Nonlinear stability issues for stochastic Runge-Kutta methods

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Abstract

The paper provides a nonlinear stability analysis for a class of stochastic Runge-Kutta methods, applied to problems generating mean-square contractive solutions. In particular, we show how this property is inherited along the solutions generated by the stochastic perturbation of an algebraically stable deterministic Runge-Kutta method. The effectiveness of the results is also confirmed by selected numerical experiments.

Keywords: Nonlinear stochastic differential equations, nonlinear stability analysis, stochastic Runge-Kutta methods.

1. Introduction

We consider a nonlinear system of stochastic differential equations (SDEs) of Itô type, assuming the form

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), \quad t \in [0,T], \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $f, g : \mathbb{R}^d \to \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $W\colon [0,T]\times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths on $(\Omega, \mathcal{F}, \mathbb{P})$. Theoretical results on the existence and uniqueness to solutions of (1.1) are discussed, for instance, in the monograph [21].

Here we focus our attention on providing a nonlinear stability analysis to a general class of stochastic Runge-Kutta methods (SRK) that, with reference to the discretized domain

$$\mathcal{I}_h = \{t_n = nh,\ n = 0, 1, \dots, N,\ N = T/h\},$$

assume the following form

$$\begin{cases} X_n = X_{n-1} + h \displaystyle\sum_{i=1}^{s} b_i f(\widehat{X}_i^{[n]}) + \Delta W_n \sum_{i=1}^{s} q_i g(\widehat{X}_i^{[n]}), \qquad n = 1, ..., N \\ \widehat{X}_i^{[n]} = X_{n-1} + h \displaystyle\sum_{j=1}^{s} a_{ij} f(\widehat{X}_j^{[n]}) + \Delta W_n \sum_{j=1}^{s} \gamma_{ij} g(\widehat{X}_i^{[n]}), \quad i = 1, ..., s. \end{cases} \tag{1.2}$$

X_n provides an approximation to $X(t_n)$, while the internal stage $\widehat{X}_i^{[n]}$ approximates $X(t_n + c_i h)$, $i = 1, \dots, s$ and the discretized Wiener increment ΔW_n is distributed as a gaussian random variable with zero mean and variance h. A usual representation for (1.2) consists in the following Butcher tableau

$$\begin{array}{c|c|c} c & A & \Gamma \\ \hline & b^{\intercal} & q^{\intercal} \end{array} = \begin{array}{c|cccc|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} & \gamma_{11} & \gamma_{12} & \dots & \gamma_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} & \gamma_{21} & \gamma_{22} & \dots & \gamma_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} & \gamma_{s1} & \gamma_{s2} & \dots & \gamma_{ss} \\ \hline & b_1 & b_2 & \dots & b_s & q_1 & q_2 & \dots & q_s \end{array}. \tag{1.3}$$

SRK methods (1.2) are then formulated as a stochastic perturbation to the well-known deterministic Runge-Kutta (RK) methods

$$\begin{cases} x_n = x_{n-1} + h \displaystyle\sum_{i=1}^{s} b_i f(\widehat{x}_i^{[n]}), \\ \widehat{x}_i^{[n]} = x_{n-1} + h \displaystyle\sum_{j=1}^{s} a_{ij} f(\widehat{x}_j^{[n]}), \quad i = 1, ..., s, \end{cases} \tag{1.4}$$

for the deterministic autonomous differential system

$$\begin{cases} x'(t) = f(x(t)), \quad t \in [0, T], \\ x(0) = x_0. \end{cases}$$

The analysis of strong and weak accuracy properties of SRK methods of various types has extensively been addressed by the existing literature; see, for instance [1, 3, 4, 5, 6, 7, 8, 10, 11, 16, 22, 23, 24, 25] and references therein. The family of SRK methods formulated as in (1.2) has been analyzed in depth in the monograph [17] and references therein, where we can infer that the strong convergence of (1.2) follows from the convergence of the underlying RK method (1.4), i.e., it occurs when $\sum_{i=1}^{s} b_i = 1$, plus the additional condition $\sum_{i=1}^{s} q_i = 1$. The extension to the multi-dimensional case has been addressed in [25], where several conditions guaranteeing its strong convergence have been considered.

In this paper, we address our investigation to the analysis of the nonlinear stability properties of SRK methods (1.2), here intended as preservation of the mean-square monotonicity property characteristic of the stochastic problems (1.1) described by the following result [19, 20].

