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High order exponentially fitted methods for Volterra integral equations with periodic solution

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Abstract

The present paper illustrates the construction of direct quadrature methods of arbitrary order for Volterra integral equations with periodic solution. The coefficients of these methods depend on the parameters of the problem, following the exponential fitting theory. The convergence of these methods is analyzed, and some numerical experiments are illustrated to confirm theoretical expectations and for comparison with other existing methods.

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1. Introduction

In this paper we consider Volterra integral equations (VIEs) with periodic solution of the type

$$\begin{aligned} &y(x) = f(x) + \int_{-\infty}^{x} k(x-s)y(s)ds, \qquad x \in [0, x_{end}] \\ &y(x) = \psi(x), \qquad -\infty < x \leq 0, \end{aligned} \tag{1.1}$$

where $k \in L^1(\mathbb{R}^+)$, f is continuous and T-periodic on $[0, x_{end}]$, ψ is continuous and bounded on $\mathbb{R}^-$. Under suitable hypotheses on the kernel k, (1.1) has a unique T-periodic solution [3]. VIEs of type (1.1) model periodic physical and biological processes with memory, like for example seasonal biological phenomena [1, 15] and the response of a nonlinear circuit to a periodic input [17]. Further examples are furnished in [3, 6, 8].

An efficient and accurate numerical solution of (1.1) may be found by means of special purpose methods, which exploit the *a priori* knowledge of the qualitative behavior of the solution. On this direction, two main numerical schemes

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have been proposed. In [2, 3] a collocation method based on a mixed interpolation technique is illustrated. In [7–9] direct quadrature (DQ) methods based on the exponential fitting technique [12, 14] were introduced and analyzed. Exponential fitting is a powerful technique which allows to derive accurate and efficient numerical schemes also for high oscillatory problems, in the context of ordinary differential equations (ODEs), partial differential equations, quadrature, interpolation [5, 10–14]. A recent review on the exponential fitting theory is [16]. In general, exponentially fitted (ef) methods have coefficients depending on some estimates of parameters of the problem itself (like the frequency of periodic solutions), and tend to the classical numerical methods when the problem is not oscillatory, thus they may be considered as a generalization of classical numerical schemes.

The aim of the paper is the construction and analysis of a family of Direct Quadrature (DQ) methods based on exponentially-fitted Gaussian quadrature formulae. Recently, an ef-DQ method based on a two-nodes Gaussian-type formula has been introduced in [8] (see also [9]). Here we go one step further, and propose a systematic approach for the construction of ef-DQ Gaussian methods of arbitrary order, to achieve a better accuracy. First, we introduce a general class of ef Gaussian quadrature rules, based on an exponential fitting space. Since a direct quadrature method based on such formulas require an approximation of the solution at points not belonging to the mesh, we introduce an interpolation approximation based on a suitable trigonometric basis. The final method may achieve an high order of convergence, which depend on the number of nodes of the quadrature rule and on the interpolation approximation accuracy.

The paper is organized as follows. In Sec. 2 we introduce and analyze the convergence of a family of ef-DQ methods based on Gaussian-type rules and on a suitable interpolation technique. In Sec. 3 we illustrate the construction of ef-DQ methods of order six. The performances of the methods are illustrated in 4. Last section contains some concluding remarks.

2. Exponentially fitted Gaussian quadrature rule

With the aim to well reproduce the problem modeled by (1.1), we approximate the whole integrand of (1.1) by suitable exponential and trigonometric functions. To this purpose we consider as *test equation* the problem (1.1) with kernel

$$k(x) = e^{\alpha x}, \tag{2.1}$$

and $f(x)$ such that

$$y(x) = \sum_{k=0}^{P-1} \left(c_{1,k} x^k \cos(\omega x) + c_{2,k} x^k \sin(\omega x) \right), \tag{2.2}$$

where $P \geq 1$, $\omega, c_{1,k}, c_{2,k} \in \mathbb{R}$, $\forall k$. It is possible to verify that, if $f(x)$ and $\psi(x)$ have the same structure as $y(x)$, and $k(x)$ is of the type (2.1), then the solution of (1.1) has the form (2.2). In the spirit of exponential fitting theory, we derive a DQ method which is exact when applied to this test equation, and will be proved to be more accurate than general-purpose methods when applied to more general periodic or oscillatory problems.

The first step to construct such DQ method is to formulate a suitable quadrature rule for the integral in (1.1), when the kernel and the solution are given by (2.1) and (2.2), respectively. We construct a Gauss quadrature formula for the integral

$$I[g](X) = \int_{X-h}^{X+h} g(x)dx,$$

where $X > 0$ and $h > 0$, which is exact on the fitting space

$$\mathcal{B} := \{x^k e^{(\alpha \pm i\omega)x}\ ,\ k = 0, \ldots, P-1\}. \tag{2.3}$$

