Long-term analysis of stochastic θ-methods for damped stochastic oscillators

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Abstract

We analyze long-term properties of stochastic θ-methods for damped linear stochastic oscillators. The presented a-priori analysis of the error in the correlation matrix allows to infer the long-time behaviour of stochastic θ-methods and their capability to reproduce the same long-term features of the continuous dynamics. The theoretical analysis is also supported by a selection of numerical experiments.

Keywords: stochastic θ-methods, stochastic oscillators, long-term analysis.
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1. Framework and scope

This paper is focused on the analysis of the long term properties of one-step discretizations of a damped linear stochastic oscillator, describing the motion of a particle driven by deterministic and stochastic forcing terms. The Ito stochastic differential equation modelling this physical problem, given in [6, 7], has the form

$$\mathrm{d}Z(t) = QZ(t)\mathrm{d}t + \epsilon q \mathrm{d}W(t), \quad t \in [0, T], \tag{1.1}$$

where

$$Z(t) = \begin{bmatrix} X(t) \\ V(t) \end{bmatrix}$$

is the vector collecting the position and velocity of the particle at time t. Q and q are defined by

$$Q = \begin{bmatrix} 0 & 1 \\ -g & -\eta \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

being g the amplitude of the deterministic forcing term and η the value of the damping. Moreover, the parameter ϵ in (1.1) provides the amplitude of the stochastic forcing term, driven by the scalar Weiner process $W(t)$.

The long-term properties of (1.1), as highlighted in [6, 7, 15, 18], can be inferred through the analysis of the stationary density

$$\Pi_\infty(x, v) = N_0 \exp\left(-\frac{\eta}{\epsilon^2}\left(gx^2 + v^2\right)\right). \tag{1.2}$$

revealing that, in the long-time, the motion described by (1.1) has a Gaussian distributed velocity, which is uncorrelated with the position of the particle. This feature can be described in compact way through the correlation matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \mu \\ \mu & \sigma_V^2 \end{bmatrix} = \frac{\epsilon^2}{2\eta} \begin{bmatrix} g^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.3}$$

where

$$\sigma_X^2 = \lim_{t\to\infty} \mathbb{E}|X(t)|^2, \quad \sigma_V^2 = \lim_{t\to\infty} \mathbb{E}|V(t)|^2, \quad \mu = \lim_{t\to\infty} \mathbb{E}|X(t)V(t)| = 0. \tag{1.4}$$

The scope of this paper is the numerical preservation of the long-term features of (1.1), by retaining the correlation matrix (1.3) along the numerical solutions generated by the family of θ-methods. For a Ito stochastic differential equation

$$\mathrm{d}Y(t) = f(Y(t))\mathrm{d}t + g(Y(t))\mathrm{d}W(t), \quad t \in [0, T],$$

the θ(-Maruyama) method is given by

$$Y_{n+1} = Y_n + (1-\theta)\Delta t f(Y_n) + \theta \Delta t f(Y_{n+1}) + g(Y_n)\Delta W_n, \tag{1.5}$$

for $\theta \in [0, 1]$, where Y_n is the approximate value for $Y(t_n)$ with reference to the discretized domain

$$\mathcal{I}_{\Delta t} = \{t_n = n\Delta T,\ n = 0, 1, \ldots, N,\ N = T/\Delta t\}$$

and the discretized Wiener increment ΔW_n is a normal random variable with zero mean and variance Δt.

The linear stability properties of ϑ-methods have extensively been analyzed in the literature (see, for instance, [2, 12, 21] and references therein) with respect to the test problem

$$\mathrm{d}Y(t) = \lambda Y(t)\mathrm{d}t + \mu Y(t)\mathrm{d}W(t), \quad \lambda, \mu \in \mathbb{C},$$

that is the stochastic perturbation of the deterministic Dahlquist test problem for the linear stability analysis of the numerical approximation to ordinary differential equations. Here, we aim to analyze the properties of θ-methods when applied to (1.1), in order to test their ability to reproduce the same long-term behaviour along the discretized dynamics. Hence, this paper follows in the spirit of establishing a theory for stochastic geometric numerical integration, along the lines drawn by several recent contributions, such as [3, 4, 6, 7, 10, 17, 22]. For further approaches on the discretization of stochastic oscillators, see [11, 16, 23, 27, 28, 29] and references therein.

The manuscript is organized as follows: Section 2 provides the long-term analysis of θ-methods applied to (1.1), while the role of the stochasticity (i.e. the role of the amplitude of ϵ in the numerical integration of (1.1)) is analyzed in Section 3. Second-moment preservation along the numerical dynamics given by (1.5) is studied in Section 4. The numerical evidence confirming the theoretical results on a selection of test problems is given in Section 5 and some concluding remarks are object of Section 6.