Theorem 1.1. *For a given nonlinear SDE* (1.1)*, let us assume the following properties for the drift f and the diffusion g, by denoting with $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$ and with $\mathbb{E}$ the mathetmatical expectation operator:*

(i) $f(0) = g(0) = 0$*;*

(ii) f satisfies a one-sided Lipschitz condition, i.e. there exists $\mu \in \mathbb{R}$ such that

$$< x - y, f(x) - f(y) > \leq \mu \, |x - y|^2, \quad \forall x, y \in \mathbb{R}^d; \tag{1.5}$$

(iii) g is a globally Lipschitz function, i.e. there exists $L > 0$ such that

$$|g(x) - g(y)|^2 \leq L \, |x - y|^2 \quad \forall x, y \in \mathbb{R}^d \tag{1.6}$$

Then, any two solutions $X(t)$ and $Y(t)$ of (1.1), with $\mathbb{E}\,|X_0|^2 < \infty$ and $\mathbb{E}\,|Y_0|^2 < \infty$, satisfy

$$\mathbb{E}\,|X(t) - Y(t)|^2 \leq \mathbb{E}\,|X_0 - Y_0|^2 \, e^{\alpha t}, \tag{1.7}$$

where $\alpha = 2\mu + L$.

We observe that, when the diffusion g is identically zero, Theorem 1.1 allows the recovery of the classical result on the contractive behaviour for the corresponding deterministic problem [18]. In this setting, the corresponding discretization of (1.1) lead to the notion of G-stability, introduced by G. Dahlquist in [12].

With reference to (1.7), if $\alpha < 0$, we have

$$\mathbb{E}|X(t) - Y(t)|^2 \leq \mathbb{E}|X(s) - Y(s)|^2, \tag{1.8}$$

if $s \leq t$, that is, the mean-square deviation of any two exact solutions of (1.1) decays as time increases. Therefore, the following definition is given.

Definition 1.1. *If* (1.8) *holds for any two solutions of* (1.1)*, problem* (1.1) *is said to generate* mean-square contractive solutions.

This issue would provide, for instance, classes of SRK methods (1.2) which may numerically inherit the stability properties of nonlinear test problems, such as those in [2, 9, 13].

In our investigation, we are interested in preserving the mean-square contractivity along the dynamics generated by SRK methods (1.2). Specifically, we propose to achieve this property from the stochastic perturbation of a deterministic algebraically stable RK method (1.4), plus some additional conditions on the coefficients of (1.2). It is worthwhile recalling that a RK method depending on the Butcher tableau (1.3) is algebraically stable if the matrix $M = BA + A^{\intercal}B - bb^{\intercal}$ is symmetric positive semi-definite and $b_i \geq 0$, for any $i = 1, ..., s$, with $B = \mathrm{diag}(b)$.

The paper is organized as follows: Section 2 describes the main result of the paper, leading to the analog of (1.8), when the exact solution is replaced by the numerical one computed by (1.2); the result also reveals the presence of a spurious term, analyzed in Section 3. In Section 4, we discuss the generalization of our results to the case of stochastic differential equations driven by a multi-dimensional Wiener process. Some numerical experiments confirming the theoretical analysis are presented in Section 5 and concluding remarks are given in Section 6.

2. Main result

In this section, we prove the main result of our investigation, i.e., we give conditions on the coefficients of the SRK method (1.2) to obtain mean-square contractivity over the numerical solutions to (1.1). We observe that the proof of the following result follows the philosophy of the proof of Theorem IV.12.4, in [18] and provides, to some extent, its stochastic extension.

Theorem 2.1. *Let us consider a stochastic differential system* (1.1) *satisfying the hypothesis (i)–(iii) of Theorem 1.1. We consider the* (c, A, Γ, b, q)*-SRK method* (1.2)*, arising from the stochastic perturbation of an algebraically-stable deterministic Runge-Kutta method* (1.4)*. Moreover, we denote by N the matrix*

$$N = Q\Gamma + \Gamma^{\intercal}Q - qq^{\intercal}, \tag{2.1}$$

where $Q = \mathrm{diag}(q)$*. If N is a symmetric positive semi-definite matrix and*

$$B\Gamma + A^{\intercal}Q = bq^{\intercal}, \tag{2.2}$$

with $B = \mathrm{diag}(b)$*, then, any two numerical solutions* X_n *and* Y_n *to* (1.1) *computed by* (1.2) *with initial values* X_0 *and* Y_0*, respectively, with* $\mathbb{E}|X_0|^2 < \infty$ *and* $\mathbb{E}|Y_0|^2 < \infty$*, satisfy the following inequality*

$$\mathbb{E}|X_n - Y_n|^2 \leq \mathbb{E}|X_{n-1} - Y_{n-1}|^2 + \phi_n(h), \tag{2.3}$$

where

$$\phi_n(h) = 2\sum_{i=1}^{s} q_i \mathbb{E}\Big(\Delta W_n \left\langle \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]}, g\left(\widehat{X}_i^{[n]}\right) - g\left(\widehat{Y}_i^{[n]}\right)\right\rangle\Big), \tag{2.4}$$

$n = 1, 2, ..., N$.