The quadrature formula is of type

$$Q[g](X) := h \sum_{k=0}^{P-1} a_k g(X + \xi_k h) \tag{2.4}$$

where the weights and nodes

$$a_k = a_k(\alpha h, \omega h), \quad \xi_k = \xi_k(\alpha h, \omega h), \qquad k = 0, 1, \ldots, P-1, \tag{2.5}$$

will be derived through the exponential fitting theory [12, 14]. To simplify the notation, we will skip the dependence of weights and nodes from αh and ωh. Following the exponential fitting formalism introduced by Ixaru, we introduce the functional $\mathcal{L}$:

$$\mathcal{L}[h, \mathbf{a}, \xi]g(X) := \int_{X-h}^{X+h} g(s)ds - h \sum_{k=0}^{P-1} a_k g(X + \xi_k h)),$$

where $\mathbf{a} = (a_0, \ldots, a_{P-1})$ and $\xi = (\xi_0, \ldots, \xi_{P-1})$ and ask that it annihilates on the fitting space. It is easy to verify that

$$\mathcal{L}[h, \mathbf{a}, \xi]X^k e^{(\alpha+i\omega)X}, \quad \text{ and } \mathcal{L}[h, \mathbf{a}, \xi]X^k e^{(\alpha-i\omega)X}$$

are complex conjugate, and so to annihilate both it is sufficient to impose

$$\mathcal{L}[h, \mathbf{a}, \xi]X^k e^{(\alpha+i\omega)X} = 0. \tag{2.6}$$

For this reason, in the following we will focus only on $\mathcal{L}[h, \mathbf{a}, \xi]X^k e^{(\alpha+i\omega)X}$. Functions $\mathcal{L}[h, \mathbf{a}, \xi]X^k e^{(\alpha+i\omega)X}$ can be expressed in the compact form:

$$\begin{aligned}
\mathcal{L}[h, \mathbf{a}, \xi]e^{(\alpha+i\omega)X} &= \ell_0 e^{(\alpha+i\omega)X}, \\
\mathcal{L}[h, \mathbf{a}, \xi]Xe^{(\alpha+i\omega)X} &= (\ell_0 X + \ell_1)e^{(\alpha+i\omega)X}, \\
\mathcal{L}[h, \mathbf{a}, \xi]X^2 e^{(\alpha+i\omega)X} &= (\ell_0 X^2 + 2\ell_1 X + \ell_2)e^{(\alpha+i\omega)X}, \\
\mathcal{L}[h, \mathbf{a}, \xi]X^3 e^{(\alpha+i\omega)X} &= \left(\ell_0 X^3 + 3\ell_1 X^2 + 3\ell_2 X + \ell_3\right) e^{(\alpha+i\omega)X}, \\
\ldots &
\end{aligned}$$

where ℓ_k are complex coefficients depending on $u = \alpha h, z = \omega h, a_k, \xi_k,\ k = 0, \ldots, P-1$. As a consequence, the quadrature rule (2.4) is exact on the fitting space $\mathcal{B}$ if and only if

$$\ell_k = 0, \quad \ell = 0, \ldots, P-1, \tag{2.7}$$

i.e. if the real and the imaginary parts of ℓ_k vanish. The system (2.7) is linear with respect to the weights $\{a_k\}_k$ and non linear with respect to the nodes $\{\xi_k\}_k$, thus the exact solution cannot be derived in a closed form and a numerical method is necessary.

Remark 1. We point out that an accurate solution of the nonlinear system (2.7) is essential to guarantee the expected order of the method, thus a stringent tolerance, equal to about the machine precision, is required when a numerical procedure is used. In our numerical experiments we found that if the residual of the nonlinear system is between 10^{-12} and 10^{-16}, the quadrature rule is exact on the fitting space (2.3) except for the round-off errors.

The following theorem analyzes the error of the quadrature formula (2.4):

$$E[g](X) := \int_{X-h}^{X+h} g(s)ds - Q[g](X).$$

Theorem 2.1. *Let assume that $g(x)$ is differentiable indefinitely many times on $[X-h, X+h]$. The error from the quadrature formula $Q[g]$ (2.4) with weights and nodes given by the system (2.7) is*

$$E[g](X) = \sum_{k=0}^{\infty} h^{2P+1+k} T_k D^k((D-\alpha)^2 + \omega^2)^P g(X), \tag{2.8}$$

where D is the derivative operator and $\{T_k\}_k$ depend on $u = \alpha h,\ z = \omega h$, and on $\{a_k, \xi_k\}_{k=0}^{P-1}$. In particular

$$T_0 = \frac{2 - \sum_{k=0}^{P-1} a_k}{(u^2+z^2)^P} \tag{2.9}$$

$$T_1 = \frac{2Puz^{2P-2}\left(2 - \sum_{k=0}^{P-1} a_k\right) - (u^2+z^2)^P \sum_{k=0}^{P-1} a_k \xi_k}{(u^2+z^2)^{2P}} \tag{2.10}$$

Proof. The fitting space $\mathcal{B}$ (2.3) is also the space of linearly indipendent solutions of the ODE

$$((D-\alpha)^2 + \omega^2)^P g(x) = 0, \tag{2.11}$$

that we call reference differential equation. Consequently, the quadrature rule (2.4) is exact whenever the integrand function g satisfies (2.11). On this basis, the thesis can be proven following the classical analysis of the discretization error of ef methods (compare [14, Ch. 1, Sec. 4.3] and [8]).
Let $\bar{x} \in [X-h, X+h]$ and write the Taylor expansion of g around $\bar{x}$

$$g(x) = \sum_{m=0}^{\infty} \frac{(x-\bar{x})^m}{m!} g^{(m)}(\bar{x}).$$

Therefore we can write

$$E[g](X) = \sum_{m=0}^{\infty} \frac{1}{m!} E[(x-\bar{x})^m](X) g^{(m)}(\bar{x}).$$

