Numerical Algorithms manuscript No.
(will be inserted by the editor)

On the numerical structure preservation of nonlinear damped stochastic oscillators

Raffaele D'Ambrosio · Carmela Scalone

Received: date / Accepted: date

Abstract The paper is focused on analyzing the conservation issues of stochastic θ-methods when applied to nonlinear damped stochastic oscillators. In particular, we are interested in reproducing the long-term properties of the continuous problem over its discretization through stochastic θ-methods, by preserving the correlation matrix. This evidence is equivalent to accurately maintaining the stationary density of the position and the velocity of a particle driven by a nonlinear deterministic forcing term and an additive noise as stochastic forcing term. The provided analysis relies on a linearization of the nonlinear problem, whose effectiveness is proved theoretically and numerically confirmed.

Keywords Stochastic differential equations; stochastic θ-methods; nonlinear damped stochastic oscillators; numerical structure-preservation.

Mathematics Subject Classification (2010) 65L07 · 60H10 · 60H35.

1 Introduction

Long-term preservation of meaningful features is a dominant topic of the numerical analysis of differential problems. In this work, we deal with a second order stochastic differential equation of the form

$$\ddot{x} = f(x) - \eta\dot{x} + \varepsilon\xi(t) \tag{1.1}$$

R. D'Ambrosio
Department of Engineering and Computer Science and Mathematics, University of L'Aquila
Via Vetoio, Loc. Coppito - 67100 L'Aquila, Italy Tel.: +390862434724
E-mail: raffaele.dambrosio@univaq.it

C. Scalone
Department of Engineering and Computer Science and Mathematics, University of L'Aquila
Via Vetoio, Loc. Coppito - 67100 L'Aquila, Italy
E-mail: carmela.scalone@univaq.it

where $\xi(t)$ satisfy $\mathbb{E}|\xi(t)\xi(t')| = \delta(t-t')$ and η is the damping parameter. The motion of a particle described by (1.1), is characterized by a deterministic force $f(x)$, which derives from a potential $V(x)$, i.e., $f(x) = -V'(x)$. The random forcing $\xi(t)$ has amplitude ε, satisfying the relation $\varepsilon^2 = 2\eta KT$, where η is the amplitude of the damping term and T is the temperature. Equation (1.1) is equivalent to the following first order system of two equations in the variables X_t (the position of the particle) and V_t (its velocity):

$$\begin{cases} dX_t = V_t dt \\ dV_t = -\eta V_t dt + f(X_t)dt + \varepsilon dW_t \end{cases} \tag{1.2}$$

At each time t, the probability density is associated, given by

$$P(x, v; t) = \frac{d}{dx}\frac{d}{dv} Prob(X_t < x, V_t < v) \tag{1.3}$$

and taking the limit of (1.3), as t goes to infinity, defines the stationary density P_∞. The essential aim of this work is to provide a study of the *attitude* of the one step methods for SDEs to preserve the stationary density. The availability for the stationary density of the analytical expression

$$P_\infty(x, v) = N\exp(-v^2/2KT - V(x)/KT), \tag{1.4}$$

which is independent from η and $s(x)$, is a crucial starting point for our analysis. The idea is to apply a general one step method to (1.2) and compare P_∞ with the obtained discrete counterpart, aiming to derive conditions for their overlapping.

The treatise is fully framed in the context of numerical structure-preservation issues in stochastic differential equations (SDEs), here intended as preservation of asymptotic invariance laws that characterize the exact dynamics. The existing literature on structure-preservation numerics for stochastic oscillators has mostly dealt with linear problems, as in [7,9,13,14,17,25,26]. Recent contributions have also regarded stochastic Hamiltonian problems [5, 6], time-dependent stochastic Schrödinger equation [1,27], energy-preserving schemes in nonlinear stochastic Hamiltonian problems [12,15], preservation of mean-square contractivity for nonlinear SDEs [2,18].

The manuscript is organized as follows: Section 2 recalls basic aspects regarding the long-term properties of (1.1); Section 3 highlights the main idea used in this paper to handle nonlinear problems by suitable linearizations; this idea is applied to the general case of stochastic one-step methods in Section 4 and applied to relevant numerical methods (Euler-Maruyama method, stochastic trapezoidal method, stochastic implicit Euler method) in Sections 5–7; some numerical experiments are given in Sections 8 and 9.

2 Related works

In [7], the authors consider the linear case for (1.1), with $f(x) = -gx$, with $g > 0$ and $s(x) = 1$, i.e., a second order equation modeling harmonic damped oscillators with additive noise. They examine how faithfully some standard numerical methods for SDEs reproduce the stationary density, varying the value of the damping. In the linear case, the stationary density becomes

$$P_\infty(x, v) = N \exp(-v^2/2KT - g/2KT) \tag{2.1}$$

The long term statistics of position and velocity are Gaussian and mutually uncorrelated. In particular, for the exact solution such quantities are given by

$$\begin{aligned} \sigma_x^2 &= \lim_{t\to\infty} \mathbb{E}|X_t|^2 = \frac{1}{g} KT, \\ \sigma_v^2 &= \lim_{t\to\infty} \mathbb{E}|V_t|^2 = KT, \\ \mu &= \lim_{t\to\infty} \mathbb{E}|X_t V_t| = 0 \end{aligned} \tag{2.2}$$

and they can be arranged in the *correlation matrix*

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \mu \\ \mu & \sigma_v^2 \end{pmatrix}.$$

Let us denote the numerical update as

$$\begin{pmatrix} X_{n+1} \\ V_{n+1} \end{pmatrix} = R \begin{pmatrix} X_n \\ V_n \end{pmatrix} + \varepsilon r \Delta W_n,$$

with ΔW_n is sampled from a Gaussian distribution with mean zero and variance Δt, independently from ΔW_m for $n \neq m$. By considering the numerical counterparts of (2.2)

$$\begin{aligned} \widetilde{\sigma}_x^2 &= \lim_{t_n \longrightarrow \infty} \mathbb{E}|X_n|^2, \\ \widetilde{\sigma}_v^2 &= \lim_{t_n \longrightarrow \infty} \mathbb{E}|V_n|^2, \\ \widetilde{\mu} &= \lim_{t_n \longrightarrow \infty} \mathbb{E} = |X_n V_n|, \end{aligned}$$

we get the *numerical* correlation matrix

$$\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\sigma}_x^2 & \widetilde{\mu} \\ \widetilde{\mu} & \widetilde{\sigma}_v^2 \end{pmatrix}.$$

The authors of [7] proves that $\widetilde{\Sigma}$ must satisfy the constraint

$$R \widetilde{\Sigma} R^T = \widetilde{\Sigma} - \varepsilon^2 r r^\intercal \Delta t, \tag{2.3}$$

thanks to which it is possible to compute $\widetilde{\sigma}_x^2$, $\widetilde{\mu}$ and $\widetilde{\sigma}_v^2$ as functions of the coefficients of the numerical method (2). A similar analysis is carried out in [17] for two step methods.