Proof: We define

$$\begin{aligned} Z_n &= X_n - Y_n, & \Delta f_i^{[n]} &= f\left(\widehat{X}_i^{[n]}\right) - f\left(\widehat{Y}_i^{[n]}\right), \\ \widehat{Z}_i^{[n]} &= \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]}, & \Delta g_i^{[n]} &= g\left(\widehat{X}_i^{[n]}\right) - g\left(\widehat{Y}_i^{[n]}\right), \end{aligned}$$

for $i = 1, ..., s$. This notation yields

$$\begin{cases} Z_n = Z_{n-1} + h\displaystyle\sum_{i=1}^{s} b_i \Delta f_i^{[n]} + \Delta W_n \sum_{i=1}^{s} q_i \Delta g_i^{[n]}, \\ \widehat{Z}_i^{[n]} = Z_{n-1} + h\displaystyle\sum_{j=1}^{s} a_{ij} \Delta f_j^{[n]} + \Delta W_n \sum_{j=1}^{s} \gamma_{ij} \Delta g_j^{[n]}, \quad i = 1, ..., s. \end{cases} \tag{2.5}$$

Squaring side by side the first relation in (2.5), applying the algebraic stability of (1.4) and hypothesis (*ii*) of Theorem 1.1 lead to

$$\begin{aligned} |Z_n|^2 \le |Z_{n-1}|^2 &+ \Delta W_n^2 \sum_{i,j=1}^{s} q_i q_j \left\langle \Delta g_i^{[n]}, \Delta g_j^{[n]} \right\rangle + 2\Delta W_n \sum_{i=1}^{s} q_i \left\langle Z_{n-1}, \Delta g_i^{[n]} \right\rangle \\ &+ 2h\Delta W_n \sum_{i,j=1}^{s} (b_i q_j - b_i \gamma_{ij}) \left\langle \Delta f_i^{[n]}, \Delta g_j^{[n]} \right\rangle. \end{aligned}$$

Applying the second relation in (2.5), we obtain

$$\begin{aligned} |Z_n|^2 \le |Z_{n-1}|^2 &- \Delta W_n^2 \sum_{i,j=1}^{s} (q_i \gamma_{ij} + q_j \gamma_{ji} - q_i q_j) \left\langle \Delta g_i^{[n]}, \Delta g_j^{[n]} \right\rangle \\ &+ 2\Delta W_n \sum_{i=1}^{s} q_i \left\langle \widehat{Z}_i^{[n]}, \Delta g_i^{[n]} \right\rangle \\ &+ 2h\Delta W_n \sum_{i,j=1}^{s} (b_i q_j - b_i \gamma_{ij} - q_i a_{ij}) \left\langle \Delta f_i^{[n]}, \Delta g_j^{[n]} \right\rangle. \end{aligned}$$

Since N is a symmetric positive definite matrix, according to [18], the term

$$\Delta W_n^2 \sum_{i,j=1}^{s} (q_i \gamma_{ij} + q_j \gamma_{ji} - q_i q_j) \left\langle \Delta g_i^{[n]}, \Delta g_j^{[n]} \right\rangle$$

is positive. Then, also taking into account assumption (2.2), we obtain

$$|Z_n|^2 \le |Z_{n-1}|^2 + 2\Delta W_n \sum_{i=1}^{s} q_i \left\langle \widehat{Z}_i^{[n]}, \Delta g_i^{[n]} \right\rangle. \tag{2.6}$$

The application, side-by-side, of the expectation operator gives the result.

□

3. Analysis of the spurious term

According to Theorem 2.1, in the numerical solution of stochastic problems generating mean-square contractive solutions via SRK methods (1.2), inequality (1.8) characterizing the continuous problem is translated into (2.3) for the discretized one. Then, the spurious term $\phi_n(h)$ defined in (2.4) may eventually corrupt the numerical preservation of the mean-square contractivity.