By putting $\bar{x} = X$, we get

$$E[g](X) = \sum_{m=0}^{\infty} \frac{1}{m!} E[x^m](0) g^{(m)}(X). \tag{2.12}$$

Then we assume that the error of the quadrature rule may be expressed as in (2.8) and we compare it with the expansion (2.12) to derive the coefficients T_k. Here we compute the first two coefficients. By equating the coefficient of $g(X)$ in (2.8) and in (2.12), we get:

$$h^{2P+1} T_0 (\alpha^2 + \omega^2)^P = E[1](0).$$

Since $E[1](0) = h\left(2 - \sum_{k=0}^{P-1} a_k\right)$, formula (2.9) arises.
We proceed similarly for T_1. The coefficient of $g'(X)$ in (2.12) is

$$E[x](0) = -h^2 \sum_{k=0}^{P-1} a_k \xi_k.$$

In (2.8), only the first two terms depend on $g'(X)$, i.e.

$$T_0 h^{2P+1} ((D-\alpha)^2 + \omega^2)^P g(X) + T_1 h^{2P+2} (\alpha^2 + \omega^2)^P g'(X).$$

The formula for the expansion of a binomial gives

$$T_0 h^{2P+1} ((D-\alpha)^2 + \omega^2)^P g(X) = T_0 h^{2P+1} \sum_{j=0}^{P} \binom{P}{j} (D-\alpha)^{2j} \omega^{2(P-j)} g(X).$$

The coefficient of $g'(X)$ is obtained by the addendum corresponding to $j = 1$, that is

$$T_0 h^{2P+1} P (D^2 - 2\alpha D + \alpha^2) \omega^{2P-2} g(X).$$

Finally we can write

$$T_1 (u^2 + z^2)^P h^2 - T_0\, 2\, P\, u\, z^{2P-2}\, h^2 = -h^2 \sum_{k=0}^{P-1} a_k \xi_k,$$

and after some simple manipulations, (2.10) follows. $\square$

2.1. Composite quadrature formula

Now we consider

$$I[g] = \int_a^b g(s)ds,$$

and introduce a set of equally spaced points in $[a, b] : a = t_0 < t_1 < ... < t_m = b$, with $h = t_{j+1} - t_j = \frac{b-a}{m}$. By applying the quadrature formula (2.4) on each subinterval $[t_j, t_{j+1}]$, we get the composite formula

$$I[g] \approx Q_m[g] := h \sum_{j=0}^{m-1} \sum_{k=0}^{P-1} \tilde{a}_k g(t_j + \tilde{\xi}_k h), \tag{2.13}$$

with

$$\tilde{a}_k = \frac{1}{2} a_k, \quad \tilde{\xi}_k = \frac{1}{2} + \frac{1}{2}\xi_k, \quad k = 0, \ldots, P-1.$$

The error $E_m[g] = I[g] - Q_m[g]$ is the sum of the errors on each subinterval, given by (2.8). By simple calculations we have, for $g \in C^{2P}([a, b])$,

$$|E_m[g]| \leq C(b-a)h^{2P}, \tag{2.14}$$

where C depends on $||((D-\alpha)^2 + \omega^2)^P g||_\infty$.

2.2. ef-DQ method

Let introduce a uniform mesh on $[0, x_{end}]$, $I_h := \{x_0 = 0 < x_1 < \cdots < x_N = x_{end}\}$, with $x_n = nh$, $\forall n$, $h = x_{end}/N$, and write the equation (1.1) at $x = x_n$ in the following form

$$y(x_n) = f(x_n) + (I\psi)(x_n) + \int_0^{x_n} k(x_n - s)y(s)ds, \tag{2.15}$$

where

$$(I\psi)(x) = \int_{-\infty}^0 k(x-s)\psi(s)ds, \quad x \in [0, x_n], \tag{2.16}$$

or is a suitable approximation of it.

By applying the composite quadrature rule (2.13), we get

$$y(x_n) = \quad f(x_n) + (I\psi)(x_n) + h \sum_{j=0}^{n-1} \sum_{i=0}^{P-1} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) y(x_j + \tilde{\xi}_i h) \tag{2.17}$$

$n = 1, ..., N$. The fully discretization of (2.17) requires a further approximation of $y(x_j + \tilde{\xi}_i h)$ in terms of the values of the solution at the time-steps. The most natural approach is to approximate the solution with the (algebraic or ef) interpolation polynomial $\mathcal{P}$ on the points

$$(x_{j+l}, y_{j+l}), \quad l = -r_-, ..., r_+ \tag{2.18}$$

with $y_n \approx y(x_n)$, $\forall n$, that is

$$y(x_j + \tilde{\xi}_k h) \approx \mathcal{P}(x_j + \tilde{\xi}_k h).$$

The resulting method is

$$y_n = f(x_n) + (I\psi)(x_n) + h \sum_{j=0}^{n-1} \sum_{i=0}^{P-1} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) \mathcal{P}(x_j + \tilde{\xi}_i h), \qquad (2.19)$$