3 Main idea

The general case of a nonlinear deterministic $f(x)$ in (1.1) is particularly challenging. The impossibility, in the most general case, to find a formal expression of σ_x^2 makes meaningless any attempt of preservation analysis. Let us consider a point x_0 in which f is defined and continuously differentiable, we consider a local linear ansatz

$$f(x) \approx x_0 + f'(x_0)(x - x_0). \tag{3.1}$$

Without loss of generality, we can take $x_0 = 0$, so that (3.1) becomes

$$f(x) \approx f'(0)x. \tag{3.2}$$

Under the conservative hypothesis on f, i.e., $V(x) = -\int f(x)dx$, the disadvantage of choosing to locally linearize f correspond to a local quadratic approximation of the potential V, since

$$V(x) = -\int f(x)dx \approx -\int f'(0)xdx \approx x^2 \frac{f'(0)}{2}.$$

This allows to think of a Gaussian-like behaviour of the stationary density, and, consequently, to suppose that

$$\sigma_x^2 \approx -\frac{KT}{f'(0)}. \tag{3.3}$$

Clearly, we must suppose that $f'(0)$ (more generally $f'(x_0)$) is strictly positive, so that (3.3) makes sense. According to this assumption, we are able to construct the correlation matrix Σ. Given a one step method for SDEs, in the general form

$$y_{n+1} = y_n + \alpha \Delta t f(y_n) + \beta \Delta t f(y_{n+1}) + \gamma \Delta W_n, \tag{3.4}$$

properly exploiting (2.3), we construct the formal discrete correlation matrix $\widetilde{\Sigma}(\alpha, \beta, \Delta t)$. We aim to provide conditions involving α and β for which the limit for Δt, which goes to zero, the distance

$$\left\| \widetilde{\Sigma}(\alpha, \beta, \Delta t) - \Sigma \right\|_F,$$

approaches zero, for all the values of the damping. This is a reasonable consistency request for a method to be a possible candidate to preserve Σ.

4 The general case

The numerical update of (3.4) applied to the system (1.2), is given by

$$\begin{cases} X_{n+1} = X_n + \alpha \Delta t V_n + \beta \Delta t V_{n+1}, \\ V_{n+1} = \alpha \Delta t f'(0) X_n + (1 - \alpha \Delta t \eta) V_n - \beta \Delta t \eta V_{n+1} + \beta \Delta t f'(0) X_{n+1} + \varepsilon \Delta W_n. \end{cases}$$

Setting $\xi = \beta \Delta t$, $\varphi = \alpha \Delta t$ and

$$I = \begin{pmatrix} 1 & -\xi \\ -\xi f'(0) & 1 - \xi\eta \end{pmatrix}, \quad E = \begin{pmatrix} 1 & \varphi \\ \varphi f'(0) & 1 - \varphi\eta \end{pmatrix},$$

we are able to recast the method as (2) taking

$$R = I^{-1}E = \begin{pmatrix} \frac{\eta\,\xi + 1 + f'(0)\,\xi\,\varphi}{-f'(0)\,\xi^2 + \eta\,\xi + 1} & \frac{\varphi\,(\eta\,\xi + 1) - \xi\,(\eta\,\varphi - 1)}{-f'(0)\,\xi^2 + \eta\,\xi + 1} \\ \frac{f'(0)\,\xi + f'(0)\,\varphi}{-f'(0)\,\xi^2 + \eta\,\xi + 1} & \frac{f'(0)\,\xi\,\varphi - \eta\,\varphi - 1}{-f'(0)\,\xi^2 + \eta\,\xi + 1} \end{pmatrix}$$

and $r = \begin{pmatrix} 0 & 1 \end{pmatrix}^\intercal$. Via relation (2.3), we compute $\widetilde{\sigma}_x^2$, $\widetilde{\mu}$ and $\widetilde{\sigma}_v^2$. They can be expressed as a rational function in the variable Δt as follows: the long-term mean-square of the position is given by

$$\widetilde{\sigma}_x^2 = \frac{q_2 \Delta t^2 + q_1 \Delta t + q_0}{p_3 \Delta t^3 + p_2 \Delta t^2 + p_1 \Delta t + p_0}, \tag{4.1}$$

where

$$\begin{aligned} q_2 &= -f'(0)\,\alpha^2\,\varepsilon^2 - f'(0)\,\beta^2\,\varepsilon^2, \\ q_1 &= \beta\,\eta\,\varepsilon^2 - \alpha\,\eta\,\varepsilon^2, \\ q_0 &= 2\,\varepsilon^2, \\ p_3 &= \alpha^4\,f'(0)^3 - 2\,\alpha^3\,\beta\,f'(0)^3 + 2\,\alpha\,\beta^3\,f'(0)^3 - \beta^4\,f'(0)^3, \\ p_2 &= 3\,\eta\,\alpha^3\,f'(0)^2 - 3\,\eta\,\alpha^2\,\beta\,f'(0)^2 - 3\,\eta\,\alpha\,\beta^2\,f'(0)^2 + 3\,\eta\,\beta^3\,f'(0)^2, \\ p_1 &= \alpha^2\,\eta^2\,f'(0) - 4\,\alpha^2\,f'(0)^2 - 2\,\beta^2\,\eta^2\,f'(0) + 4\,\beta^2\,f'(0)^2, \\ p_0 &= -4\,\alpha\,\eta\,f'(0) - 4\,\beta\,\eta\,f'(0). \end{aligned}$$

The long-term expected product of position and velocity assumes the form

$$\widetilde{\mu} = \frac{a_3 \Delta t^3 + a_2 \Delta t^2 + a_1 \Delta t}{b_3 \Delta t^3 + b_2 \Delta t^2 + b_1 \Delta t + b_0}, \tag{4.2}$$

with

$$
\begin{aligned}
a_3 &= \alpha^2\,\beta\,f'(0)\,\varepsilon^2 - \alpha\,\beta^2\,f'(0)\,\varepsilon^2\\
a_2 &= -2\,\alpha\,\beta\,\eta\,\varepsilon^2\\
a_1 &= \alpha\,\varepsilon^2 - \beta\,\varepsilon^2\\
b_3 &= \alpha^4\,f'(0)^2 - 2\,\alpha^3\,\beta\,f'(0)^2 + 2\,\alpha\,\beta^3\,f'(0)^2 - \beta^4\,f'(0)^2\\
b_2 &= 3\,\eta\,f'(0)\,\alpha^3 - 3\,\eta\,f'(0)\,\alpha^2\,\beta - 3\,\eta\,f'(0)\,\alpha\,\beta^2 + 3\,\eta\,f'(0)\,\beta^3\\
b_1 &= 2\,\alpha^2\,\eta^2 - 4\,f'(0)\,\alpha^2 - 2\,\beta^2\,\eta^2 + 4\,f'(0)\,\beta^2\\
b_0 &= -4\;\eta(\alpha+\beta).
\end{aligned}
$$