The first result of this section shows that the spurious term $\phi_n(h)$ is small for sufficiently small values of the stepsize.

Theorem 3.1. *Under condition (iii) of Theorem* (1.1)*, we have*

$$\lim_{h\to 0} \max_n \phi_n(h) = 0. \tag{3.7}$$

Proof: For any $n = 1, 2, ..., N$, applying the Hölder inequality yields

$$\begin{aligned} |\phi_n(h)| &\leq 2\mathbb{E}|\Delta W_n| \sum_{i=1}^{s} q_i \left\| \left\langle \widehat{Z}_i^{[n]}, \Delta g_i^{[n]} \right\rangle \right\|_\infty \\ &= \sqrt{\frac{2h}{\pi}} \sum_{i=1}^{s} q_i \inf_{A\in\mathcal{F}, \mathbb{P}(A)=1} \max_{\omega\in\Omega} \left| \left\langle \widehat{Z}_i^{[n]}(\omega), \Delta g_i^{[n]}(\omega) \right\rangle \right|, \end{aligned}$$

where $\mathbb{P}$ denote the probability measure and

$$\Delta g_i^{[n]}(\omega) = g\left(\widehat{X}_i^{[n]}(\omega)\right) - g\left(\widehat{Y}_i^{[n]}(\omega)\right).$$

The Lipschitz continuity of g and Cauchy-Schwarz inequality lead to

$$|\phi_n(h)| \leq \sqrt{\frac{2hL}{\pi}} \sum_{i=1}^{s} q_i \inf_{A\in\mathcal{F}, \mathbb{P}(A)=1} \max_{\omega\in\Omega} \left| \widehat{X}_i^{[n]}(\omega) - \widehat{Y}_i^{[n]}(\omega) \right|^2.$$

According to [17] (see Equation (1.22), Chapter 7), SRK methods (1.2) are numerically-stable in the quadratic mean-squared sense, i.e., there exists $\varepsilon > 0$ such that

$$\mathbb{E}\left| \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]} \right|^2 \leq \varepsilon,$$

for any $i = 1, ..., s$ and for any $n = 1, 2, ..., N$. Since

$$\mathbb{E}\left| \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]} \right|^2 = \int_{\omega\in\Omega} \left| \widehat{X}_i^{[n]}(\omega) - \widehat{Y}_i^{[n]}(\omega) \right|^2 d\mathbb{P},$$

then for any $n = 1, 2, ..., N$,

$$\mathbb{P}\left(\left| \widehat{X}_i^{[n]}(\omega) - \widehat{Y}_i^{[n]}(\omega) \right|^2 < \infty \right) = 1, \quad i = 1, .., s,$$

leading to the result.

$\square$

The following theorem analyzes the long-term behavior of the function $\phi_n(h)$.

Theorem 3.2. *Under condition (iii) of Theorem* (1.1)*, for any fixed $h > 0$, we have*

$$\lim_{n\to\infty} \phi_n(h) = 0. \tag{3.8}$$

Proof: We denote

$$\xi_n = 2\sum_{i=1}^{s} q_i \left\langle \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]}, g\left(\widehat{X}_i^{[n]}\right) - g\left(\widehat{Y}_i^{[n]}\right)\right\rangle.$$

We consider an auxiliary random variable $\widehat{\xi_n}$ such that

$$\widehat{\xi_n} = \mathbb{E}\left(\xi_n | \mathcal{F}_{t_{n-1}}\right)$$

and denote by φ_n and $\widehat{\varphi_n}$ the random variables

$$\varphi_n = \Delta W_n \xi_n, \qquad \widehat{\varphi_n} = \Delta W_n \widehat{\xi_n}.$$

Then,

$$\mathbb{E}(\Delta W_n \widehat{\xi_n}) = \mathbb{E}(\Delta W_n)\mathbb{E}(\widehat{\xi_n}) = 0. \tag{3.9}$$

Therefore, in agreement with (3.9), it is sufficient to show that

$$\mathbb{P}\left(\lim_{n\to\infty} \varphi_n = \widehat{\varphi_n}\right) = 1, \tag{3.10}$$

where the limit in (3.10) is the pointwise limit. Indeed, if (3.10) holds, then φ_n and $\widehat{\varphi_n}$, for $n \to \infty$, must have the same expectation and, because of (3.9), the result holds true.

