 3. $(0,H_0^2)$ *is a periodicity interval for* (1) *if,* $\forall z^2 \in (0,H_0^2)$, $r_1(z^2)$ *and* $r_2(z^2)$ *are complex conjugate and* $|r_{1,2}(z^2)| = 1$, *while* $|r_{3,4}(z^2)| \leq 1$.

For a one-step RKN method, the interval of periodicity [10] is then defined by

$$(0,H_0^2) := \{z^2 \;:\; P(z^2) \equiv 1,\; |S(z^2)| < 2\}.$$

Definition 4. *The method* (1) *is P-stable if its periodicity interval is* $(0,+\infty)$.

RUNGE-KUTTA-NYSTRÖM STABILITY

In the context of the numerical integration of first order ODEs, Runge-Kutta methods possess strong stability properties which are, in particular, superior to the ones of linear multistep methods. For this reason, in recent times, the attention of many authors has been devoted to the construction of general linear methods for first order ODEs having the same stability properties of Runge-Kutta methods (see [1, 3] and references therein). If $\mathbf{M}(z) \in \mathbb{R}^{s\times s}$ is the stability matrix of a certain GLM, this method is said to be *Runge-Kutta stable* if its stability polynomial $p(\omega,z)$ takes the form

$$p(\omega,z) = \omega^{s-1}(\omega - R(z)),$$

where $R(z)$ is a rational function. Butcher and Wright (2003; see also [1] and references therein) characterized Runge-Kutta stability in terms of algebraic conditions on the coefficient matrices of the method, introducing the concept of *inherent Runge-Kutta stability*.

In the case of second order ODEs, indirect RKN methods generated by highly stable Runge-Kutta methods inherit those stability properties: for example, Gaussian Runge-Kutta methods are *A*-stable and generate *A*-stable indirect RKN methods; Runge-Kutta methods based on Radau IIA collocation points are *L*-stable and generate *L*-stable indirect RKN methods (see [11]). Therefore, following the lines drawn in the literature in the context of GLMs for first order ODEs [1, 3], we aim for TSRKN methods having the same stability properties of highly stable RKN methods, according to the following definition.

Definition 5. *A TSRKN method (1) is said to be* ***Runge-Kutta-Nyström stable*** *if its stability polynomial exhibits the form*

$$p(\omega,z^2) = \omega^2\big(q_2(z)\omega^2 + q_1(z)\omega + q_0(z)\big), \tag{5}$$

where $q_2(z)\omega^2+q_1(z)\omega+q_0(z)$ *is the stability polynomial of a certain Runge-Kutta-Nyström method.*

In other words, the stability properties of TSRKN methods having Runge-Kutta-Nyström stability (*RKN-stability*) are determined by the polynomial

$$q_2(z)\omega^2+q_1(z)\omega+q_0(z),$$

which is exactly the stability polynomial of a RKN method. Therefore TSRKN methods with *RKN-stability* on Gaussian points are A-stable and, in particular, P-stable, while TSRKN method with *RKN-stability* on Radau IIA points are L-stable. In order to derive TSRKN methods with *RKN-stability*, we use the following procedure:

- the stability polynomial of the TSRKN method has, in general, degree 4 and its coefficients are rational functions of z, depending on the parameters of the method. It is transformed into a polynomial $\tilde{p}(\omega,z)$ of the type (5), i.e.

$$\tilde{p}(\omega,z^2)=\omega^2\left(\alpha_2(z)\omega^2+\alpha_1(z)\omega+\alpha_0(z)\right);$$

- if we want to reproduce the stability properties of a certain RKN method whose stability function is $q_2(z)\omega^2+q_1(z)\omega+q_0(z)$, we have to solve the system of equations

$$\alpha_2(z)=q_2(z),\quad \alpha_1(z)=q_1(z),\quad \alpha_0(z)=q_0(z). \tag{6}$$

Even if the system (6) is not solvable, the first step of this procedure (i.e. the reduction of the degree of the stability polynomial) would produce a polynomial having only two nonzero roots and this property is very similar to that of RKN methods. This leads to the following definition.