$n = 1, ..., N$.
Both in the case of algebraic and ef interpolation (which will be specified later), it results

$$\mathcal{P}(x_j + sh) = \sum_{l=-r_-}^{r_+} p_l(s) y_{j+l} \qquad (2.20)$$

where $p_l(s)$ does not depend on x_j but only on r_-, r_+. Therefore we have:

$$y_n = f(x_n) + (I\psi)(x_n) + h \sum_{j=0}^{n-1} \sum_{i=0}^{P-1} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) \sum_{l=-r_-}^{r_+} p_l(\tilde{\xi}_i) y_{j+l}, \qquad (2.21)$$

$n = 1, ..., N$.
To avoid the use of the approximate solution in future mesh points, the following condition must be satisfied:

$$r_+ \leq 1. \qquad (2.22)$$

The method is explicit for $r_+ = 0$, and implicit for $r_+ = 1$.

In the case of classical algebraic interpolation, $p_l(s)$ is the l-th fundamental Lagrange polynomial on nodes $-r_-, \dots, r_+$, namely,

$$p_l(s) = \prod_{\substack{i=-r_- \\ i \neq l}}^{r_+} \frac{s - i}{l - i}, \qquad l = -r_-, \dots, r_+. \qquad (2.23)$$

In the case of ef-interpolation, the function $\mathcal{P}$ is

$$\mathcal{P}(x_j + sh) = \sum_{l=-r_-}^{r_+} b_{l+r_-}(s) y_{j+l}, \qquad (2.24)$$

where functions $b_l(s)$ are derived in the following subsection.

2.3. ef-interpolation

Here we illustrate the constructive procedure for the interpolation polynomial $\mathcal{P}$ at the points (2.18) on the fitting space $\{x^k \cos(\omega x), x^k \sin(\omega x)\}_{k=0}^{P-1}$, that is the space of the solution of the test equation (2.2). For ease of computation we write the fitting space as

$$\mathcal{B}_y = \left\{ x^k exp(\pm i\omega x) \right\}_{k=0}^{P-1}. \qquad (2.25)$$

Let us consider a smooth function y in a suitable neighborhood of x, and take the points

$$(x + lh, y(x + lh)), \qquad l = -r_-, \dots, r_+,$$

with $r_- + r_+ = 2P - 1$. The interpolation polynomial $\mathcal{P}$ on these points can be written as

$$\mathcal{P}(x + sh) = b_0(s)y(x - r_-h) + b_1(s)y(x - r_-h + h) + \\ \dots + b_{r_-+r_+}(s)y(x + r_+h). \quad (2.26)$$

We introduce the functional

$$\mathcal{L}[h, s, \mathbf{b}]g(x) = g(x + sh) - \\ b_0(s)y(x - r_-h) - b_1(s)y(x - r_-h + h) - \dots - b_{r_-+r_+}(s)y(x + r_+h),$$

with $\mathbf{b} = \big(b_0(s), b_1(s), \dots, b_{r_-+r_+}(s)\big)$. By imposing that the interpolation polynomial is exact on the fitting space $\mathcal{B}_y$, the following system arises:

$$\mathcal{L}[h, s, \mathbf{b}]x^k e^{\pm i\omega x} = 0, \qquad k = 0, \dots, P - 1. \quad (2.27)$$

Since $\mathcal{L}[h, s, \mathbf{b}]x^k e^{i\omega x}$ and $\mathcal{L}[h, s, \mathbf{b}]x^k e^{-i\omega x}$ are complex conjugate, it is sufficient to annihilate $\mathcal{L}[h, s, \mathbf{b}]x^k e^{i\omega x}$. Functions $\mathcal{L}[h, s, \mathbf{b}]x^k e^{i\omega x}$ can be written as

$$\begin{aligned}
&\mathcal{L}[h, s, \mathbf{b}]e^{i\omega x} = e^{i\omega x}\hat{\ell}_0 \\
&\mathcal{L}[h, s, \mathbf{b}]xe^{i\omega x} = e^{i\omega x}\left(\hat{\ell}_0 x + \hat{\ell}_1\right) \\
&\mathcal{L}[h, s, \mathbf{b}]x^2 e^{i\omega x} = e^{i\omega x}\left(\hat{\ell}_0 x^2 + 2\hat{\ell}_1 x + \hat{\ell}_2\right) \\
&\dots
\end{aligned}$$

where $\hat{\ell}_k$ are complex coefficients depending on $\omega h, \mathbf{b}, s$. Thus the system (2.27) is equivalent to

$$\hat{\ell}_k = 0, \qquad k = 0, \dots, P - 1.$$

By annihilating the real and the imaginary part of $\hat{\ell}_k$, we get the linear system of dimension $2P$:

$$\mathbf{A}\mathbf{b} = \mathbf{c}, \quad (2.28)$$

where the matrix $\mathbf{A}$ and the vector $\mathbf{c}$ depend on $z = \omega h$, namely