Using the definition on conditional expectation, we have that

$$\int_A \widehat{\xi_n}\, \mathrm{d}P = \int_A \xi_n\, \mathrm{d}P, \tag{3.11}$$

for any $A \in \mathcal{F}_{t_{n-1}}$. Then, (3.10) holds true; indeed, the sequence $\{\mathcal{F}_{t_i}\}_{i\geq 0}$ is a filtration and, as a consequence, (3.11) holds true, for $n \to \infty$, for any $A \subset \Omega$.

□

In summary, Theorem 3.2 shows us that the spurious term (2.4) vanishes on long time windows, so the numerical method (1.2) reproduces the same long-time behavior as the exact solution.

4. Extension to multidimensional systems with multiple Wiener processes

In this section, we aim to generalize the aforementioned results to the case of SDEs driven by multidimensional Wiener processes, according to the formulation of the methods given by [25]. Specifically, we consider the problem

$$dX(t) = f(X(t))dt + \sum_{\ell=1}^{m} g^{\ell}(X(t))dW^{\ell}(t), \tag{4.12}$$

where $f, g^{\ell} : \mathbb{R}^d \to \mathbb{R}^d$, $\ell = 1, ..., m$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $W^{\ell}\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths on

$(\Omega, \mathcal{F}, \mathbb{P})$, where $\ell = 1, ..., m$. Theoretical results on the existence and uniqueness to solutions of (4.12) are discussed, for instance, in the monograph [21]. A more compact notation for (4.12) is the following

$$dX(t) = f(X(t))dt + G(X(t))dW(t), \tag{4.13}$$

where

$$G(X(t)) = \left[g^1(X(t)),\ g^2(X(t)),\ \cdots,\ g^m(X(t))\right]$$

is a matrix of dimension $d \times m$ and $W(t)$ is a m dimensional vector.

In correspondence to Bucher tableau (1.3), we consider the following s-stages SRK methods

$$\begin{cases} X_n = X_{n-1} + h \displaystyle\sum_{i=1}^{s} b_i f(\widehat{X}_i^{[n]}) + \sum_{\ell=1}^{m} \Delta W_n^\ell \sum_{i=1}^{s} q_i g^\ell(\widehat{X}_i^{[n]}), & n = 1, ..., N, \\ \widehat{X}_i^{[n]} = X_{n-1} + h \displaystyle\sum_{j=1}^{s} a_{ij} f(\widehat{X}_j^{[n]}) + \sum_{\ell=1}^{m} \Delta W_n^\ell \sum_{j=1}^{s} \gamma_{ij} g^\ell(\widehat{X}_j^{[n]}), & i = 1, ..., s. \end{cases} \tag{4.14}$$

In the sequel, we state the theorems generalizing the results of Sections 2 and 3. Their proof straightforwardly descend from the proofs given in the previous sections.

Theorem 4.1. *Let us consider a stochastic differential system* (4.12) *satisfying the hypothesis (i)–(iii) of Theorem 1.1. We consider the* (c, A, Γ, b, q)*-SRK method* (4.14)*, arising from the stochastic perturbation of an algebraically-stable deterministic Runge-Kutta method* (1.4)*. If*

$$Q\Gamma + \Gamma^{\intercal} Q = qq^{\intercal}, \tag{4.15}$$

where $Q = \mathrm{diag}(q)$ *and*

$$B\Gamma + A^{\intercal} Q = bq^{\intercal}, \tag{4.16}$$

with $B = \mathrm{diag}(b)$*, then, any two numerical solutions* X_n *and* Y_n *to* (4.12) *computed by* (4.14) *with initial values* X_0 *and* Y_0*, respectively, with* $\mathbb{E}|X_0|^2 < \infty$ *and* $\mathbb{E}|Y_0|^2 < \infty$*, satisfy the following inequality*

$$\mathbb{E}|X_n - Y_n|^2 \leq \mathbb{E}|X_{n-1} - Y_{n-1}|^2 + \psi_n(h),$$

where

$$\psi_n(h) = 2 \sum_{i=1}^{s} q_i \sum_{j=1}^{m} \mathbb{E}\left[\Delta W_n^j \left\langle \widehat{X}_i^{[n]} - \widehat{Y}_i^{[n]}, g^j\left(\widehat{X}_i^{[n]}\right) - g^j\left(\widehat{Y}_i^{[n]}\right)\right\rangle\right], \tag{4.17}$$

$n = 1, 2, ..., N$.