2. Long-term analysis

Applying the θ-method (1.5) to (1.1) leads to

$$Z_{n+1} = R(\theta, \Delta t)Z_n + \epsilon r(\theta, \Delta t)\Delta W_n, \tag{2.1}$$

where

$$R(\theta, \Delta t) = \Big(I - \theta\Delta t Q\Big)^{-1}\Big(I + (1-\theta)\Delta t Q\Big), \quad r(\theta, \Delta t) = \Big(I - \theta\Delta t Q\Big)^{-1} q,$$

being $I \in \mathbb{R}^{2\times 2}$ the identity matrix.

In order to analyze the long-time features of the θ-method (1.5), we aim to a-priori compute the numerical correlation matrix

$$\widetilde{\Sigma}(\theta, \Delta t) = \begin{bmatrix} \widetilde{\sigma}_X^2 & \widetilde{\mu} \\ \widetilde{\mu} & \widetilde{\sigma}_V^2 \end{bmatrix}, \tag{2.2}$$

with

$$\widetilde{\sigma}_X^2 = \lim_{t_n\to\infty} \mathbb{E}|X_n|^2, \quad \widetilde{\sigma}_V^2 = \lim_{t_n\to\infty} \mathbb{E}|V_n|^2, \quad \widetilde{\mu} = \lim_{t_n\to\infty} \mathbb{E}|X_n V_n|,$$

where X_n and V_n are the numerical solutions of (1.1) computed by (1.5). The following results hold true.

Theorem 2.1. *The numerical correlation matrix* (2.2) *corresponding to the θ-method* (1.5) *assumes the form*

$$\widetilde{\Sigma}(\theta, \Delta t) = \frac{\epsilon^2}{\beta g}\begin{bmatrix} g(2\theta-1)^2\Delta t^2 + \eta(2\theta-1)\Delta t + 2 & g(2\theta-1)\Delta t \\ g(2\theta-1)\Delta t & 2g \end{bmatrix}, \tag{2.3}$$

with

$$\beta = g^2(2\theta-1)^3\Delta t^3 + 3\eta g(2\theta-1)^2\Delta t^2 + 2(\eta^2+2g)(2\theta-1)\Delta t + 4\eta.$$

Proof: According to [6], $\widetilde{\Sigma}(\theta, \Delta t)$ satisfies to the following matrix equation

$$\widetilde{\Sigma}(\theta, \Delta t) = R(\theta, \Delta t)\widetilde{\Sigma}(\theta, \Delta t)R(\theta, \Delta t)^{\intercal} + \epsilon^2 r(\theta, \Delta t)r(\theta, \Delta t)^{\intercal}\Delta t. \tag{2.4}$$

Solving last equation with respect to $\widetilde{\Sigma}$ leads to the thesis.

□

This result leads to the following straightforward corollary highlighting two relevant long-term properties of θ-methods.

Corollary 2.1. *For the θ-method* (1.5)*, we have that*

$$\lim_{\Delta t\to 0} \widetilde{\Sigma}(\theta, \Delta t) = \Sigma, \tag{2.5}$$

for any value of $\theta \in [0,1]$. Moreover, the θ-method (1.5) *with $\theta = 1/2$, i.e. the stochastic trapezoidal rule, exactly preserves the correlation matrix* (1.3).

Another relevant property, also highlighted in [6, 7, 15], is the behaviour of the numerical discretization with respect to η. The following result, again straightforwardly arising from Theorem 2.1, shows that the preservation through one-step methods (1.5) is valid also for severely damped oscillators.

Corollary 2.2. *For the θ-method* (1.5)*, we have that*

$$\lim_{\eta\to\infty} \widetilde{\Sigma}(\theta, \Delta t) = \Sigma, \tag{2.6}$$

for any value of $\theta \in [0, 1]$, of the stepsize Δt and of the parameters ϵ and g.

3. ϵ-expansion of the solution

It is worth highlighting the behaviour of θ-methods when ϵ grows, i.e. when the stochastic term becomes more dominant in the right-hand side of (1.1). One can see, through (2.3), that

$$\lim_{\epsilon\to\infty} \widetilde{\Sigma}(\theta, \Delta t) \neq \Sigma.$$

In order to give an idea of the gap between Σ and $\widetilde{\Sigma}(\theta, \Delta t)$, let us refer to Table 1, reporting the value of $\|\Sigma - \widetilde{\Sigma}(\theta, \Delta t)\|_\infty$, for fixed values of θ, Δt, η, g and for varying ϵ. We can observe that, the more ϵ grows, the more the deviation between Σ and $\widetilde{\Sigma}(\theta, \Delta t)$ becomes larger. In other terms, if the stochastic term becomes dominant, θ-methods may not preserve Σ accurately, unless a small enough stepsize is chosen, according to Corollary 2.1.

