Improved ϑ-methods for stochastic Volterra integral equations

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Abstract

The paper introduces improved stochastic ϑ-methods for the numerical integration of stochastic Volterra integral equations. Such methods, compared to those introduced by the authors in [14], have better stability properties. This is here made possible by inheriting the stability properties of the corresponding methods for systems of stochastic differential equations. Such a superiority is confirmed by a comparison of the stability regions.

Key words: Stochastic Volterra integral equations, ϑ-method, mean-square stability, convolution test problem.
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1. Introduction

We focus our attention on the discretization of stochastic Volterra integral equations (SVIEs)

$$X_t = X_0 + \int_0^t a(t, s, X_s)ds + \int_0^t b(t, s, X_s)dW_s, \quad t \in [0, T], \tag{1.1}$$

where the functions a and b are assumed to have suitable regularity properties in such a way that existence and uniqueness of solutions can be admitted [24, 27, 31]. As regards the right-hand side of (1.1), we assume that the second integral is an Itô integral taken with respect to the Brownian motion W_s [21]. The problem is relevant in many applications, especially those concerning stochastic dynamical systems with memory, such as in economy (general stock, insurance, portfolio, financial markets) [1, 2, 32, 40] and engineering [37]. Their numerical approximation has raised a significant interest in the recent literature, through various numerical techniques: for instance, stochastic collocation [9, 36] and spline interpolation methods [28], wavelets based numerical schemes [19, 25, 26], Petrov-Galerkin methods [20], direct quadrature methods via rectangular rule [33, 34]. For most of the methods in the existing literature, the stability issues were almost unexplored.

In order to further improve direct quadrature methods, the authors have recently introduced in [10, 14] the so-called stochastic ϑ-method for (1.1), defined as follows

$$Y_n = Y_0 + h\sum_{i=0}^{n-1}(\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta)a(t_n, t_i, Y_i)) + \sqrt{h}\sum_{i=0}^{n-1} b(t_n, t_i, Y_i)V_i, \quad (1.2)$$

where $Y_0 = X_0$, $h = t_{n+1} - t_n$, $n = 0, 1, \ldots, N$ and V_i is a standard Gaussian random variable, i.e., it is $\mathcal{N}(0, 1)$-distributed. Under suitable regularity assumptions on the coefficients a and b of (1.1), the stochastic ϑ-method (1.2) is convergent of order $1/2$, i.e., there exists a real constant C such that

$$\mathbb{E}[(X(t_n) - Y_n)] \leq Ch^{\frac{1}{2}},$$

for any fixed $t_n = nh \in [0, T]$ and sufficiently small values of h, where $\mathbb{E}$ denotes the expected value. In particular, the stability analysis of the ϑ-method and some proposed variants has been carried out in [14]. This analysis also showed the stability properties of existing direct quadrature methods [33, 34] and the improvement gained by ϑ-methods (1.2).

Here we propose a class of revised ϑ-methods in order to further improve their stability properties, by inheriting the mean-square stability properties of the corresponding ϑ-method for stochastic differential equations (SDEs).

The paper is organized as follows: Section 2 provides the stability analysis of the exact solution to the proposed test equations and recalls the main stability results given in [14] for (1.2); Section 3 analyzes the connections among ϑ-methods for stochastic integral and differential equations when applied to the same test equation; in Section 4 we introduce an improved version of (1.2) for SVIEs inheriting the mean-square stability properties from the analogous methods for SDEs and provide examples of stability regions. The effectiveness of the improved methods is discussed in Section 5 through selected numerical experiments on given nonlinear problems; some conclusions are given in Section 6.

2. Stability issues

The stability analysis is here provided with respect to the following test equations introduced in [14]: the *basic test equation*

$$X_t = X_0 + \int_0^t \lambda X_s\, ds + \int_0^t \mu X_s\, dW_s, \quad \lambda, \mu \in \mathbb{R} \quad (2.1)$$

and the *convolution test equation*

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s))\, X_s\, ds + \int_0^t \mu X_s\, dW_s, \quad \lambda, \mu, \sigma \in \mathbb{R}. \quad (2.2)$$

Such equations derive from the basic and convolution test equations employed in the stability analysis of numerical methods for deterministic VIEs [3, 15, 16], by including an additional stochastic term.