We note that condition (4.15) is more restrictive than (2.1), since it should imply the matrix N to be null. Indeed, in case of multi-dimensional Wiener process, cross products between the m brownian increments appear and a restriction (4.15) of (2.1) is needed to make them null. In addition, we can rewrite the term in (4.17) as

$$\psi_n(h) = \sum_{j=1}^{m} \phi_n^j(h), \tag{4.18}$$

where for any $j = 1, .., m$, the term $\phi_n^j(h)$ are defined as in (2.4). The expression (4.18) allows us to obtain direct generalization to the results given in Section 3.

Theorem 4.2. *Under condition (iii) of Theorem 1.1, we have*

$$\lim_{h\to 0}\max_n \psi_n(h) = 0.$$

Theorem 4.3. *Under condition (iii) of Theorem 1.1, for any fixed $h > 0$, we have*

$$\lim_{n\to\infty} \psi_n(h) = 0.$$

5. Numerical experiments

In this section, we confirm the given theoretical analysis through the numerical evidence arising from the application of SRK methods (1.2) to nonlinear problems generating mean-square contractive solutions, according to Definition 1.1 and we show a comparison with respect to some well-known methods, whose nonlinear stability analysis has already been performed in [13]. The expected values computed in the remainder rely on the numerical solutions over 1000 paths.

Problem 1. We consider the SDE (1.1) with

$$f(X(t)) = -4X(t) - X(t)^3, \qquad g(X(t)) = X(t) \tag{5.19}$$

for $t \in [0, 100]$ and initial values $X_0 = 1$ and $Y_0 = 0$, used as test example in [19]. For this problem, the constants appearing in conditions (*ii*) and (*iii*) of Theorem 1.1 are given by $L = 1$ and $\mu = -4$, so $\alpha = -7$. So, according to Definition 1.1, the problem generates mean-square contractive solutions.

We consider the following numerical discretizations to (5.19).

- The stochastic perturbation of the deterministic implicit midpoint method, whose Butcher tableau (1.3) is given by

$$\begin{array}{c|c|c} 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & 1 & 1 \end{array}. \tag{5.20}$$

- The stochastic perturbation of the two-stage Gaussian Runge-Kutta method, whose Butcher tableau (1.3) is given by

$$\begin{array}{c|cc|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \tag{5.21}$$

- The well-known Euler-Maruyama method, that is,

$$X_{n+1} = X_n + hf(X_n) + g(X_n)\Delta W_n. \tag{5.22}$$

- The stochastic trapezoidal method, that is,

$$X_{n+1} = X_n + \frac{1}{2}h\,(f(X_n) + f(X_{n+1})) + g(X_n)\Delta W_n. \tag{5.23}$$

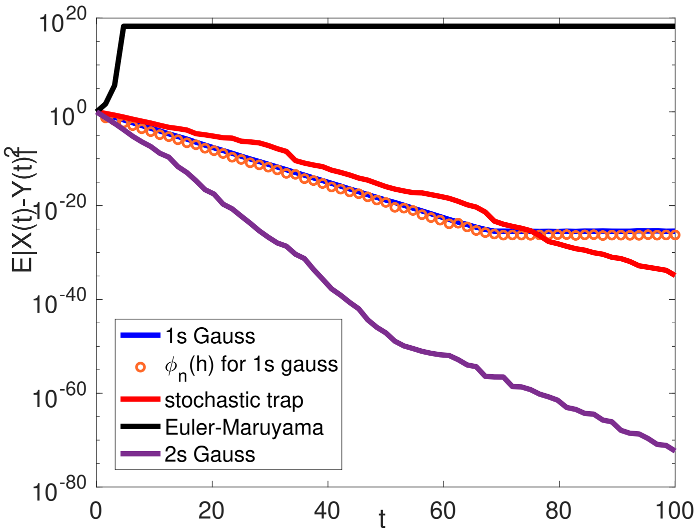

Figure 1: Numerical results arising applying the SRK methods (5.20)-(5.21) and methods (5.22)-(5.23) to Problem 1, with $h = 100/2^6$. Blue line: mean-square deviation over 1000 paths for method (5.20); circles: pattern of $\phi_n(h)$ for method (5.20); Violet line: mean-square deviation over 1000 paths for method (5.21); red line: mean-square deviation over 1000 paths for method (5.23); black line: mean-square deviation over 1000 paths for method (5.22).