The long-term mean-square of the velocity is given by

$$
\widetilde{\sigma}_v^2 = \frac{u_4\Delta t^4 + u_3\Delta t^3 + u_2\Delta t^2 + u_1\Delta t + u_0}{v_3\Delta t^3 + v_2\Delta t^2 + v_1,\Delta t + v_0} \tag{4.3}
$$

with

$$
\begin{aligned}
u_4 &= -\alpha^2\,\beta^2\,f'(0)^2\,\varepsilon^2 + 2\,\alpha\,\beta^3\,f'(0)^2\,\varepsilon^2 - \beta^4\,f'(0)^2\,\varepsilon^2,\\
u_3 &= 3\,\beta^3\,\eta\,f'(0)\,\varepsilon^2 - 3\,\alpha\,\beta^2\,\eta\,f'(0)\,\varepsilon^2,\\
u_2 &= -2\,\beta^2\,\eta^2\,\varepsilon^2 + 4\,f'(0)\,\beta^2\,\varepsilon^2 - 2\,\alpha\,f'(0)\,\beta\,\varepsilon^2,\\
u_1 &= -4\,\beta\,\eta\,\varepsilon^2,\\
u_0 &= -2\,\varepsilon^2,\\
v_3 &= \alpha^4\,f'(0)^2 - 2\,\alpha^3\,\beta\,f'(0)^2 + 2\,\alpha\,\beta^3\,f'(0)^2 - \beta^4\,f'(0)^2,\\
v_2 &= 3\,\eta\,f'(0)\,\alpha^3 - 3\,\eta\,f'(0)\,\alpha^2\,\beta - 3\,\eta\,f'(0)\,\alpha\,\beta^2 + 3\,\eta\,f'(0)\,\beta^3,\\
v_1 &= 2\,\alpha^2\,\eta^2 - 4\,f'(0)\,\alpha^2 - 2\,\beta^2\,\eta^2 + 4\,f'(0)\,\beta^2,\\
v_0 &= -4\,\eta\,(\alpha+\beta).
\end{aligned}
$$

Letting Δt go to zero, the limit correlation matrix is is given by

$$
\begin{pmatrix} -\dfrac{K\,T}{(\alpha+\beta)\;f'(0)} & 0\\ 0 & \dfrac{KT}{\alpha+\beta} \end{pmatrix}. \tag{4.4}
$$

Clearly, the condition to hold preservation is given by $\alpha + \beta = 1$. Such a condition makes the family of θ methods suitable for studying the conservation of the correlation matrix.

In [7], the authors study the correlation matrix of some numerical methods and show that the Implicit Midpoint Rule is the unique Runge-Kutta method able to perfectly preserve the exact correlation matrix of the linear case. For the linearized problem, we proceed in terms of determining conditions on the parameters of the problem (1.2), for which the distance $||\Sigma - \widetilde{\Sigma}||_F$ remains of the order of magnitude of the chosen Δt, when a selected method does not guarantee a perfect preservation of Σ. More precisely, if we impose a condition such as

$$|\sigma_x^2 - \widetilde{\sigma}_x^2| \leq k_x \Delta t, \qquad |\sigma_v^2 - \widetilde{\sigma}_v^2| \leq k_v \Delta t, \qquad |\mu - \widetilde{\mu}| \leq k_\mu \Delta t, \tag{4.5}$$

with k_x, k_v and k_μ constant, then

$$||\Sigma - \widetilde{\Sigma}||_F < \kappa \Delta t, \tag{4.6}$$

where $\kappa = \sqrt{{k_x}^2 + {k_v}^2 + 2{k_\mu}^2}$. k_x, k_v, k_μ and, consequently, κ should be chosen in a way to not exceed the order of magnitude of Δt for the distance. We make our analysis considering homogeneous bounds, i.e. $k_x = k_v = k_\mu = k$. With such choice we impose an homogeneous threshold to the error on each element of the matrix.

A study of the type (4.5) for a selected methods is, in general, quite difficult because of the dependence by several parameters, we proceed to simplify such a study in the most reasonable way possible, providing bounds for the step-size and for the damping parameter, in order to satisfy constraints such as (4.5).

Remark 1 Let us consider a generic explicit one-step method

$$y_{n+1} = y_n + \alpha \Delta t f(y_n) + \Delta W_n, \tag{4.7}$$

which applied to the system (1.2), assuming the ansatz (3.2), reads as

$$\begin{cases} X_{n+1} = X_n + \alpha \Delta t V_n, \\ V_{n+1} = \alpha \Delta t f'(0) X_n + (1 - \alpha \Delta t \eta) V_n + \varepsilon \Delta W_n. \end{cases}$$

Following the notation (2) of [7], we have

$$R = \begin{pmatrix} 1 & \alpha \Delta t \\ \alpha \Delta t f'(0) & 1 - \alpha \Delta t \eta \end{pmatrix}, \quad \text{and} \quad r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Via relation (2.3), we compute

$$\begin{aligned} \widetilde{\sigma}_x^2 &= -(\zeta \alpha f'(0))^{-1} \varepsilon^2 \left(f'(0)\, \Delta t^2\, \alpha^2 + \eta\, \Delta t\, \alpha - 2 \right), \\ \widetilde{\mu} &= \zeta^{-1} \Delta t\, \varepsilon^2, \\ \widetilde{\sigma}_v^2 &= -2(\zeta \alpha)^{-1} \varepsilon^2, \end{aligned}$$

with

$$\zeta = \Delta t^3\, \alpha^3\, f'(0)^2 + 3\, \Delta t^2\, \alpha^2\, \eta\, f'(0) + 2\, \Delta t\, \alpha\, \eta^2 - 4\, \Delta t\, \alpha\, f'(0) - 4\, \eta.$$

Letting Δt go to zero, the limit correlation matrix is is given by

$$\begin{pmatrix} -\dfrac{KT}{\alpha\, f'(0)} & 0 \\ 0 & \dfrac{KT}{\alpha} \end{pmatrix}. \tag{4.8}$$

5 Euler-Maruyama method

We study the particular case of the Euler method, whose numerical correlation matrix, similarly to [7], is given by

$$\widetilde{\Sigma}_E = \frac{KT}{1+\dfrac{f'(0)}{\eta}\Delta t}\left(2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2\right)^{-1}\begin{pmatrix} -\dfrac{1}{f'(0)}\left(2-\eta\Delta t-f'(0)\Delta t^2\right) & -\Delta t \\ -\Delta t & 2 \end{pmatrix}.$$

We start ensuring the positivity of $\widetilde{\sigma}_x^2$ and $\widetilde{\sigma}_v^2$. Since

$$\widetilde{\sigma}_x^2 = -\frac{KT}{f'(0)\left(1+\dfrac{f'(0)}{\eta}\Delta t\right)}\left(2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2\right)^{-1}\left(2-\eta\Delta t-f'(0)\Delta t^2\right)$$

and

$$\widetilde{\sigma}_v^2 = -\frac{2KT}{1+\dfrac{1}{f'(0)}\dfrac{f'(0)}{\eta}\Delta t}\left(2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2\right)^{-1},$$

we get positivity imposing

$$\frac{KT}{1+\dfrac{f'(0)}{\eta}\Delta t} > 0, \tag{5.1}$$

$$2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2 > 0, \tag{5.2}$$

$$2-\eta\Delta t-f'(0)\Delta t^2 > 0. \tag{5.3}$$

Notice that condition 5.2 implies 5.3.