ϵ	$\|\Sigma - \widetilde{\Sigma}(3/4, 10^{-1})\|_\infty$	$\|\Sigma - \widetilde{\Sigma}(3/4, 10^{-2}))\|_\infty$	$\|\Sigma - \widetilde{\Sigma}(3/4, 10^{-3}))\|_\infty$
0	0	0	0
0.1	$4.73 \cdot 10^{-4}$	$4.97 \cdot 10^{-5}$	$5.00 \cdot 10^{-6}$
0.5	$1.18 \cdot 10^{-2}$	$1.24 \cdot 10^{-3}$	$1.25 \cdot 10^{-4}$
1	$4.73 \cdot 10^{-2}$.	$4.97 \cdot 10^{-3}$	$5.00 \cdot 10^{-4}$
10	4.73	$6.02 \cdot 10^{+1}$	$5.00 \cdot 10^{-2}$

Table 1: Deviation between Σ and $\widetilde{\Sigma}(\theta, \Delta t)$ for θ=3/4, $\eta = g = 1$ and various values of Δt and ϵ.

We now aim to analyze which is the effect of this issue on the numerical solution computed by (1.5). To this purpose, we perform an ϵ-expansion to the solution of (1.1), i.e., we assume as ansatz that the exact solution can be represented as a power series of ϵ and, as a consequence, the numerical solution computed by (1.5) can be seen as a truncation of this expansion up to a certain power of ϵ. Such a technique is quite common in deterministic numerics; we refer, for instance, to [19] and references therein.

To perform the ϵ-expansion, we directly act on the matrix formulation (1.1) of the problem and assuming, as ansatz, that

$$Z(t) = \sum_{i\geq 0} Z_i(t)\epsilon^i, \tag{3.1}$$

where the coefficients $Z_i(t)$ are vectors in $\mathbb{R}^2$. Replacing the ansatz in (1.1) leads to

$$\mathrm{d}\left(\sum_{i\geq 0} Z_i(t)\epsilon^i\right) = Q\sum_{i\geq 0} Z_i(t)\epsilon^i \mathrm{d}t + \epsilon q \mathrm{d}W(t).$$

It is now sufficient to isolate the terms up to the linear one, obtaining the stochastic differential equations

$$\mathrm{d}Z_0(t) = QZ_0(t)\mathrm{d}t$$

and

$$\mathrm{d}Z_1(t) = QZ_1(t)\mathrm{d}t + \epsilon q \mathrm{d}W(t),$$

in the unknowns $Z_0(t)$ and $Z_1(t)$. In particular, solving the second equation reveals the presence in $Z_1(t)$ of $\epsilon\sqrt{t}$, known in the literature as *secular term*. Clearly, a small enough value of ϵ makes the secular term less dominant in the long-time; on the contrary, if the stochastic part is dominant in the right-hand side of (1.1), the secular term becomes dominant and compromise the accurate preservation of Σ, unless a really small value of Δt is chosen.

To confirm our analysis, we solve numerically (1.1) by the stochastic θ-method, that exactly preserves the correlation matrix (1.3), according to Corollary 2.1. As visible from Table 2, the more ϵ grows, the more the method loses the excellent preservation properties achieved for more moderate values of ϵ. This is not surprising, according to the theoretical arguments given in this section. Clearly, in order to be more accurate when ϵ is bigger, we need to balance the presence of the secular term with a smaller stepsize, as also highlighted in the previous section.

ϵ	$\left\lvert\sigma_X^2 - \widetilde{\sigma}_X^2\right\rvert$	$\left\lvert\sigma_V^2 - \widetilde{\sigma}_V^2\right\rvert$
10^{-6}	$1.78 \cdot 10^{-14}$	$1.83 \cdot 10^{-15}$
10^{-5}	$2.94 \cdot 10^{-12}$	$2.32 \cdot 10^{-12}$
10^{-4}	$7.00 \cdot 10^{-11}$	$4.92 \cdot 10^{-11}$
10^{-3}	$4.74 \cdot 10^{-09}$	$1.64 \cdot 10^{-08}$
10^{-2}	$1.34 \cdot 10^{-06}$	$6.08 \cdot 10^{-07}$
10^{-1}	$5.07 \cdot 10^{-05}$	$2.40 \cdot 10^{-04}$
1	$1.13 \cdot 10^{-02}$	$3.99 \cdot 10^{-02}$

Table 2: Deviations on the mean-squares of position and velocity for the stochastic θ-method applied to (1.1) in [0,100], with $\eta = g = 1$, $\Delta t = 100/2^{12}$ and for various values of ϵ.

4. Second-order moment preservation

We now aim to investigate the ability of θ-methods (1.5) to preserve the character of the second-order moment $\mathbb{E}[X_n^2 + V_n^2]$. As proved in [28] for a simplified version of (1.1), the exact second-moment behaves as follows

$$\mathbb{E}[X(t)^2 + V(t)^2] \leq \alpha + \epsilon^2 t,$$

where α is a real constant. For the θ-method (2.1), the following result holds true.