$$\mathbf{A} = \begin{bmatrix}
1 & \cos(z) & \cos(2z) & \cdots & \cos((2P-1)z) \\
0 & \sin(z) & \sin(2z) & \cdots & \sin((2P-1)z) \\
r_- & (r_- - 1)\cos(z) & (r_- - 2)\cos(2z) & \cdots & (r_- - 2P + 1)\cos((2P-1)z) \\
0 & (r_- - 1)\sin(z) & (r_- - 2)\sin(2z) & \cdots & (r_- - 2P + 1)\sin((2P-1)z) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_-^{P-1} & (r_- - 1)^{P-1}\cos(z) & (r_- - 2)^{P-1}\cos(2z) & \cdots & (r_- - 2P + 1)^{P-1}\cos((2P-1)z) \\
0 & (r_- - 1)^{P-1}\sin(z) & (r_- - 2)^{P-1}\sin(2z) & \cdots & (r_- - 2P + 1)^{P-1}\sin((2P-1)z)
\end{bmatrix}$$

$$
\mathbf{c} \quad = \quad \begin{bmatrix} \cos((r_- + s)z) \\ \sin((r_- + s)z) \\ s\cos((r_- + s)z) \\ s\sin((r_- + s)z) \\ \vdots \\ s^{P-1}\cos((r_- + s)z) \\ s^{P-1}\sin((r_- + s)z) \end{bmatrix}.
$$

The following theorem analyzes the error of the interpolation polynomial (2.26)

Theorem 2.2. *Assume that y is differentiable infinitely many times in a suitable neighborhood of x. Then*

$$
\mathcal{L}[h, s, \mathbf{b}]y(x) = \sum_{k=0}^{\infty} h^{2P+k} T_k D^k (D^2 + \omega^2)^P y(x), \tag{2.29}
$$

where

$$
T_0 = (-1)^P \frac{1 - b_0(s) - b_1(s) - \ldots - b_{r_- + r_+}(s)}{z^{2P}},
$$

with $z = \omega h$.

Proof. Observe that the interpolation polynomial (2.26) can be also derived by applying the six step procedure introduced by Ixaru to find approximation formulae and illustrated in [14, Cap. 3, Sec. 3]. Thus, following the exponential fitting theory, and taking into account that the fitting space is (2.25) and that the functional $\mathcal{L}$ is dimensionless, the error is given by (2.29), where the leading term of the error is (compare [14, Cap. 3, Sec. 3, eq. (3.57)]):

$$
\ell te = (-1)^P h^{2P} \frac{L^*[h, s, \mathbf{b}]}{z^{2P}} (D^2 + \omega^2)^P y(x) \tag{2.30}
$$

where $L^*[h, t, \mathbf{b}] := \mathcal{L}[h, s, \mathbf{b}]1|_{x=0} = 1 - b_0(s) - b_1(s) - \ldots - b_{r_- + r_+}(s)$. $\square$

2.4. Convergence analysis

The analysis of the ef DQ method (2.21) has as its central ingredients the error of the ef Gaussian quadrature rule and the interpolation error studied in Th. 2.2.

Theorem 2.3. *Assume that the equation* (1.1) *satisfies the hypotheses for the existence and uniqueness of solution, and assume that $y(x) \in C^{2P}([0, x_{end}])$. Let $\{y_n\}_{n=1}^N$ be the numerical solution of (1.1) obtained by the ef DQ method (2.21) with $r_+ + r_- = 2P - 1$, where the polynomial $\mathcal{P}$ is either the Lagrange polynomial* (2.20)(2.23) *or the ef-based interpolation polynomial* (2.26). *Then, the error $e_n = y(x_n) - y_n$ satisfies:*

$$
\max_{1 \le n \le N} |e_n| = \mathcal{O}(h^{2P}) \quad as \quad h \to 0.
$$

Proof. By (2.19) and (2.15), we obtain

$$e_n = \sum_{j=0}^{n-1} \left[\int_0^h k(x_{n-j} - s) y(x_j + s) ds - h \sum_{i=0}^{P-1} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) \mathcal{P}(x_j + \tilde{\xi}_i h) \right]$$

The error can be expressed as $e_n = \sum_{j=0}^{n-1} [A_{nj} + B_{nj}]$, where (cfr. (2.21))

$$A_{nj} = \int_0^h k(x_{n-j} - s) \left[y(x_j + s) - \sum_{l=-r_-}^{r_+} p_l(s/h) y_{j+l} \right] ds,$$

$$B_{nj} = \int_0^h k(x_{n-j} - s) \mathcal{P}(x_j + s) ds - h \sum_{i=0}^{P-1} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) \mathcal{P}(x_j + \tilde{\xi}_i h).$$