Remark 2 In principle, even imposing

$$\begin{cases} \dfrac{KT}{1+\dfrac{f'(0)}{\eta}\Delta t} < 0, \\ 2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2 < 0, \\ 2-\eta\Delta t-f'(0)\Delta t^2 > 0, \end{cases} \tag{5.4}$$

we would have guaranteed the positivity of both $\widetilde{\sigma}_x^2$ and $\widetilde{\sigma}_v^2$. We do not consider this possibility, since we would have had instability, as said in [7].

Since (5.2) implies (5.3), then we must study

$$\frac{KT}{1+\dfrac{f'(0)}{\eta}\Delta t} > 0, \tag{5.5}$$

$$2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2 > 0. \tag{5.6}$$

If we solve with respect to η, we get the condition

$$-f'(0)\Delta t < \eta < \frac{2}{\Delta t} - \frac{1}{2}f'(0)\Delta t, \tag{5.7}$$

which holds by verifying the inequality

$$-f'(0)\Delta t < \frac{2}{\Delta t} - \frac{1}{2}f'(0)\Delta t. \tag{5.8}$$

From (5.8), we get a limitation on the step-size

$$\Delta t < \frac{2}{\sqrt{-f'(0)}}. \tag{5.9}$$

A very advantageous situation occurs whenever $f'(0)$ is small in modulus. Since we expect $KT\left(1+\frac{f'(0)}{\eta}\Delta t\right)^{-1}$ close to KT, if $\eta > (-f'(0))/k$, then $\frac{f'(0)}{\eta}\Delta t$ does not exceed the order of magnitude of Δt. Instead when $\eta < k$, the term $\eta\Delta t$ in $2-\eta\Delta t-\frac{1}{2}f'(0)\Delta t^2$ holds of the same order of Δt. We suppose to choose the step size in a way that the quantity $-f'(0)\Delta t$ does not exceed the order of magnitude of Δt, so that we can consider it negligible. Therefore we can say that

$$\widetilde{\sigma}_x^2 \approx -\frac{KT}{f'(0)},$$

which is exactly the value of σ_x^2. By detecting the terms involving $-f'(0)\Delta t$ in $\widetilde{\sigma}_v^2$ we get

$$\widetilde{\sigma}_v^2 \approx \frac{2KT}{2-\eta\Delta t},$$

so that $\left|\sigma_v^2 - \widetilde{\sigma}_v^2\right| < k\Delta t$ if and only if

$$\eta \lesssim \frac{2k}{KT + k\Delta t}.$$

Finally

$$\widetilde{\mu} \approx -\frac{KT\eta\Delta t}{2-\eta\Delta t},$$

therefore, imposing

$$|\widetilde{\mu}| < k\Delta t,$$

we get again $\eta < \dfrac{2k}{KT + k\Delta t}$.

Remark 3 With the hypothesis on the negligible term, the limitation on η to ensure the positivity of $\widetilde{\sigma}_x^2$ and $\widetilde{\sigma}_v^2$ is

$$2 - \eta\Delta t > 0 \Longleftrightarrow \eta < \frac{2}{\Delta t}.$$

The limitation found for bounding the error among the entries of the correlation matrices is tighter since

$$\frac{2k}{KT + k\Delta t} < \frac{2}{\Delta t} \Longleftrightarrow \frac{KT}{\Delta t(KT + k\Delta t)} > 0$$

which is satisfied for all $KT > 0$.

We conclude defining the critical value for η for the Euler-Maruyama method as

$$\eta_c = \frac{2k}{KT + k\Delta t}. \tag{5.10}$$

6 Trapezoidal Rule

In this section we dedicate a special attention to the theta method with $\theta = \frac{1}{2}$, i.e., the implicit trapezoidal rule. The motivation arises studying the general form of the correlation matrix and trying to understand the best value of $\theta \in [0, 1]$, corresponding to which, we get a correlation matrix as close as possible to (2). We start considering the expression of $\widetilde{\mu}$ (4.2) for a generic theta method, aiming to have the variables X_n and V_n closer to be uncorrelated as possible. $\widetilde{\mu}$ is expressed in (4.2) as a ratio of polynomials in the variable Δt, with coefficients depending by theta. In order to get a satisfactory integration, in general Δt is chosen of small amplitude, hence, we may think that the terms of greater amplitude in the numerator of (4.2) are the those involving the powers of Δt with exponent one. We find the value of θ which annihilates the coefficients of such term, i.e.,

$$a_1 = (\alpha - \beta)\varepsilon^2 = (1 - 2\theta)\varepsilon^2 = 0 \Longleftrightarrow \theta = \frac{1}{2}. \tag{6.1}$$

Then we observe that $\widetilde{\mu}$ reduces to

$$\widetilde{\mu} = \frac{\Delta t^2 \eta KT}{4}$$

and we have

$$\widetilde{\sigma}_x^2 = -\frac{KT}{f'(0)}\left(1 - \frac{f'(0)\Delta t^2}{4}\right),$$

$$\widetilde{\sigma}_v^2 = \frac{KT}{4}\left(\Delta t^2\eta^2 - f'(0)\Delta t^2 + 4\Delta t\eta + 4\right).$$

Therefore, $\widetilde{\mu} < k\Delta t$ if and only if

$$\eta < \frac{4k}{KT\Delta t}$$

. The distance between $\widetilde{\sigma}_x^2$ and σ_x^2 is independent by η, since

$$\begin{aligned}|\widetilde{\sigma}_x^2 - \sigma_x^2| &= \left|\frac{KT}{-f'(0)}\left(1 - \frac{f'(0)\Delta t^2}{4}\right) - \frac{KT}{-f'(0)}\right|\\ &= \frac{KT}{-f'(0)}\left|1 - \frac{f'(0)\Delta t^2}{4} - 1\right|\\ &= \frac{KT\Delta t^2}{4},\end{aligned}$$

so the condition $|\widetilde{\sigma}_x^2 - \sigma_x^2| < k\Delta t$ is satisfied whenever $KT < \dfrac{4k}{\Delta t}$.
Finally we consider

$$\begin{aligned}|\widetilde{\sigma}_v^2 - \sigma_v^2| &= \left|\frac{KT}{4}\left(\Delta t^2\eta^2 - f'(0)\Delta t^2 + 4\Delta t\eta + 4\right) - KT\right|\\ &= \frac{KT}{4}\left(\Delta t^2\eta^2 - f'(0)\Delta t^2 + 4\Delta t\eta\right),\end{aligned}$$