Theorem 4.1. *The second moment associated to the numerical solution of* (1.1) *computed by* (2.1) *satisfies the following estimate*

$$\mathbb{E}\left[X_n^2 + V_n^2\right] \leq \mathbb{E}\left[X_0^2 + V_0^2\right] + \epsilon^2 t_n.$$

Proof: Let us perform a single step from t_{n-1} to t_n via (2.1). This leads to the following componentwise representation of the advancing law described by (2.1)

$$\begin{aligned} X_n &= r_{11}X_{n-1} + r_{12}V_{n-1} + \epsilon r_1 \Delta W_{n-1}, \\ V_n &= r_{21}X_{n-1} + r_{22}V_{n-1} + \epsilon r_2 \Delta W_{n-1}. \end{aligned}$$

Squaring, summing and passing to the expectations leads to

$$\mathbb{E}\left[X_n^2 + V_n^2\right] = \left(r_{11}^2 + r_{21}^2\right)\mathbb{E}\left[X_n^2\right] + \left(r_{12}^2 + r_{22}^2\right)\mathbb{E}\left[V_n^2\right] + \epsilon^2 \Delta t\left(r_1^2 + r_2^2\right).$$

Since

$$\begin{aligned} r_{11}^2 + r_{21}^2 &= 1 + O(\Delta t^2), \\ r_{12}^2 + r_{22}^2 &= 1 - 2\eta\Delta t + O(\Delta t^2), \\ r_1^2 + r_2^2 &= 1 - 2\eta\theta\Delta t + O(\Delta t^2), \end{aligned}$$

we have

$$\mathbb{E}\left[X_n^2 + V_n^2\right] \leq \mathbb{E}\left[X_{n-1}^2 + V_{n-1}^2\right] + \epsilon^2 \Delta t,$$

that, recursively applied, leads to

$$\mathbb{E}\left[X_n^2 + V_n^2\right] \leq \mathbb{E}\left[X_0^2 + V_0^2\right] + n\epsilon^2 \Delta t,$$

that gives the thesis.

□

In other terms, the numerical second-order moment computed by the stochastic θ-method (2.1) has the same character of the exact second moment, since it is upper-bounded by a linear term in t and quadratic in ϵ.

5. Numerical experiments

In this section we present a selection of numerical tests arising from the application of the θ-methods (1.5) to the linear oscillator (1.1), as well as on a suitable nonlinear variant.

We apply the stochastic trapezoidal method to solve Equation (1.1) in $[0, 100]$ with

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

initial value $Z_0 = [0 \ \ 0]^{\intercal}$ and $\Delta t = 100/2^{15}$. We choose various values for the amplitude ϵ of the stochastic forcing term. As visible from Figures 1 to 3, coherently with the theoretical issues proved in the previous sections, the stochastic trapezoidal method shows