Let us now analyze the stability properties of the solutions to these test problems and their numerical counterpart on the approximate solutions computed by the ϑ-method (1.2).

2.1. Stability of the exact solutions to the test problems

The basic test equation (2.1) is the integral representation of the linear test equation for SDEs [5, 21–23, 29]

$$dX_t = \lambda X_t dt + \mu X_t dW_t. \tag{2.3}$$

According to [22], solutions of (2.1) are mean-square stable, i.e.,

$$\lim_{t\to\infty} \mathbb{E}|X_t|^2 = 0,$$

if and only if

$$\mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0.$$

As regards the convolution test equation (2.2), it can be written as a 2 by 2 linear system of SDEs, having the form

$$d\begin{bmatrix} X_t \\ Z_t \end{bmatrix} = A\begin{bmatrix} X_t \\ Z_t \end{bmatrix} dt + B\begin{bmatrix} X_t \\ Z_t \end{bmatrix} dW_t, \tag{2.4}$$

with

$$A = \begin{bmatrix} \lambda & \sigma \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$Z_t = \int_0^t X_s ds.$$

Therefore, the mean-square stability of solutions to (2.2) is equivalent to the mean-square stability of solutions to (2.4) , which are now analyzed by applying the results in [6]. In particular, the condition for the mean-square stability of solutions to (2.4) is given by

$$\alpha(S) < 0,$$

where

$$S = I_2 \otimes A + A \otimes I_2 + B \otimes B, \tag{2.5}$$

I_2 is the 2 by 2 identity matrix and $\alpha(S)$ is the spectral abscissa of S, i.e.,

$$\alpha(S) = \max_i \mathrm{Re}(\lambda_i),$$

with λ_i, $i = 1, 2$, as the eigenvalues of S. In (2.5), $\otimes$ is the standard Kronecker tensor product.

The computations in the right-hand side of (2.5) lead to the block matrix

$$S = \left[\begin{array}{cc|cc} \mu^2 + 2\lambda & \sigma & \sigma & 0 \\ 1 & \lambda & 0 & \sigma \\ \hline 1 & 0 & \lambda & \sigma \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Since, according to [30], the determinant of a block matrix

$$\left[\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array}\right].$$

with $S_3 S_4 = S_4 S_3$ can be computed as $\det(S_1 S_4 - S_2 S_3)$, the characteristic polynomial $p(s)$ of S requires the computation of

$$\begin{aligned} p(s) &= \det\left(\begin{bmatrix} \mu^2 + 2\lambda - s & \sigma \\ 1 & \lambda - s \end{bmatrix} \begin{bmatrix} \lambda - s & \sigma \\ 1 & -s \end{bmatrix} - \sigma I_2 \right) \\ &= -(\lambda - s) q(s), \end{aligned}$$

where

$$q(s) = s^3 - (3\lambda + \mu^2)s^2 + (2\lambda^2 + \lambda\mu^2 - 4\sigma)s + 2\sigma\mu^2 + 4\sigma\lambda.$$

Then, the solutions of (2.4) are mean-square stable if and only if

$$\mathrm{Re}(\lambda) < 0, \quad \mathrm{Re}(s_1) < 0, \quad \mathrm{Re}(s_2) < 0, \quad \mathrm{Re}(s_3) < 0,$$

being s_1, s_2 and s_3 the roots of $q(s)$.

2.2. Stability of the ϑ-method

The method (1.2) is mean-square stable with respect to a given test equation if

$$\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right] = 0,$$

where Y_n is the numerical solution obtained by applying the method to the chosen test equation. The stability analysis of the ϑ-method (1.2) relies on the following results, proved in [14].