so that

$$|\widetilde{\sigma}_v^2 - \sigma_v^2| < k\Delta t$$

is equivalent to

$$\Delta t\eta^2 + 4\eta - f'(0)\Delta t - \frac{4}{KT}k < 0. \tag{6.2}$$

We compute the quantity

$$\frac{\Delta}{4} = 4 + f'(0)\Delta t^2 + \frac{4k\Delta t}{KT},$$

in which the unique negative addend is $f'(0)\,\Delta t^2$. However, it is supposedly to be small since it involves the factor Δt^2. Therefore we have two real roots for the equation associated to (6.2), which are given by

$$\eta_1 = -\frac{1}{\Delta t}\left(2 - \sqrt{4 + f'(0)\Delta t^2 + \frac{4k\Delta t}{KT}}\right),$$

$$\eta_2 = -\frac{1}{\Delta t}\left(2 + \sqrt{4 + f'(0)\Delta t^2 + \frac{4k\Delta t}{KT}}\right).$$

Since

$$\sqrt{4 + f'(0)\Delta t^2 + \frac{4k\Delta t}{KT}} > 2,$$

η_2 is negative so that (6.2) is satisfied for

$$0 < \eta < \eta_1.$$

We can conclude that (4.5) is satisfied when

$$0 < \eta < \min\left\{\eta_1, \frac{4k}{KT\Delta t}\right\}.$$

We can define the critical value for the Trepezoidal method as

$$\eta_c = \min\left\{\eta_1, \frac{4k}{KT\Delta t}\right\}. \tag{6.3}$$

7 Implicit Euler

The entries of the correlation matrix obtained via Implicit Euler method ($\theta = 1$) are

$$\widetilde{\sigma}_x^2 = -2KT\eta\left(-f'(0)\,\Delta t^2 + \eta\,\Delta t + 2\right)(f'(0)\tau)^{-1}, \tag{7.1}$$

$$\widetilde{\mu} = 2\Delta t\tau^{-1}\eta KT, \tag{7.2}$$

$$\widetilde{\sigma}_v^2 = 2KT\,\eta(\Delta t + 2\tau^{-1}), \tag{7.3}$$

with

$$\tau = \Delta t^3\, f'(0)^2 - 3\,\Delta t^2\,\eta\, f'(0) + 2\,\Delta t\,\eta^2 - 4\,\Delta t\, f'(0) + 4\,\eta.$$

We suppose that the term $-f'(0)\Delta t^2$ is negligible for the chosen Δt, then we truncate the terms involving it so that

$$\widetilde{\sigma}_x^2 \approx -\frac{2KT\eta\,\Delta t\,(\eta\,\Delta t + 2)}{f'(0)\,\left(2\,\Delta t^2\,\eta^2 + 4\,\Delta t\eta\right)} = -\frac{KT}{f'(0)},$$

which is the exact value of ${\sigma_x}^2$. With the same considerations, we get

$$\widetilde{\mu} \approx \frac{2\Delta t^2\,\eta KT}{2\,\Delta t^2\,\eta^2 + 4\,\eta\Delta t} = \frac{KT\Delta t}{\Delta t\eta + 2},$$

therefore

$$\widetilde{\mu} < k\Delta t \Longleftrightarrow \eta > \frac{KT - 2k}{k\Delta t}.$$

As we can see in [7], the values of $KT < 2k$ (for reasonable k), therefore the condition is verified for all $\eta > 0$.

Finally, we consider the value of $|\widetilde{\sigma}_v^2 - \sigma_v^2|$, given by

$$\left|\tau^{-1}\Delta tKT\left(2\eta\tau - 3\Delta t\eta f'(0) - 2\eta^2 - \Delta t^2 f'(0)^2 + 4f'(0)\right)\right|.$$

We truncate the terms involving powers of Δt until the second, except the term $4\Delta t^2\eta^3$ in the numerator, which is not, in general, negligible. We approximate the term $4\Delta t^2\eta^3$ by $4\Delta t^2\eta^2$, in order to be able to solve the inequality to find our bounds. The presence of Δt^2 should limit the effect of such a substitution. Therefore,

$$|\widetilde{\sigma}_v^2 - \sigma_v^2| \approx \left| \frac{\Delta t\, KT\, \left(4\,\Delta t\,\eta^3 + 6\,\eta^2 + 4\,f'(0)\right)}{2\,\Delta t\,\eta^2 + 4\,\eta - 4\,\Delta t\,f'(0)} \right| = \frac{\Delta t\, KT|(2\,\Delta t + 3\,)\eta^2 + 2\,f'(0)|}{\Delta t\,\eta^2 + 2\,\eta - 2\,\Delta t\,f'(0)}.$$

For $0 < \eta < \sqrt{\dfrac{-2\,f'(0)}{2\Delta t + 3}}$, we study

$$-\frac{\Delta t\, KT((2\,\Delta t + 3\,)\eta^2 + 2\,f'(0))}{\Delta t\,\eta^2 + 2\,\eta - 2\,\Delta t\,f'(0)} < k\Delta t. \tag{7.4}$$

otherwise

$$\frac{\Delta t\, KT((2\,\Delta t + 3\,)\eta^2 + 2\,f'(0))}{\Delta t\,\eta^2 + 2\,\eta - 2\,\Delta t\,f'(0)} < k\Delta t, \tag{7.5}$$

Since the denominator in (7.5) is positive, (7.5) is equivalent to

$$(2KT\Delta t + 3KT - k\Delta t)\eta^2 - 2k\Delta t\eta + 2KTf'(0) + 2k\Delta tf'(0) < 0,$$

we have that the quantity

$$\frac{\Delta}{4} = (-2KT + k)2\,f'(0)k\Delta t^2 - (KT + k)4\,f'(0)\,KT\,\Delta t + 4k^2 - 6KT^2 f'(0)$$

is generally positive since the only negative term is proportional to Δt^2. The roots of the equation associated to (7.5) are

$$\eta_1 = \frac{k\Delta t - \sqrt{\Delta/4}}{\Delta t(2KT\Delta t + 3KT - k\Delta t)}, \quad \eta_2 = \frac{k\Delta t + \sqrt{\Delta/4}}{\Delta t(2KT\Delta t + 3KT - k\Delta t)}$$

but η_1 is negative, therefore (7.5) is satisfied for

$$\eta > \max\left\{\eta_2, \sqrt{\frac{-2\,f'(0)}{2\Delta t + 3}}\right\}$$

When $0 < \eta < \sqrt{\dfrac{-2f'(0)}{2\Delta t + 3}}$, (6.2) is equivalent to

$$(2KT\Delta t + 3KT + k\Delta t)\eta^2 + 2k\eta + 2KTf'(0) - 2k\Delta tf'(0) > 0.$$