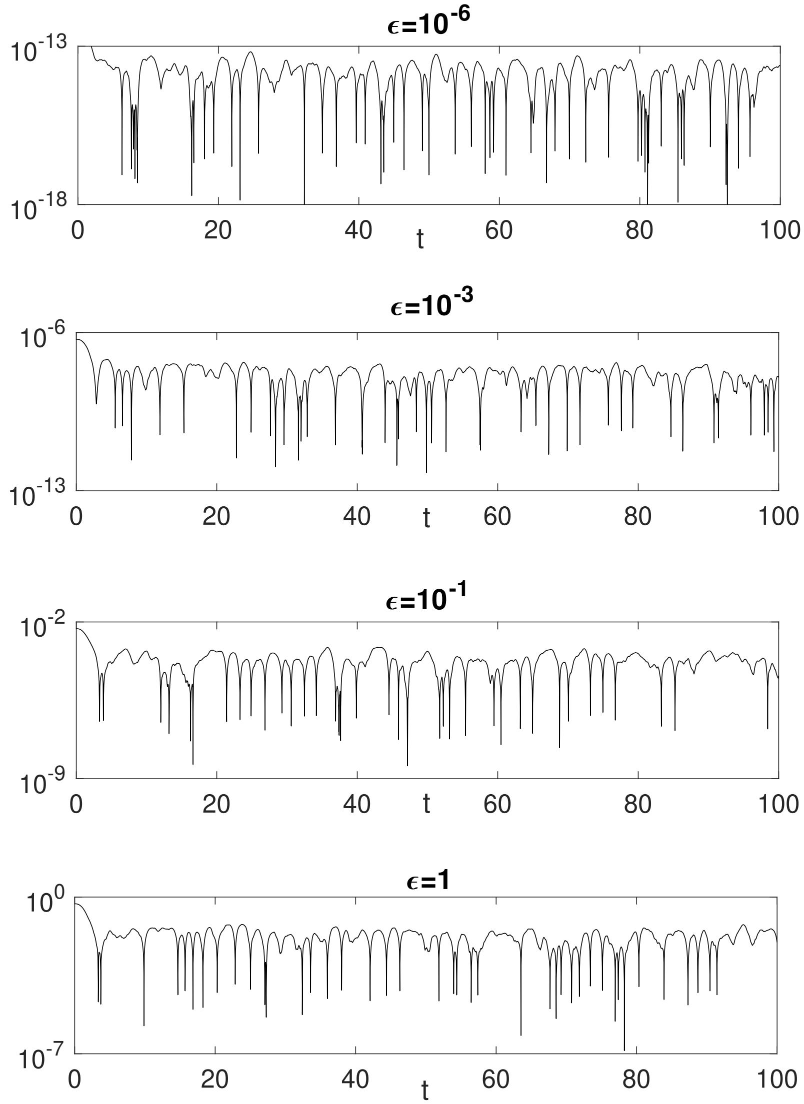

Figure 1: Observed deviation between the numerical and theoretical values of σ_X^2 in (1.4) for the damped oscillator (1.1) with $\eta = g = 1$ and various values of ϵ. The mean has been computed over 1000 trajectories. The initial value is $Z_0 = [0 \;\; 0]^{\mathsf{T}}$ and the chosen stepsize is $\Delta t = 100/2^{15}$.

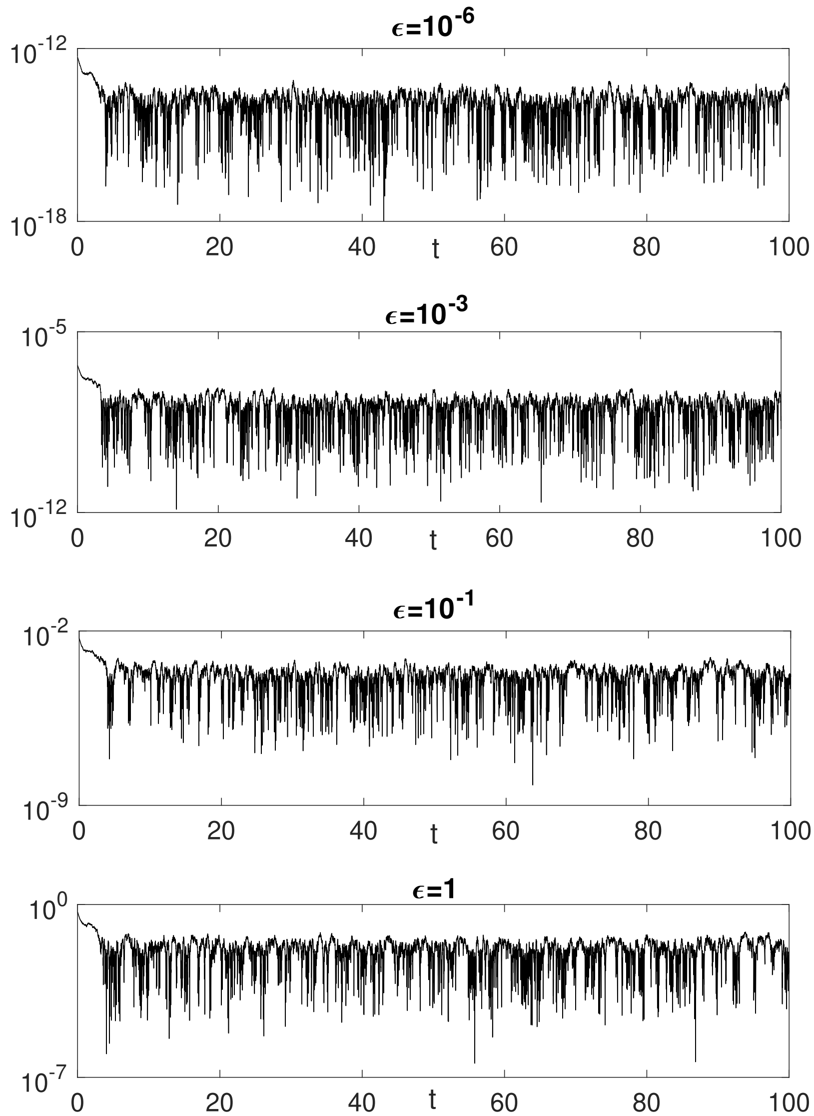

Figure 2: Observed deviation between the numerical and theoretical values of σ_V^2 in (1.4) for the damped oscillator (1.1) with $\eta = g = 1$ and various values of ϵ. The mean has been computed over 1000 trajectories. The initial value is $Z_0 = [0 \;\; 0]^{\mathsf{T}}$ and the chosen stepsize is $\Delta t = 100/2^{15}$.

