A spectral method for stochastic fractional differential equations

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Abstract

The paper provides a spectral collocation numerical scheme for the approximation of the solutions of stochastic fractional differential equations. The discretization of the operator leads to a system of nonlinear algebraic equations, whose coefficient matrix can be computed by an automatic procedure, consisting of linear steps. A selection of numerical experiments confirming the effectiveness of the approach is given, with respect to various sets of function bases and of collocation points.

Keywords: Stochastic fractional differential equations, spectral methods.

1. Introduction

The treatise is devoted to the numerical discretization of stochastic fractional differential equations

$$\begin{aligned} D^\alpha u &= f(t,u) + \int_0^t g(t,s)dW_s, \quad t \in [0,T] \\ u(t_0) &= u_0, \end{aligned} \tag{1.1}$$

$0 < \alpha < 1$, where D^α denotes the Caputo fractional derivative, defined as [16]:

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)ds, \quad 0 < \alpha < 1,\ t > 0.$$

In (1.1) W_s, $s \in [0,T]$ is standard Wiener process and the integral with respect to it is assumed to be the so-called Itô integral, defined as the following limit of a quadrature formula [11]

$$\int_0^t g(t,s)dW_s = \lim_{h\to 0} \sum_{i=0}^{M-1} g(\tau_i, s)\left(W(\tau_{i+1}) - W(\tau_i)\right),$$

where $\tau_0 = 0 < \tau_1 < \ldots < \tau_M = T$ are equidistant points of steplength δt. We observe that Weiner increments $W(\tau_{i+1}) - W(\tau_i)$ are distributed as $\sqrt{\delta t}\,\mathcal{N}(0,1)$ where $\mathcal{N}(0,1)$ is a standard normal random variable.

Existence of the uniqueness of the solution, under suitable regularity assumptions, have been discussed in [1, 14, 17] and references therein, while fields of applications where stochastic fractional models are relevant are given, for instance, in [15] and references therein.

On the side of numerical treatment, a few methods have been proposed so far for fractional stochastic differential models. In [12] Kamrani applied to (1.1) a Galerkin method. In [1] Ahmadi *et al.* introduced a collocation method based on radial basis functions, to solve problem (1.1). In [14] Mohammadi considered a stochastic fractional integro-differential equation (SFIDE) and introduced a Galerkin method based on Chebychev wavelets. In [18] Tahari *et al.* the solution of a SFIDE is expanded by shifted Legendre polynomials, while the noise term is discretized by Newton-Cotes formulas, after a suitable transformation. In [13] Mirzaee and Samadyar expand the solution of a SFIDE and other functions involved in the model by orthogonal Bernstein polynomials

The scope of this paper is the description of a spectral numerical scheme for the computation of approximate solutions of the stochastic problem (1.1). As a matter of fact, the deterministic version of problem (1.1) has been handled by the literature with various numerical methods (see [4, 6, 9] and references therein). The proposed approach for the stochastic problem is described in Section 2, while Section 3 provides a selection of experiments confirming the effectiveness of the approach.

2. Discretization of the operator

We approximate the solution of (1.1) by means of a linear combination of chosen basis functions $\{P_j(x),\ j = 0, 1, \ldots, N\}$

$$u(t) \approx u_N(t) = \sum_{j=0}^{N} \hat{u}_j P_j(t), \tag{2.1}$$

which is usually denoted as *modal expansion* of the numerical solution. For the actual computation, it will be useful to express the numerical solution using the *nodal expansion* [2]

$$u_N(t) = \sum_{j=0}^{N} \varphi_j(t) u_N(t_j), \tag{2.2}$$

where $t_0 = 0 < t_1 < \ldots < t_N = T$ is the set of collocation points. Though the basis functions $\varphi_j(t)$ are not known, we can give their complete characterization, as described in the remainder of the treatise. Let us set

$$\psi_j(t) = D^\alpha \varphi_j(t)$$

leading to

$$D^\alpha u_N(t_k) = \psi_0(t_k) u_0 + \sum_{j=1}^{N} \psi_j(t_k) u_j, \tag{2.3}$$

where $u_j = u_N(t_j) \approx u(t_j)$, $j = 1, 2, \ldots, N$. We evaluate (1.1) in the collocation points and, taking into account (2.3), we obtain

$$\psi_0(t_k)u_0 + \sum_{j=1}^{N} \psi_j(t_k)u_j = f(t_k, u_k) + \int_0^{t_k} g(t_k, s)dW_s, \quad k = 1, 2, \ldots, N.$$