The quantity

$$\frac{\Delta}{4} = (2KT + k)2\,f'(0)k\Delta t^2 + (k - KT)4\,f'(0)\,KT\,\Delta t + 4k^2 - 6KT^2 f'(0)$$

may be assumed positive. In fact, the negative terms are negligible, since they involves the factor Δt^2 and Δt, and the dominant term is positive. The real roots of the equation associated to (7.4) are

$$\eta_1 = \frac{-k\Delta t - \sqrt{\Delta/4}}{\Delta t(2KT\Delta t + 3KT - k\Delta t)}, \quad \eta_2 = \frac{-k\Delta t + \sqrt{\Delta/4}}{\Delta t(2KT\Delta t + 3KT - k\Delta t)}$$

Finally, since η_1 is negative, (7.4) is satisfied for

$$0 < \eta < \min\left\{\eta_2, \sqrt{\frac{-2f'(0)}{2\Delta t + 3}}\right\}.$$

We define the critical values for η in the Implicit Euler case

$$\eta_1 = \max\left\{\eta_2, \sqrt{\frac{-f'(0)}{2\Delta t + 3}}\right\}, \quad \eta_2 = \min\left\{\eta_2, \sqrt{\frac{-f'(0)}{2\Delta t + 3}}\right\}. \tag{7.6}$$

8 Numerical experiments: scalar case

We now present the numerical evidence arising from selected nonlinear scalar problems and systems of equations. In our numerical experiments, we set all the constants k_x, k_v and k_μ equal to 5, which certainly ensures $\kappa < 10$.

We first consider the equation of the damped pendulum

$$\begin{cases} dX_t = V_t dt, \\ dV_t = -\eta V_t dt - \omega \sin(X_t)dt + \varepsilon dW_t, \end{cases} \tag{8.1}$$

in this case $f(x) = -\omega \sin(x)$, with $\omega > 0$, therefore $f'(0) = -\omega$. Setting $KT = 1$ and $\Delta t = 0.01$, we plot the function $d(\eta) = ||\Sigma(\eta) - \widetilde{\Sigma}(\eta)||_F$ for the Euler-Maruyama method in Figure 1, for $\omega = 1$ (left side) and $\omega = 10$ (right side). When $\omega = 1$, the assumption made in Section 5 of choosing Δt such that $f'(0)\Delta t$ is negligible is satisfied. Coherently with the theoretical considerations in Section 5, $d(\eta)$ remains under the threshold $5\Delta t$ for $\eta < \eta_c$, where the critical value $\eta_c = 9.5238$ is computed according to (5.10). As we expect by construction of our analysis , η_c is a severe limitation since the error d(η) does not exceed the order of magnitude of Δt (i.e., it is less than $10\Delta t$) for values $\eta \lessapprox 16$. For $\omega = 10$, it is clear that the limitation analysis does not work anymore.

Figures 2 and 3 are dedicated to describe the study of the Trapezoidal Rule. We can observe that $d(\eta)$ increases almost linearly and it crosses the value $5\Delta t$ in the corresponding critical value η_c given by (6.3). In Figure 3, we aim to show that passing by ω to $\omega = 100$, the behaviours of the corresponding $d(\eta)$and the corresponding critical values of η_c are very close, although Δt remains unchanged. This is a deep difference between Euler-Maruyama and Trapezoidal method. Notice that, for any method considered, $d(\eta)$ is always

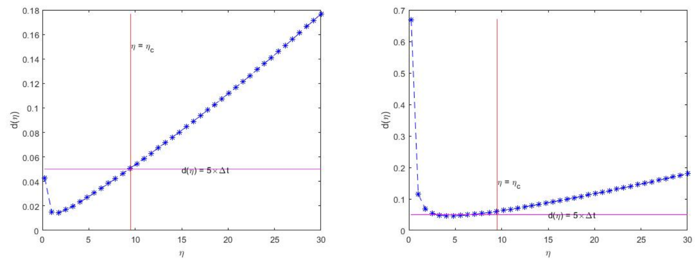

Fig. 1: The figures show the behaviour of the function $d(\eta)$ of the Euler-Maruyama method. In the figure on the left, we suppose $\omega = 1$ and set $\Delta t = 10^{-2}$ and $KT = 1$. In the figure on the right, we suppose $\omega = 10$.

proportional to the constant KT; this means that small values of KT are advantageous. By the left Figure 2, we highlight dependence of $d(\eta)$ of the Trapezoidal Rule from the value of KT; in fact, setting $KT = 1$ we get $\eta_c \approx 44.95$, while instead for $KT = 0.1$, $\eta_c \approx 4.495$, fro the same values of Δt and ω. In the right Figure 3, we plot $d_x(\eta)$, which accordingly to the theoretical analysis of Section 6, is perfectly preserved for all the values of the damping. In other words, the main term in $d(\eta)$ is the error on $\widetilde{\sigma}_v^2$, i.e., $d_v(\eta) = |\sigma_v^2 - \widetilde{\sigma}_v^2|$. The same situation occurs for the Implicit Euler method, so that we directly plot $d_v(\eta)$ in Figure 4 and 5.

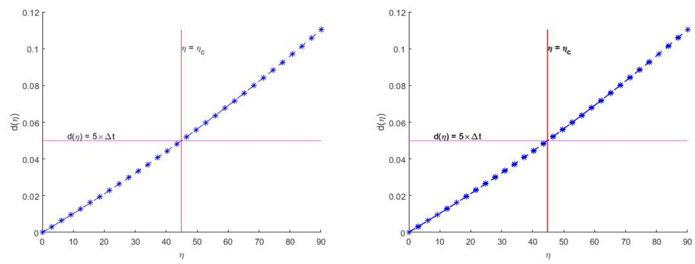

Fig. 2: The figures show the behaviour of the function $d(\eta)$, for the Trapezoidal Rule. In both cases we set $KT = 0.1$ and $\Delta t = 0.01$. In the figure on the right $\omega = 1$ and $\eta_c \approx 44.95$. In the figure on the right $\omega = 100$ and $\eta_c \approx 44.74$.

The range of the critical values of η is essentially translated passing by $\omega = 1$ to $\omega = 10$, Figure 4. In the left Figure 5, we set $\omega = 100$, maintaining $\Delta t = 0.01$, in order to show that the hypothesis $\Delta t^2 f'(0)$ negligible (Section 7) must be satisfied, in order to guarantee the validity of our analysis.

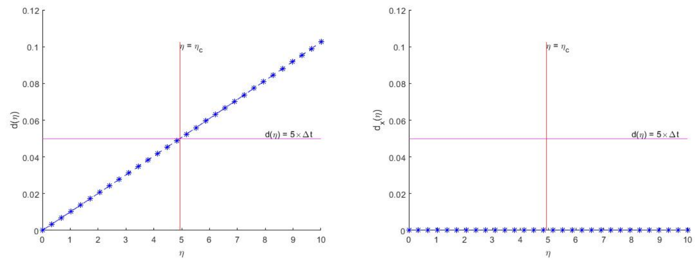

Fig. 3: The figures are referred to the Trapezoidal Rule. We set $\Delta t = 10^{-2}$ and $KT = 1$ and $\omega = 1$. In the figure on the left, we show the behaviuor of $d(\eta)$. The critical point is $\eta_c = 4.95$. In the figure on the right is represented the function $d_x(\eta)$.