It results

$$|A_{nj}| \quad \leq hK \left(\max_{[0,h]} |I_j(s)| + W \sum_{l=-r_-}^{r_+} |e_{j+l}| \right), \tag{2.31}$$

where $K = \|k(x)\|_{[0,x_{end}]}$, $W = \max_{l=-r_-,\ldots,r_+} \|p_l(x)\|_{[0,1]}$. $I_j(s)$ is the interpolation error at $x_j + s$, thus by hypotheses we have (compare (2.30) and the known formula for the error of Lagrange intepolation)

$$|I_j(s)| \leq C_P h^{2P}, \quad \forall s \in [0, h]. \tag{2.32}$$

From (2.31) and (2.32) it comes out:

$$|A_{nj}| \leq K C_P h^{2P+1} + KWh \sum_{l=-r_-}^{r_+} |e_{j+l}|. \tag{2.33}$$

On the other hand, from the quadrature error formula (2.8), we have

$$|B_{nj}| \leq D_{nj} h^{2P+1}, \tag{2.34}$$

with $D_{nj} > 0$. Now, by using (2.33) and (2.34), we can estimate the error e_n:

$$\begin{aligned} |e_n| &\leq \sum_{j=0}^{n-1} \left[K C_P h^{2P+1} + KWh \sum_{l=-r_-}^{r_+} |e_{j+l}| + D_{nj} h^{2P+1} \right] \\ &\leq x_{end} \bar{C}_P h^{2P} + KW x_{end} \sum_{j=0}^{n} |e_j|, \end{aligned}$$

where $\bar{C}_P = 2 \max\{K C_P, \bar{D}\}$ and $\bar{D} = \max_{n,j} D_{nj}$. Therefore,

$$|e_n| \leq \frac{\bar{C}_P x_{end}}{1 - KW x_{end}} h^{2P} + \frac{KW x_{end}}{1 - KW x_{end}} \sum_{j=0}^{n-1} |e_j|.$$

The thesis is then proven by applying the Gronwall-type inequality [4, p. 41] and considering that there are no starting errors:

$$|e_n| \leq \frac{\bar{C}_P x_{end}}{1 - KWx_{end}} \exp\left(\frac{KWx_{end}}{1 - KWx_{end}}\right) h^{2P}.$$

□

Observe that a possible discretization of $(I\psi)(x)$ will not affect the order of convergence, as long as the error of this further discretization behaves like h^{2P} as $h \to 0$.

3. Examples of methods

Here we show how to derive ef DQ methods of type (2.21) of order six. The first step consists in deriving the Gauss rule (2.4) with $P = 3$, i.e.

$$Q_3[g] = \sum_{k=0}^{2} a_k g(\xi_k),$$

where the weights and nodes are solution of the nonlinear system (2.7). Here we explicitly write the system:

$$\begin{cases} \displaystyle\sum_{i=0}^{2} a_i \left(z\sin(z\xi_i) - u\cos(z\xi_i)\right) e^{u\xi_i} = -2\sinh(u)\cos(z), \\ \displaystyle\sum_{i=0}^{2} a_i \left(z\cos(z\xi_i) + u\sin(z\xi_i)\right) e^{u\xi_i} = 2\cosh(u)\sin(z), \\ \displaystyle\sum_{i=0}^{2} a_i \left((z^2 - u^2)\cos(z\xi_i) + 2uz\sin(z\xi_i)\right) \xi_i e^{u\xi_i} = \\ \qquad 2u\cosh(u)\cos(z) - 2\sinh(u)(z\sin(z) + \cos(z)), \\ \displaystyle\sum_{i=0}^{2} a_i \left((z^2 - u^2)\sin(z\xi_i) - 2uz\cos(z\xi_i)\right) \xi_i e^{u\xi_i} = \\ \qquad -2(u\sinh(u)\sin(z) - \cosh(u)(z\cos(z) - \sin(z))), \\ \displaystyle\sum_{i=0}^{2} a_i \left((u^3 - 3uz^2)\cos(z\xi_i) + (z^3 - 3u^2 z)\sin(z\xi_i)\right) \xi_i^2 e^{u\xi_i} = \\ \quad 2\sinh(u)\left(\left(u^2 - z^2 + 2\right)\cos(z) + 2z\sin(z)\right) - 4u\cosh(u)(z\sin(z) + \cos(z)), \\ \displaystyle\sum_{i=0}^{2} a_i \left((-u^3 + 3uz^2)\sin(z\xi_i) + (z^3 - 3u^2 z)\cos(z\xi_i)\right) \xi_i^2 e^{u\xi_i} = \\ \quad 2\cosh(u)\left(\left(-u^2 + z^2 - 2\right)\sin(z) + 2z\cos(z)\right) + 4u\sinh(u)(\sin(z) - z\cos(z)). \end{cases}$$

Once we fix the values of u and z, we solve the system by a numerical procedure. See Remark 1 for further details.
The ef DQ method is then (2.19), where the weights and nodes are the solution of the above system and the polynomials p_l are obtained by the Lagrange polynomial (2.20)(2.23) or by the ef interpolation polynomial (2.24) where the b_ℓ are the solution of the linear system (2.28) with $P = 3$, while $r_- = 2P - 1 = 5$ in the explicit case and $r_- = 2P - 2 = 4$ in the implicit case.

4. Numerical experiments

In this section we illustrate the performances of our methods, starting from the ef-Gaussian rule (2.13) and then considering the ef-DQ method (2.21) based on such rule. We consider the explicit method constructed in Sec.3, corresponding to $P = 3$ and $r_- = 5$. The numerical experiments have been carried out by Matlab®. The nonlinear system (2.7) has been solved by Matlab routine fsolve with maximal accuracy (see also Remark 1).