From [13], we know that the stability region of methods (5.22)-(5.23) is bounded. Therefore, it makes sense to compare the behavior of these methods with the SRK methods (5.20)-(5.21) that satisfy the hypothesis of Theorem 2.1. The patterns displayed in Figure 1 confirms the numerical preservation of mean-square contractive character of Problem 1 for methods (5.20)-(5.21), the convergence to 0 of the function $\phi_n(h)$ in (2.4), as $n \to \infty$ for SRK method (5.20) and shows a comparison between SRK methods (5.20)-(5.21) with methods (5.22)-(5.23). Moreover, Figure 2 shows that for smaller and smaller values of Δt, the rate of exponential decay visible in the experiments gets closer and closer to the exact one.

Problem 2. We consider the SDE (1.1) with

$$f(X(t)) = -5X(t), \qquad g(X(t)) = \sin(X(t))$$

$t \in [0, 100]$ and initial data $X_0 = 1$ and $Y_0 = 0$. The constants L and μ appearing in Theorem 1.1 are given by $L = 1$ and $\mu = -5$, so $\alpha = -9$ and Problem 2 generates mean-square contractive solutions, according to Definition (1.1).

We consider SRK method (5.21), methods (5.22)-(5.23) and an explicit SRK method arising from the perturbation of explicit trapezoidal rule, whose Butcher tableau (1.3) is given by

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \hline & 0 & 1 & 0 & 1 \end{array} \tag{5.24}$$

Since method (5.24) is an explicit SRK method, it can not satisfy the hypothesis of Theorem 2.1. The graph displayed in Figure 3 confirms the accurate numerical preservation of the mean-square contractive

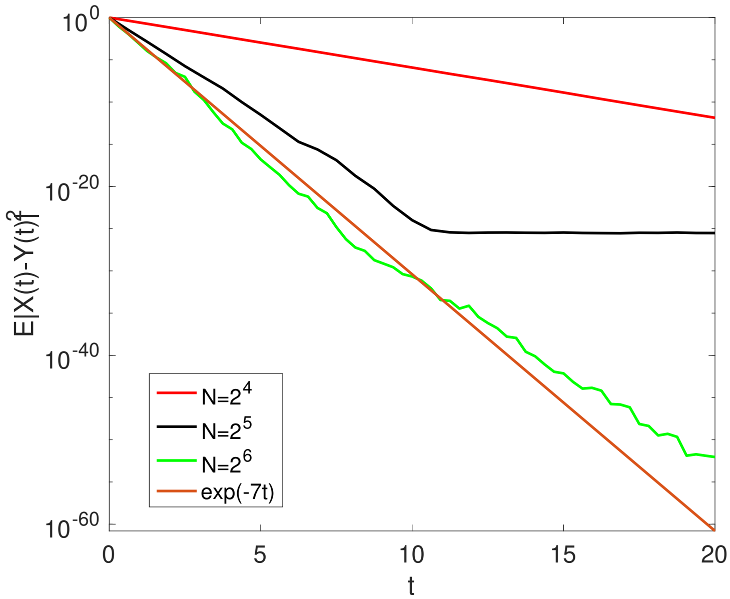

Figure 2: Numerical results arising applying the SRK method (5.20) to Problem 1, with $h = 20/N$, for several values of N. The figure also reports the slope $\exp(-7t)$ associated to the mean-square contractive behaviour of the exact solution to Problem 1.

character of Problem 2 for method (5.21), the non-preservation of the mean-square contractive character of Problem 2 for method (5.24) and shows a comparison with methods (5.22) and (5.23). In this case, for method (5.21), the function $\phi_n(h)$ in (2.4) rapidly converges to 0 and, in the endpoint of the integration interval, its value is $-6.74 \cdot 10^{-65}$.

Problem 3. We finally consider the nonlinear system of SDEs [13], with

$$f(X(t)) = -4\begin{bmatrix}\sin(X_1(t))\\ \sin(X_2(t))\end{bmatrix}, \qquad G(X(t)) = \frac{1}{7}\begin{bmatrix} X_1(t) & \frac{3}{2}X_2(t) \\ \frac{5}{2}X_1(t) & -\frac{1}{2}X_2(t)\end{bmatrix}.$$

and initial data $X_0 = [1 \quad 1]^\intercal$ and $Y_0 = [0 \quad 0]^\intercal$. For this problem the constants L and μ are estimated as $L = 0.148$ and $\mu = -3.56$, so $\alpha \approx -7.5$ and, as a consequence, the problem generates mean-square contractive solutions according to Definition 1.1. The pattern in Figure 4 confirms the theoretical results shown in Section 4.