Once the integral in the above right-hand side is suitably discretized by a chosen quadrature formula, we obtain a nonlinear system of dimension N is the unknowns $u_1, u_2, \ldots, u_N$. This leads to

$$\mathbf{D}\,\mathbf{u} + \mathbf{d}\,u_0 = f(\mathbf{u}) + \mathbf{I}, \tag{2.4}$$

where

$$\mathbf{D} = \left(\psi_j(t_k)\right)_{k,j=1}^{N}, \ \mathbf{d} = (\varphi_0(t_k))_{k,j=1}^{N}, \ \mathbf{u} = \left(u_j\right)_{j=1}^{N}, \ f(\mathbf{u}) = (t_k, f(u_k))_{k=1}^{N}, \ \mathbf{I} = (I_k)_{k=1}^{N}.$$

I_k is an approximation of the Itô integral appearing in (1.1). The formalism for the fully discretized problem (2.4) is independent on the definition of the integral approximated by I_k. In the specific case of Itô integrals,

$$I_k = \sum_{i=0}^{k} \sqrt{\delta t} V_i,$$

where V_i is a standard normal random variable. For the simulation of V_i, in the experiments presented in Section 3, we have employed the Matlab built-in function `randn`.

Matrix $\mathbf{D}$ and vector $\mathbf{d}$ rely on the knowledge of the fractional derivates ψ_j in the collocation points. Their computation can be performed by an automatical procedure based on the following result.

Theorem 2.1. *Let define the matrices*

$$\mathbf{A} = \begin{bmatrix} P_0(t_0) & \cdots & P_N(t_0) \\ \vdots & \ddots & \vdots \\ P_0(t_N) & \cdots & P_N(t_N) \end{bmatrix}, \ \mathbf{B} = \mathbf{A}^{-1}, \ \mathbf{P} = \begin{bmatrix} D^\alpha P_0(t_1) & \cdots & D^\alpha P_N(t_1) \\ \vdots & \ddots & \vdots \\ D^\alpha P_0(t_N) & \cdots & D^\alpha P_N(t_N) \end{bmatrix}.$$

Then,

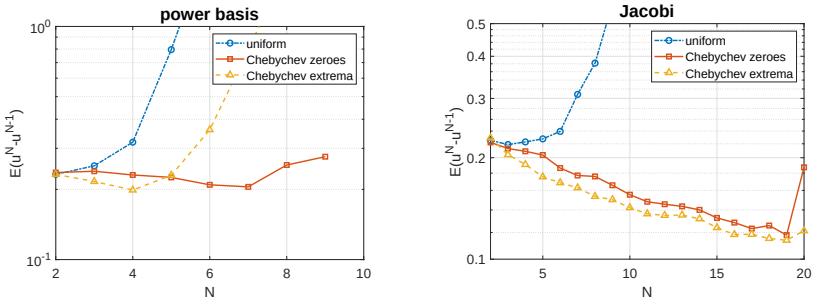
$$[\mathbf{d}, \mathbf{D}] = \mathbf{P} \cdot \mathbf{B}. \tag{2.5}$$

Proof. We evaluate (2.1) in the collocation points $t_0, t_1, \ldots, t_N$ and obtain the $N + 1$-dimensional linear system

$$\mathbf{A}\hat{\mathbf{u}} = \bar{\mathbf{u}},$$

in the unknown $\hat{\mathbf{u}} = [\hat{u}_0, \ \hat{u}_1, \ \ldots, \ \hat{u}_N]^\intercal$, with $\bar{\mathbf{u}} = [u_N(t_0), \ u_N(t_1), \ \ldots, \ u_N(t_N)]^\intercal$. Therefore, $\hat{\mathbf{u}} = \mathbf{B}\bar{\mathbf{u}}$. We now recall the modal expansion (2.1) and replace $u_N(t_i)$ by

$$u_N(t_i) = \sum_{j=0}^{N} \mathbf{B}_{ij} u_N(t_j), \quad i = 0, \ldots, N,$$



(a) (b)

Figure 1: Mean of estimated error for problem (3.3). path=1000 for power basis, path=2000 for Jacobi basis. There is a logarithmic scale on the y-axis.

obtaining

$$u_N(t) = \sum_{\ell=0}^{N} \hat{u}_\ell P_\ell(t) = \sum_{j=0}^{N} \left(\sum_{\ell=0}^{N} \mathbf{B}_{\ell j} P_\ell(t) \right) u_N(t_j).$$