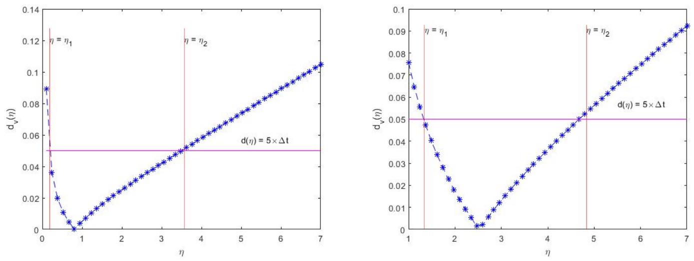

Fig. 4: The figures shows the function $\mathrm{d}_v(\eta)$ of the Implicit Euler method. In the figure on the left, we set $KT = 1$, $\Delta t = 0.01$, $\omega = 1$. The critical values for η are $\eta_1 \approx 0.18$ and $\eta_1 \approx 3.57$. In the fiugre on the right, the values differ just for $\omega = 10$. The critical values for η are $\eta_1 \approx 1.34$ and $\eta_1 \approx 4.83$.

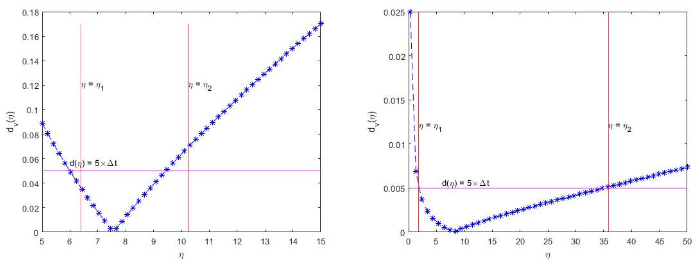

Fig. 5: The figure on the left shows the function $d_v(\eta)$ of the Implicit Euler method, setting $KT = 1$, $\Delta t = 0.01$ and $\omega = 100$. In the figure on the right, we set $KT = 0.1$, $\Delta t = 0.001$ and $\omega = 100$. The critical values for η are $\eta_1 = 1.8$ and $\eta_2 = 35.86$.

8.1 Comparison of the deviations in the correlation matrix

In this section, we aim to compare how the linearized study of (8.1) behaves with respect to the fully implicit approach. For a considered θ method, we construct the matrix $\overline{\Sigma}$, taking the long term behaviours of the position and velocity and evaluating what happens to their correlation for long times. we compare

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.5	0.0932	0.0977	0.0048
1	0.0909	0.0925	0.0063
2	0.0925	0.0909	0.0080

Table 1: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Euler-Maruyama method with $\Delta t = 0.01$.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.5	0.0017	0.0463	0.0469
1	0.0078	0.0715	0.0897
2	0.0031	0.0613	0.0636

Table 2: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Euler-Maruyama method with $\Delta t = 0.001$.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.5	0.0941	0.0921	0.0051
1	0.0950	0.0867	0.0052
2	0.0942	0.0935	0.0050

Table 3: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Trapezoidal method with $\Delta t = 0.01$.

the obtained $\overline{\Sigma}$ with the corresponding $\widetilde{\Sigma}$ of the linearized approach for different values of η. Clearly we analyse the cases $\theta = 0, 0.5, 1$ for $\eta = 0.5, 1, 2$. We set $KT = 0.1$ and $\omega = 1$. Tables 1, 3 and 5 show the results for Euler-Maruyama, Trapezoidal Rule and Implicit Euler, respectively, for $\Delta t = 0.01$. Tables 2, 4 and 6 show the results for Euler-Maruyama, Trapezoidal Rule and Implicit Euler method, respectively, for $\Delta t = 0.001$. We can observe that choosing such good values for η (with respect to our linearized analysis) the difference among the long term statistics of X and V never exceed the order of magnitude of $c \times 10^{-2}$ and they are particularly satisfactory in many cases. Clearly the comparison highlights that a reduction of the step-size corresponds to a reduction of the errors $\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$ and $\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$. Looking at the difference among $\overline{\mu}$ and $\widetilde{\mu}$ we can not say that X and V are perfectly long term uncorrelated, but the correlation is not strong.

η	$\lvert\overline{\sigma}_x^2-\widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2-\widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu}-\widetilde{\mu}\rvert$
0.5	0.0442	0.0025	0.0427
1	0.0188	0.0348	0.0470
2	0.0241	0.0198	0.0908

Table 4: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Trapezoidal method with $\Delta t = 0.001$.

η	$\lvert\overline{\sigma}_x^2-\widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2-\widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu}-\widetilde{\mu}\rvert$
0.5	0.0971	0.0908	0.0014
1	0.0983	0.0895	0.0983
2	0.0990	0.0940	9.6748×10^{-4}

Table 5: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Implicit Euler method with $\Delta t = 0.01$.

η	$\lvert\overline{\sigma}_x^2-\widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2-\widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu}-\widetilde{\mu}\rvert$
0.5	0.0105	0.0083	0.0717
1	0.0074	0.0247	0.0800
2	0.0119	0.0446	0.0556

Table 6: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Implicit Euler method with $\Delta t = 0.001$.

9 Numerical experiments: a nonlinear system

We consider the following system

$$\begin{cases} dX = V dt \\ dV = -\eta V dt + f(X)dt + \varepsilon dW \end{cases} \tag{9.2}$$

where $X = [X_1, X_2]^\intercal$, $V = [V_1, V_2]^\intercal$ and $dW = [dW_1, dW_2]^\intercal$. We choose

$$f(X) = \left[\cos\left(X_1 + \frac{\pi}{2}\right), \quad -\frac{\sin(X_2)}{X_2^2+1}\right]^\intercal.$$

We suppose independence between the particles (X_1, V_1) and (X_2, V_2), coherently with the theory followed in this article. We test the theta methods, with $\theta = 0, 0.5, 1$, for values of $\eta = 0.7, 1.2.$. Tables 7 and 8 show the results for Euler-Maruyama, while Tables 9 and 10 are referred to the Trapezoidal Rule. Finally, the comparisons for the implicit Euler method are exhibited in Tables 11and 12. In our experiments, we fix $\Delta t = 0.001$ and $KT = 0.1$.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	0.0107	0.0189	0.0806
1.2	0.0074	0.0291	0.0476

Table 7: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Explicit- Euler, for the variables X_1 and V_1.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	0.0928	0.0458	0.0544
1.2	0.0355	0.0106	0.0848

Table 8: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Explicit- Euler, for the variables X_2 and V_2.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	8.3160×10^{-4}	0.0300	0.0627
1.2	0.0612	0.0215	0.0689

Table 9: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Trapezoidal method, for the variables X_1 and V_1.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	0.0578	0.0137	0.0694
1.2	0.0971	0.0477	0.0544

Table 10: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Explicit- Euler, for the variables X_2 and V_2.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	0.0108	0.0188	0.0803
1.2	0.0167	0.0235	0.0557

Table 11: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Explicit- Euler, for the variables X_1 and V_1.