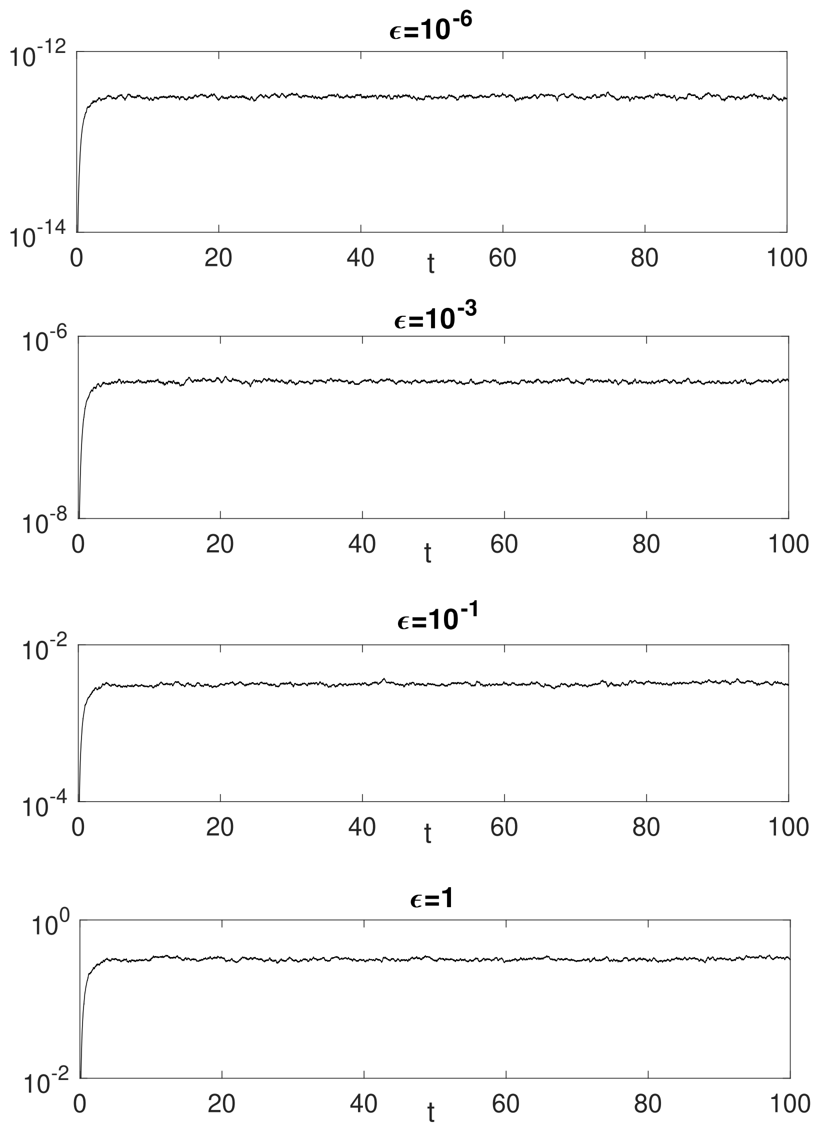

Figure 3: Observed numerical values of μ in (1.4) for the damped oscillator (1.1) with $\eta = g = 1$ and various values of ϵ. The mean has been computed over 1000 trajectories. The initial value is $Z_0 = [0 \;\; 0]^{\mathsf{T}}$ and the chosen stepsize is $\Delta t = 100/2^{15}$.

an excellent conservation behaviour for small values of ϵ. The accurate conservation deteriorates when ϵ increases, as proved in Section 3. Clearly, a better conservation would be possible for smaller values of Δt, as shown in Table 1.

We now conclude by giving a glance to the nonlinear case

$$\begin{aligned} \mathrm{d}X(t) &= V(t)\mathrm{d}t, \\ \mathrm{d}V(t) &= -\left(\eta V(t) - f(X(t)\right)\mathrm{d}t + \varepsilon \mathrm{d}W(t), \end{aligned} \tag{5.1}$$

for $t \in [0, 1000]$, where $f(X)$ is associated to a nonlinear potential $\mathcal{P}(X)$ through $f(X) = -\mathcal{P}'(X)$. As highlighted in [6, 7, 15], the stationary density in the nonlinear case is given by

$$\Pi_\infty(x, v) = N_0 \exp\left(-\frac{\eta}{\varepsilon^2}(v^2 + 2\mathcal{P}(x))\right), \tag{5.2}$$

where the constant N_0 can be computed by the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi_\infty(x, v)\mathrm{d}x\mathrm{d}v = 1.$$

As a test case, we consider the double-well potential

$$\mathcal{P}(X) = -\frac{1}{2}X^2 + \frac{1}{4}X^4$$

and solve the corresponding problem (5.1) with $Z_0 = [0 \;\; 0]^\intercal$ and $\eta = 1$, by means of the stochastic θ-method with $\Delta t = 1000/2^6$, for various values of ϵ. It is known from [6] that, in the nonlinear case, the stationary density is no longer Gaussian, but position and velocity still appear independent. Then, we aim to see if such a long-term independency is preserved by the stochastic trapezoidal method also in this nonlinear case.