4.1. Tests on the ef-Gaussian rule (2.13) *with* $P = 3$*.*

We consider the integral

$$\int_1^5 e^x \cos(\bar{\omega}x)dx \tag{4.1}$$

with the exact values (up to 16 figures) -2.2684781432379239 when $\bar{\omega} = 10$, -2.852120449004837 when $\bar{\omega} = 50$. When we apply the ef-Gauss formula (2.13) with $\omega = \bar{\omega}$, the the error must be zero within the machine precision. This is confirmed by results listed in Tables 1-2. In the same tables, we reported the errors obtained when the parameter ω is an approximation of the exact frequency $\bar{\omega}$. We observe that even in this case, the error of the ef-Gaussian

Table 1: h dependence of the errors from the ef-based Gauss rule (2.13) and from the classical Gauss-Legendre rule for integral (4.1) with $\bar{\omega} = 10$.

	ef-Gauss rule (2.13)		class. Gauss-Legendre
h	$\omega = 10$	$\omega = 9$	
1/8	-3,47E-18	3,19E-06	1,67E-03
1/16	3,47E-17	3,35E-08	2,12E-05
1/32	1.46E-16	4,75E-10	3,16E-07
1/64	-4,16E-17	7,91E-12	4,87E-09
1/128	-5,20E-17	1,12E-12	7,59E-11
1/256	-1,08E-16	4,97E-13	1,19E-12

Table 2: h dependence of the errors from the ef-based Gauss rule (2.13) and from the classical Gauss-Legendre rule for integral (4.1) with $\bar{\omega} = 50$, for the problem Test 1.

	ef-Gauss rule (2.13)			class. Gauss-Legendre
h	$\omega = 50$	$\omega = 49$	$\omega = 45$	
1/4	1,04E-13	1,16E-03	2,19E-01	3,05E+01
1/8	4,04E-13	1,17E-04	1,26E-02	1,15E+00
1/16	2,62E-14	1,27E-07	1,39E-05	1,96E-03
1/32	6,66E-15	1,42E-09	1,56E-07	2,30E-05
1/64	4,89E-15	2,05E-11	2,26E-09	3,36E-07
1/128	6,22E-15	3,14E-13	3,47E-11	5,18E-09
1/256	7,55E-15	9,41E-14	1,20E-12	8,05E-11

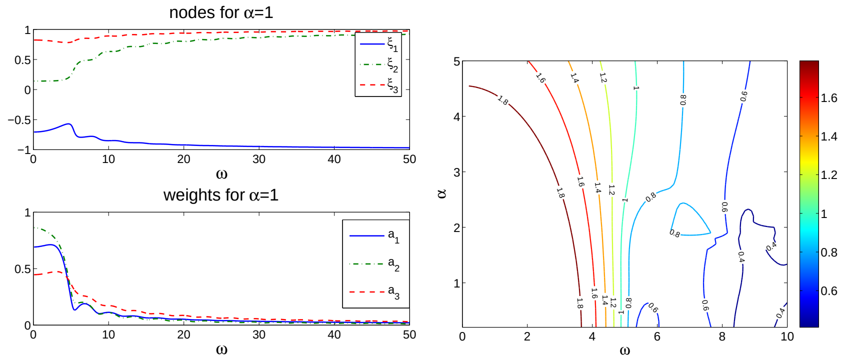

Figure 1: Left: weights and nodes for $\alpha = h = 1$ and $\omega \in [0, 50]$. Right: contour plots of $|a_1| + |a_2| + |a_3|$, generated for $h = 1$.

rule is smaller than the error of the classical Gaussian quadrature rule, and the difference is considerable especially for large values of h.

As an illustration of the behavior of weights and nodes, we plot in Fig. 1 their values for the case $\alpha = 1$, $h = 1$ and ω variable in $[0, 50]$. We observe that the weights are positive, moreover they go to zero as ω goes to infinity, and this behavior is consistent with the construction of the quadrature rule. Moreover, we plot in the right part of Fig. 1 the contour plot of the sum $|a_1|+|a_2|+|a_3|$ with respect to $\omega \in [0, 10]$ and $\alpha \in [0, 5]$, as a measure of the stability of the formula. We observe that this sum is always not greater than 2, and is decreasing with ω, thus we limited the plot to the interval $[0, 10]$.

4.2. Tests on the efDQ method (2.21)

4.2.1. Test 1

The first test we consider is a problem of type (1.1), with $X = 10$,

$$k(x) = e^{\bar{\alpha}x},$$

and $f(x)$ is such that

$$y(x) = x(\sin(\bar{\omega}x) + \cos(\bar{\omega}x)),$$

provided the same expression is adopted for $\psi(x)$. We take $\bar{\alpha} = -1$.