Comparing the above expression with the nodal expansion (2.2) yields

$$\varphi_j(t) = \sum_{\ell=0}^{N} \mathbf{B}_{\ell j} P_\ell(t), \qquad \psi_j(t) = D^\alpha \varphi_j(t) = \sum_{\ell=0}^{N} \mathbf{B}_{\ell j} D^\alpha P_\ell(t)$$

and (2.5) follows. □

3. Numerical experiments

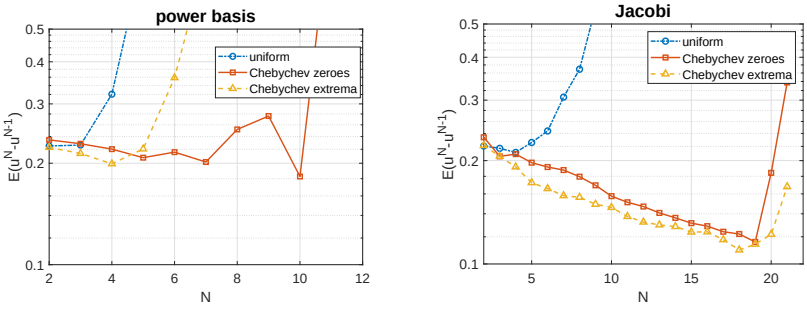
To define our numerical scheme, we have to choose a basis of functions and a set of collocation nodes. Here we consider two classes of functions, both satisfying the requirement of having fractional derivatives. The first one is the power basis

$$1, t^\alpha, t^{2\alpha}, \ldots, t^{N\alpha}. \tag{3.1}$$

This class is suggested by the observation that the FDE $D^\alpha y(t) = \lambda y(t)$, $\lambda \in \mathbb{R}$, has the solution $y(t) = E_\alpha(\lambda t^\alpha)$, where E_α is the Mittag-Leffler function $E_\alpha(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(1+\alpha j)}$ [10, 16]. The second basis consists of a class of Jacobi polynomials shifted in the interval $[0, T]$:

$$P_{n,T}^{(0,1)}(t) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k+1)!}{(k+1)!(n-k)!k!T^k} t^k. \tag{3.2}$$

This class plays a relevant role in the solution of the fractional Sturm Liouville problem [19]. As regards the collocation points, we considered equidistant points in $[0, T]$, Chebychev zeroes and extrema shifted in $[0, T]$.



(a) (b)

Figure 2: Mean of estimated error for problem (3.4). path=2000 for both bases. There is a logarithmic scale on the y-axis.

To validate our approach, we consider this selection of test problems:

$$D^{\frac{1}{2}}u(t) = -u(t) + t + \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{0.5} + \int_0^t dW_s,\ t \in [0,1], \quad u(0) = 0. \tag{3.3}$$

$$D^{\frac{1}{2}}u(t) = -u(t) + t^2 + \frac{2t^{\frac{3}{2}}}{\Gamma(5/2)} + \int_0^t dW_s,\ t \in [0,1], \quad u(0) = 0. \tag{3.4}$$

In both cases the solution is not known, thus we measured convergence by computing the mean value of the estimated error, defined as follows:

$$est.error = |u^N(T) - u^{N-1}(T)|.$$

We employed a large number of paths to obtain significant results (compare also [11]), usually 1000 or more.

Figures 1 and 2 illustrate the behavior of the error on test problems (3.3) and (3.4). In both cases, power basis has poor results, since a very slow reduction of the error is registered only for Chebychev zeroes. On the other hand, the method with Jacobi polynomial basis exhibits exponential convergence, when Chebychev zeroes or extrema are used. We observe that the accuracy of our scheme is affected also by the condition number of matrix $\mathbf{D}$. In our experiments, high condition numbers of $\mathbf{D}$ were registered in the case of power basis and in the case of uniform nodes (both for power basis and for Jacobi basis), and in any case for large N. This was responsible for the increase of the error for large N, in Figures 1(b) and 2(b).

4. Conclusions

We have presented a spectral collocation scheme for the discretization of stochastic fractional differential equations (1.1). The introduced method relies on the numerical solution of a nonlinear system of algebraic equations providing the fully discretized version of the continuous operator. The numerical evidence, carried out on a selection of test problems, confirms the effectiveness of the approach, provided that a suitable

choice of the function basis and of the collocation points are given. Further developments of this research will be oriented to the analysis of the discretization of other stochastic operators, such as stochastic Volterra integral equations [5] and stochastic differential equations, that inherits the main qualitative features of the problem along the numerical solutions [7, 8]. In the case of stochastic Volterra integral equations we will also consider the numerical technique employed in [3].

Acknowledgements. The authors are members of the INdAM Research group GNCS. The work is supported by GNCS-Indam project.

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