Definition 6. *A TSRKN method (1) is said to be* ***almost Runge-Kutta-Nyström stable*** *if its stability polynomial exhibits the form*

$$p(\omega,z^2)=\omega^2\left(p_2(z)\omega^2+p_1(z)\omega+p_0(z)\right), \tag{7}$$

where $p_0(z)$, $p_1(z)$, *and* $p_2(z)$ *are rational functions in* z.

Almost Runge-Kutta-Nyström stability (*almost RKN-stability*) constitute a desirable tool for the practical derivation of highly stable TSRKN methods, because it requires the investigation of a quadratic polynomial instead of a polynomial of degree 4. This idea will be explored in a paper in preparation.

RKN stability on Gaussian points

We now consider the derivation of TSRKN methods having the same stability properties of indirect RKN methods based on Gauss-Legendre points, which are P-stable. Paternoster in [5] already provided a complete characterization of P-stable indirect methods of order 1 and 2 in the class (1): we now analyse if some of those methods possess the *RKN-stability* property inherited from the indirect RKN method based on the Gauss-Legendre point, i.e.

$$\begin{array}{c|c} 1/2 & 1/4 \\ \hline & 1/2 \\ & 1 \end{array} \tag{8}$$

The stability polynomial of one-stage TSRKN of order 1 with *almost RKN-stability* is

$$p(\omega,z^2)=\omega^2\left(\omega^2+\frac{-2+(-2a+c+\bar{w})z^2}{1+az^2}\omega+\frac{1+(1+a-c-\bar{w})z^2}{1+az^2}\right),$$

and it is obtained imposing $w=1$ and $v=\bar{v}=\theta=0$. This means that TSRKN with *almost RKN-stability* or *RKN-stability* fall inside the class of RKN methods. The stability polynomial of the indirect one-stage Gauss-Legendre RKN method is

$$q(\omega,z^2)=\omega^2+\frac{-8+2z^2}{4+z^2}\omega+1,$$

and, therefore, *RKN-stability* is forced by solving the linear system

$$\begin{cases} -4+4c+4\bar{w}=0 \\ -4a+c+\bar{w}=0 \\ 1-c-\bar{w}=0 \end{cases}$$

As a consequence, we obtain the following family of P-stable methods

$$\begin{array}{c|c} c & 1/4 \\ \hline & 0 \\ 0 & 1-c \\ & 0 \\ & 1 \end{array} \qquad (9)$$

In particular, when $c = \frac{1}{2}$, order 2 is achieved and we obtain again the RKN method on the Gaussian point. These considerations lead to the following result.

Theorem 1. *One-stage TSRKN methods with order 1 and having RKN-stability are RKN methods themselves. The only one-stage TSRKN method of order 2 and having RKN-stability is the Gauss-Legendre RKN method itself.* □

In the case of two-stage TSRKN methods, we have been able to derive a family of P-stable methods of order 2 and with *RKN-stability* inherited from Gauss-Legendre RKN methods. The derived family of methods is reported in the following result.

Theorem 2. *Two-stage TSRKN methods with order 2 and having RKN-stability are*

$$\begin{array}{c|cc} 1/2 & (1-4a_{12})/4 & a_{12} \\ 1/2 & (1-4a_{12})/4 & a_{22} \\ \hline & 0 & 0 \\ 0 & (1-2\bar{w}_2)/2 & \bar{w}_2 \\ & -v_2 & v_2 \\ & 1-w_2 & w_2 \end{array} \qquad (10)$$

□

CONCLUSIONS AND FUTURE WORK

We have shown a procedure to construct new TSRKN methods for $y" = f(x,y)$, starting from the stability properties of the best RKN methods. We strongly believe that our analysis is a good and practical starting point for the construction of GLM methods for $y" = f(x,y)$. It is under development the analysis and construction of GLMs having the same stability properties of indirect collocation Radau methods discussed in [11].

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