The efDQ method (2.21) with $P = 3$ and $r_- = 5$, $r_+ = 0$ (illustrated in Sec.3) must give the exact solution, within the round-off errors, when $\omega = \bar{\omega}$ and $\alpha = \bar{\alpha}$. This is confirmed by results listed in Table 3. Our intent is to apply our methods, when the solution is linear combination of the fitting space functions, and $\bar{\alpha} = \bar{\alpha}(x)$ and $\bar{\omega} = \bar{\omega}(x)$ are functions slowly varying around constant values. For this reason, we test the accuracy of the efDQ method when only an approximation of the parameters is available. To measure the gain in accuracy we introduce the parameter (similarly as in [8])

$$acc.gain = err^{G_{class}}/err^{G},$$

Table 3: h dependence of the errors from the ef-DQ method (2.21) and from the classical Gauss-Legendre DQ method for test 1 with $\bar{\omega} = 10$.

h	ef-DQ method (2.21)	class. Gauss-Legendre
1/8	4,44E-16	1,20E+00
1/16	8,88E-16	5,76E-03
1/32	1,07E-14	3,43E-04
1/64	2,10E-12	6,25E-06
1/128	1,00E-13	9,88E-08
1/256	1,22E-13	1,53E-09
1/512	6,47E-13	2,37E-11

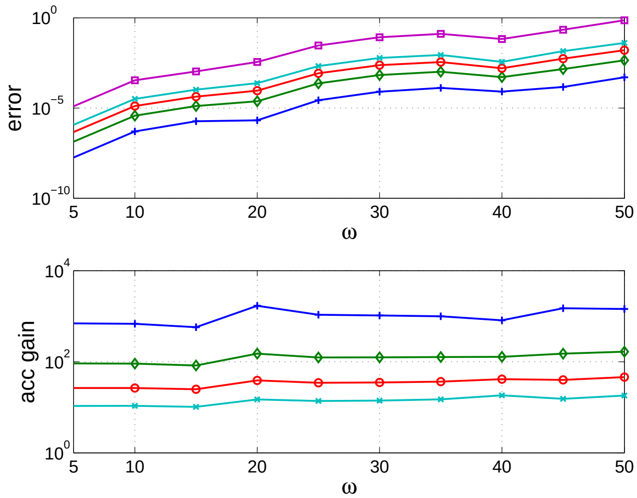

Figure 2: Errors and accuracy gain values computed by classical DQ method and by efDQ method (2.21) with $\omega = \bar{\omega}(1+\delta)$ and $\alpha = \bar{\alpha}(1+\delta)$, $h = 1/32$. Logarithmic scale on the y-axis. Legend: classical DQ method, efDQ with $\delta = 0.05$, efDQ with $\delta = 0.10$, efDQ with $\delta = 0.15$, efDQ with $\delta = 0.20$

where G stands for the efDQ method and G_{class} is the DQ method based on classical 3-nodes Gaussian rule. In Fig. 2 we plot the error and the accuracy gain when α and ω are approximations of $\bar{\alpha}$ and $\bar{\omega}$. We observe that in any case, the gain in accuracy is considerably large, greater than 10 even when the frequency is known with an error of 20%.

4.2.2. Test 2

Now we consider the problem of type (1.1), with $X = 5$,

$$k(x) = e^{\bar{\alpha} x},$$

and $f(x)$ is such that

$$y(x) = x^3 \cos(\bar{\omega} x),$$

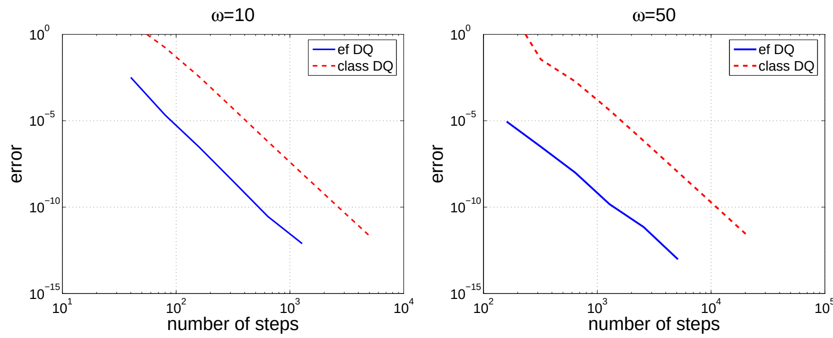

Figure 3: Error versus number of steps of the classical DQ method and by efDQ method (2.21) on the test problem 2, with $\bar{\omega} = 10$ on the left and $\bar{\omega}$ on the right. Logarithmic scale on the x- and y-axis.

provided the same expression is adopted for $\psi(x)$. We take $\bar{\alpha} = -1$.
We apply the efDQ method (2.21) with $P = 3$ and $r_- = 5$, $r_+ = 0$ (illustrated in Sec.3), which is convergent of order six. Although the solution is oscillatory, is not of the type (2.2) with $P = 3$, thus the method is not exact on this equation. We plotted in Fig. 3 the error of the efDQ method and of the classical DQ method of the same order six. We observe that the error of the efDQ method is considerably smaller, at the same computational cost.

5. Conclusions

We illustrated the construction of a family of ef DQ methods for Volterra integral equations with coefficients which depend on some parameters of the problem itself. We carried out the error analysis and the convergence analysis, and proved the effectiveness of our methods by some numerical experiments.

An interesting possible future development may regard an appropriate numerical treatment of the error of quadrature and interpolation formula will allow to derive a good approximations for the parameters of the methods in the cases these are not a priori given, following the lines of [10].

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