η	$\lvert\overline{\sigma}_x^2 - \widetilde{\sigma}_x^2\rvert$	$\lvert\overline{\sigma}_v^2 - \widetilde{\sigma}_v^2\rvert$	$\lvert\overline{\mu} - \widetilde{\mu}\rvert$
0.7	0.0122	0.0340	0.0460
1.2	0.0946	0.0110	0.0842

Table 12: Element-wise comparison among $\overline{\Sigma}$ and $\widetilde{\Sigma}$ for the Explicit- Euler, for the variables X_2 and V_2.

10 Conclusions

In this work, the conservation properties of the correlation matrix of the stochastic θ-methods for the SDE (1.1) are analyzed. At the best of our knowledge, this is the first attempt to rigorously analyze the properties of stochastic θ-methods for the nonlinear oscillator (1.1). In addition, the presented approach also substantially enriches the analysis of the linear case studied in [7]. A particular attention has been devoted to the Euler-Maruyama method, widely used in the discretization of SDEs. Moreover, the study of the trapezoidal and implicit Euler methods is also extremely useful in order to assess the knowledge of their properties. Indeed, these methods show excellent preservation properties, such as the unconditional contractivity of the mplicit Euler method, proved in [18], and the mean square A-stability of the trapezoidal method, analyzed in [22].

Acknowledgments
The authors thank the anonymous referee, for valuable remarks; in particular, for suggesting further comparisons in Section 9. This work is supported by GNCS-INDAM project and by PRIN2017-MIUR project. The first author is member of the INdAM Research group GNCS.

References

1. Anton, R., Cohen, D.: Exponential integrators for stochastic Schrödinger equations driven by Ito noise, J. Comput. Math. 36(2), 276–309 (2019).
2. Buckwar, E., D'Ambrosio, R.: Exponential mean-square stability properties of stochastic multistep methods, submitted.
3. Buckwar, E.; Sickenberger, T.: A comparative linear mean-square stability analysis of Maruyama- and Milstein-type methods. Math. Comput. Simul. 81, 1110–1127 (2011).
4. Bryden, A.; Higham, D.J.: On the boundedness of asymptotic stability regions for the stochastic theta method. BIT 43, 1–6 (2003).
5. Burrage, P.M.; Burrage, K.: Structure-preserving Runge-Kutta methods for stochastic Hamiltonian equations with additive noise. Numer. Algor. 65, 519–532 (2012).
6. Burrage, P.M.; Burrage, K.: Low rank Runge-Kutta methods, symplecticity and stochastic Hamiltonian problems with additive noise. J. Comput. Appl. Math. 236, 3920–3930 (2014).
7. Burrage, K.; Lenane, I.; Lythe, G.: Numerical methods for second-order stochastic differential equations. SIAM J. Sci. Comput. 29(1), 245–264 (2007).
8. Burrage, K.; Lythe, G.: Accurate stationary densities with partitioned numerical methods for stochastic differential equations. SIAM J. Numer. Anal. 47, 1601–1618 (2009).
9. Burrage, K.; Lythe, G.: Accurate stationary densities with partitioned numerical methods for stochastic partial differential equations. Stochastic Partial Differential Equations: Analysis and Computations. 2(2), 262–280 (2014).
10. Cardone, A., D'Ambrosio, R., Paternoster, B.: A spectral method for stochastic fractional differential equations, Appl. Numer. Math. 139, 115–119 (2019).
11. Cardone, A., Conte, D., D'Ambrosio, R., Paternoster, B.: Stability Issues for Selected Stochastic Evolutionary Problems: A Review, Axioms 7(4), 91 (2018).
12. Chen, C., Cohen, D., D'Ambrosio, R., Lang, A.: Drift-preserving numerical integrators for stochastic Hamiltonian systems, arXiv:1907.08804, submitted.
13. Citro, V., D'Ambrosio, R.: Long-term analysis of stochastic θ-methods for damped stochastic oscillators, doi: 10.1016/j.apnum.2019.08.011, Appl. Numer. Math. (2019).

14. Citro, V., D'Ambrosio, R., Di Giovacchino, S.: A-stability preserving perturbation of Runge–Kutta methods for stochastic differential equations, Appl. Math. Lett. 102,106098 (2020).
15. Cohen, D., Gauckler, L., Hairer, E., Lubich, C. Long-term analysis of numerical integrators for oscillatory Hamiltonian systems under minimal non-resonance conditions, BIT Numer. Math. 55(3), 705–732 (2015).
16. Conte, D.; D'Ambrosio, R.; Paternoster, B. On the stability of θ-methods for stochastic Volterra integral equations. Discret. Cont. Dyn. Syst. B 2018, 23, 2695–2708.
17. D'Ambrosio, D.; Moccaldi, M.; Paternoster, B. Numerical preservation of long-term dynamics by stochastic two-step methods. Discrete and Continuous Dynamical Systems Series B. 23(7), 2763–2773 (2018).
18. D'Ambrosio, R., Di Giovacchino, S.: Mean-square contractivity of stochastic θ-methods, submitted.
19. Gard, T.C.: Introduction to Stochastic Differential Equations, Marcel Dekker Inc., New York-Basel (1988).
20. Gardiner, C.W. Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences, 3rd ed., Springer-Verlag, Berlin (2004).
21. Hairer, L.; Lubich, C.; Wanner, G. Geometric Numerical Integration; Springer: Berlin/Heidelberg, Germany (2006).
22. Higham, D.J. Mean-square and asymptotic stability of the stochastic theta method. SIAM J. Numer. Anal. 38, 753–769 (2000).
23. Kloeden, P.E.; Platen, E.: The Numerical Solution of Stochastic Differential Equations. Springer-Verlag (1992).
24. Milstein, G.N., Tretyakov M.V.: Stochastic Numerics for Mathematical Physics. Scientific Computation. Springer-Verlag, Berlin (2004).
25. Schurz, H. The invariance of asymptotic laws of linear stochastic systems under discretization. Z. Angew. Math. Mech. 6, 375–382 (1999).
26. Strömmen Melbö, A.H.; Higham, D.J. Numerical simulation of a linear stochastic oscillator with additive noise. Appl. Numer. Math. 51, 89–99 (2004).
27. Vilmart, G. Weak second order multirevolution composition methods for highly oscillatory stochastic differential equations with additive or multiplicative noise. SIAM J. Sci. Comput. 36(4), A1770–A1796 (2014).