As visible from Figure 4, the stochastic trapezoidal method is still able to catch the independency of position and velocity in an accurate way. Similarly as in the linear case, the smaller is ϵ, the more the observed value of the expectation is accurate.

6. Conclusions

The investigation has been devoted to the analysis of long-time features of stochastic θ-methods (1.5), applied to a linear damped oscillator (1.1). The analysis has emphasized the role of the stepsize and of the amplitude ϵ of the diffusion term in the preservation of the correlation matrix (1.3) along the numerical solutions. The numerical evidence has confirmed the theoretical analysis. A glance to the nonlinear version (5.1) of the oscillator has also been experimentally given. It seems that, to some extent, the conclusions of the linear analysis here provided can be extended to the nonlinear case; this issue will object of future contributions, in the spirit of inheriting properties of nonlinear problems over their discretizations [1, 5, 9, 10, 13, 14].

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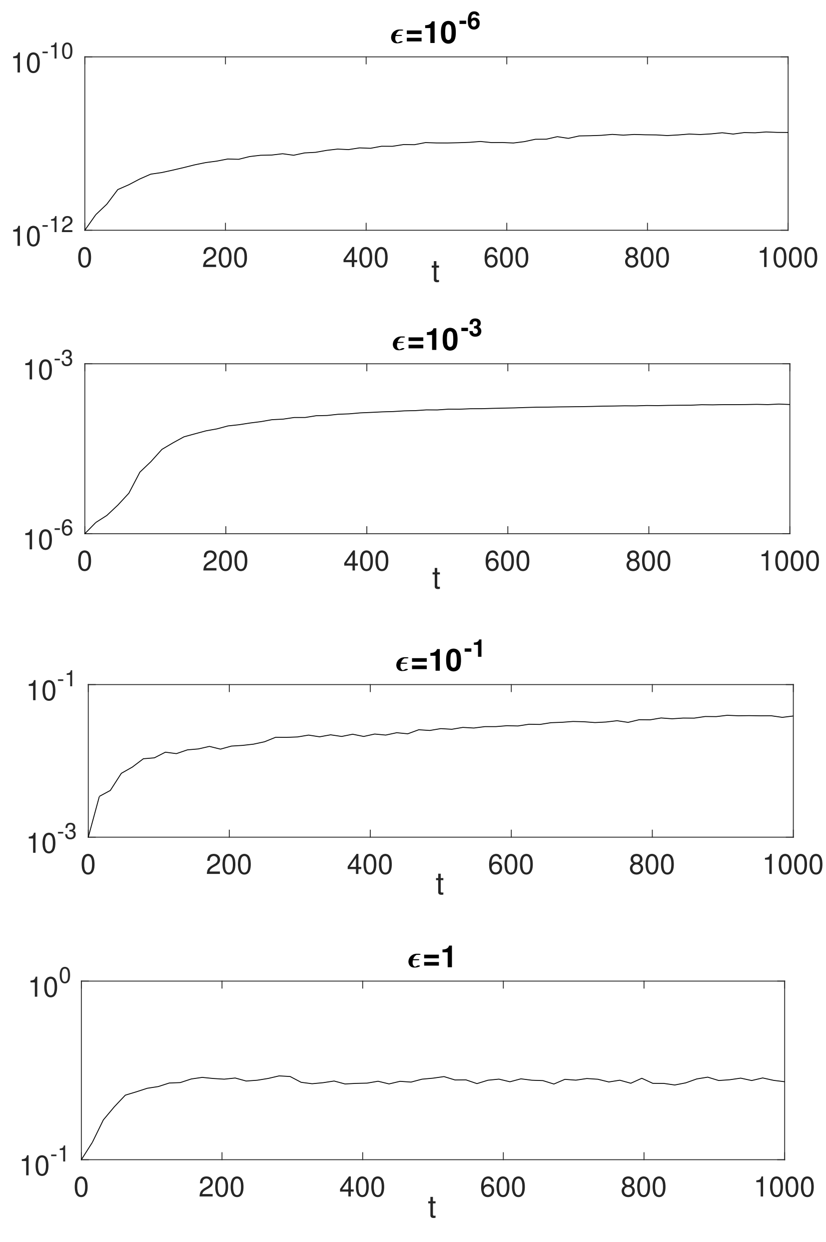

Figure 4: Observed numerical values of μ in (1.4) for the nonlinear damped oscillator (5.1) with $\eta = 1$ and various values of ϵ. The mean has been computed over 1000 trajectories. The initial value is $Z_0 = [0 \;\; 0]^{\mathsf{T}}$ and the chosen stepsize is $\Delta t = 1000/2^6$.

References

[1] E. Buckwar, R. D'Ambrosio, Exponential mean-square stability properties of stochastic multistep methods, submitted.