6. Conclusions

In this paper, we have studied nonlinear stability properties of SRK methods (1.2) applied to nonlinear problems (1.1) generating mean-square contractive solutions, according to Definition 1.1. In particular, the methods under investigation are stochastic perturbations of algebraically stable deterministic RK methods, according to Theorem 2.1. Our analysis has shown, both theoretically and experimentally, that it is possible

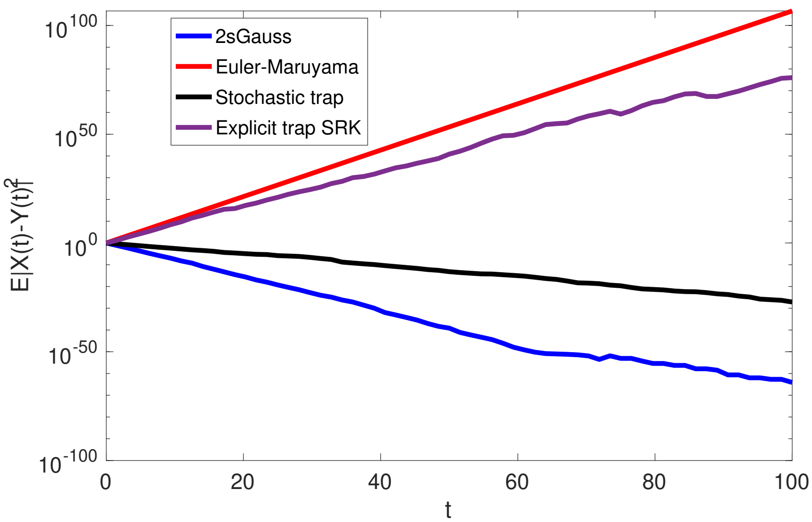

Figure 3: Mean-square deviations over 1000 paths for the SRK methods (5.21)-(5.24) and methods (5.22)-(5.23) applied to Problem 2, with $h = 100/2^6$.Blue line: mean-square deviation over 1000 paths for method (5.21); black line: mean-square deviation over 1000 paths for method (5.23); violet line: mean-square deviation over 1000 paths for method (5.24); red line: mean-square deviation over 1000 paths for method (5.22)

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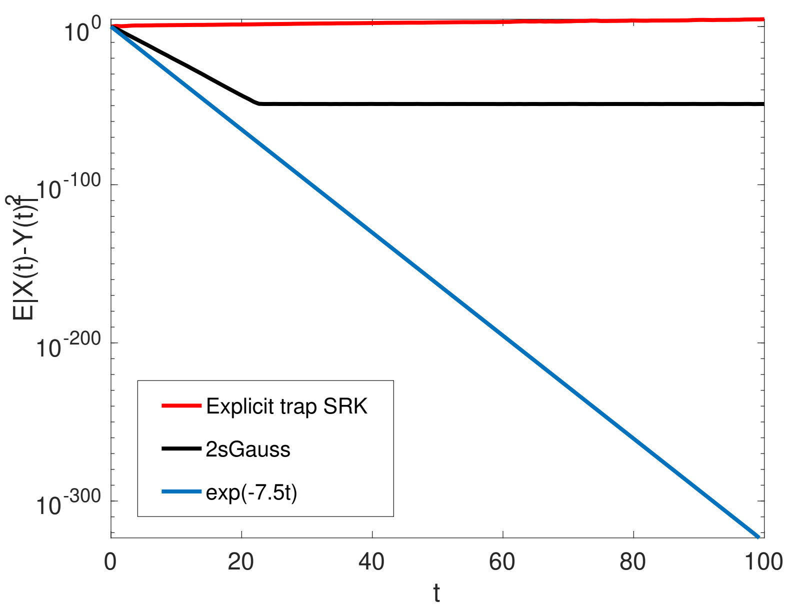

Figure 4: Numerical results arising applying the SRK methods (5.21) and methods (5.24) to Problem 3, with $h = 100/2^7$. Black line: mean-square deviation over 1000 paths for method (5.21); red line: mean-square deviation over 1000 paths for method (5.24).

to inherit the contractive character of algebraically stable RK methods under suitable stochastic perturbations, when applied to SDEs. A generalization to multi-dimensional Wiener process has been also discussed.

Further developments of this research will still be oriented in the direction of studying suitable stochastic perturbations to numerical methods for deterministic ordinary differential equations, in order to directly obtain corresponding stochastic methods with relevant accuracy, stability and, eventually, conservation properties [9, 13, 14, 15].

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