[2] E. Buckwar, T. Sickenberger, A comparative linear mean-square stability analysis of Maruyama- and Milstein-type methods. Math. Comput. Simul. 81, 1110–1127 (2011).

[3] P.M. Burrage, K. Burrage, Structure-preserving Runge-Kutta methods for stochastic Hamiltonian equations with additive noise. Numer. Algor. 65, 519–532 (2014).

[4] P.M. Burrage, K. Burrage, Low rank Runge-Kutta methods, symplecticity and stochastic Hamiltonian problems with additive noise. J. Comput. Appl. Math. 236, 3920–3930 (2012).

[5] K. Burrage, A. Cardone, R. D'Ambrosio, B. Paternoster, Numerical solution of time fractional diffusion systems, Appl. Numer. Math. 116, 82-94 (2017).

[6] K. Burrage, I. Lenane, G. Lythe, Numerical methods for second-order stochastic differential equations. SIAM J. Sci. Comput. 29, 245–264 (2007).

[7] K. Burrage, G. Lythe, Accurate stationary densities with partitioned numerical methods for stochastic differential equations. SIAM J. Numer. Anal. 47, 1601–1618 (2009).

[8] A. Cardone, D. Conte, R. D'Ambrosio, B. Paternoster, Stability Issues for Selected Stochastic Evolutionary Problems: A Review. Axioms 7(4), 91 (2018).

[9] A. Cardone, R. D'Ambrosio, B. Paternoster, A spectral method for stochastic fractional differential equations, Appl. Numer. Math. 139, 115–119 (2019).

[10] C. Chen, D. Cohen, R. D'Ambrosio, A. Lang, Drift-preserving numerical integrators for stochastic Hamiltonian systems, arXiv:1907.08804, submitted.

[11] D. Cohen, On the numerical discretisation of stochastic oscillators. Math. Comput. Simul. 82(8), 1478–1495 (2012).

[12] D. Conte, R. D'Ambrosio, B. Paternoster, On the stability of ϑ-methods for stochastic Volterra integral equations. Discr. Cont. Dyn. Sys. - B 23(7), 2695–2708 (2018).

[13] R. D'Ambrosio, S. Di Giovacchino, Mean-square contractivity of stochastic θ-methods, submitted.

[14] R. D'Ambrosio, G. Izzo, Z. Jackiewicz, Search for highly stable two-step Runge-Kutta methods for ODEs, Appl. Numer. Math. 62(10), 1361-1379 (2012).

[15] R. D'Ambrosio, M. Moccaldi, B. Paternoster, Numerical preservation of long-term dynamics by stochastic two-step methods. Discr. Cont. Dyn. Sys. - B 23(7), 2763–2773 (2018).

[16] H. de la Cruz, J.C. Jimenez, J.P. Zubelli, Locally linearized methods for the simulation of stochastic oscillators driven by random forces. BIT 57(1), 123–151 (2017).

[17] E. Faou, T. Lelièvre, Conservative stochastic differential equations: mathematical and numerical analysis, Math. Comp. 78, 2047–2074 (2009).

[18] C. W. Gardiner, Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences, 3rd ed., Springer-Verlag, Berlin (2004).

[19] E. Hairer, G. Wanner, Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems, Springer-Verlag, Berlin (1996).

[20] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. 43, 525–546 (2001).

[21] D.J. Higham, Mean-square and asymptotic stability of the stochastic theta method. SIAM J. Numer. Anal. 38, 753–769 (2000).

[22] J. Hong, S. Zhai, J. Zhang, Discrete gradient approach to stochastic differential equations with a conserved quantity. SIAM J. Numer. Anal. 49(5), 2017–2038 (2011).

[23] J. Hong, R. Scherer, L. Wang, Midpoint rule for a linear stochastic oscillator with additive noise. Neural Parallel Sci. Comput. 14(1), 1–12 (2006).

[24] P.E. Kloeden, E. Platen, The Numerical Solution of Stochastic Differential Equations, Springer-Verlag (1992).

[25] G.N. Milstein, M.V. Tretyakov, Stochastic Numerics for Mathematical Physics, Springer-Verlag, Berlin (2004).

[26] Y. Saito, T. Mitsui, Stabilty analysis of numerical schemes for stochastic differential equations. SIAM J. Numer. Anal. 33, 333–344 (1996).

[27] M.J. Senosiain, A. Tocino, A review on numerical schemes for solving a linear stochastic oscillator. BIT 55(2), 515–529 (2015).

[28] A.H. Strømmen Melbö, D.J. Higham, Numerical simulation of a linear stochastic oscillator with additive noise. Appl. Numer. Math. 51, 89–99 (2004).

[29] G. Vilmart, Weak second order multi-revolution composition methods for highly oscillatory stochastic differential equations with additive or multiplicative noise. SIAM J. Sci. Comput. 36, 1770–1796 (2014).