Theorem 2.1. *Let $x = h\lambda$ and $y = h\mu^2$. The recurrence relation for the stochastic ϑ-method* (1.2) *applied to the basic test equation* (2.1) *assumes the form*

$$Y_{n+1} = (\alpha + \beta V_n) Y_n, \tag{2.6}$$

with

$$\alpha = \frac{1 + x(1-\vartheta)}{1 - \vartheta x}, \quad \beta = \frac{\sqrt{y}}{1 - \vartheta x}.$$

Theorem 2.2. *The stochastic ϑ-method* (1.2) *is mean-square stable with respect to the basic test equation* (2.1) *if and only if*

$$\left|\alpha^2 + \beta^2\right| < 1.$$

Theorem 2.3. *Let $x = h\lambda$, $y = h\mu^2$ and $z = h^2\sigma$. The recurrence relation for the stochastic ϑ-method* (1.2) *applied to the convolution test equations* (2.2) *assumes the form*

$$(1 - \vartheta x) Y_{n+2} = (\mu + \sqrt{y} V_{n+1}) Y_{n+1} - (\nu + \sqrt{y} V_n) Y_n, \tag{2.7}$$

where

$$\mu = 2 + (1 - 2\vartheta)x + z, \quad \nu = 1 + (1 - \vartheta)x$$

and

$$(1 - \vartheta x) Y_1 = (1 + (1 - \vartheta)x + (1 - \vartheta)z + \sqrt{y} V_0) Y_0.$$

Theorem 2.4. *The stochastic ϑ-methods* (1.2) *is mean-square stable with respect to the convolution test equation* (2.2) *if* $\rho(K) < 1$, *with*

$$K = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{y}{(1-\vartheta x)^2} & -\dfrac{\nu}{1-\vartheta x} & \dfrac{\mu}{1-\vartheta x} \\ \dfrac{(\nu^2+y)(1-\vartheta x)-2\mu y}{(1-\vartheta x)^3} & -\dfrac{2\nu\mu}{(1-\vartheta x)^2} & \dfrac{\mu^2+y}{(1-\vartheta x)^2} \end{bmatrix}.$$

3. Connection with SDEs

There is a deep connection among test equations for SDEs and for SVIEs. Indeed, as observed in the previous section, the basic test equation (2.1) is equivalent to the linear test equation for SDEs (2.3). As concerns the convolution test equation (2.2), it can be written as the 2 by 2 linear system of SDEs (2.4).

We observe that the recurrence relation (2.6) corresponds to the recurrence relation of stochastic ϑ-method for SDEs applied to the linear test equation (2.3), see [22]. However, if we apply the ϑ-method for SDEs to the system (2.4), we obtain the scheme

$$\begin{aligned} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} + \vartheta h \begin{bmatrix} \lambda & \sigma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} + (1-\vartheta) h \begin{bmatrix} \lambda & \sigma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} \\ + V_n \sqrt{h} \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} \end{aligned} \tag{3.1}$$

with $Z_0 = 0$, i.e., by setting $S_n = Z_n/h$,

$$\begin{bmatrix} 1-\vartheta x & -\vartheta z \\ -\vartheta & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ S_{n+1} \end{bmatrix} = \begin{bmatrix} \nu + V_n\sqrt{y} & (1-\vartheta) z \\ 1-\vartheta & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ S_n \end{bmatrix}, \tag{3.2}$$

with $S_0 = 0$, which is distinct from (2.7), due to the presence of the implicit z-term on the left-hand side of (3.2), which is not present in (2.7). The presence of this additional term is made more evident in the following result.

Theorem 3.1. *Let* $x = h\lambda$, $y = h\mu^2$ *and* $z = h^2\sigma$. *The recurrence relation* (3.2) *is equivalent to*

$$(1-\vartheta x-\vartheta^2 z)Y_{n+2} = (\tilde{\mu} + \sqrt{y}V_{n+1})Y_{n+1} - (\tilde{\nu} + \sqrt{y}V_n)Y_n, \tag{3.3}$$

where

$$\tilde{\mu} = 2 + (1-2\vartheta)x + 2\vartheta(1-\vartheta)z, \quad \tilde{\nu} = 1 + (1-\vartheta)x - (1-\vartheta)^2 z$$

and

$$(1-\vartheta x-\vartheta^2 z)Y_1 = (1 + (1-\vartheta)x + \vartheta(1-\vartheta)z + \sqrt{y}V_0)Y_0.$$

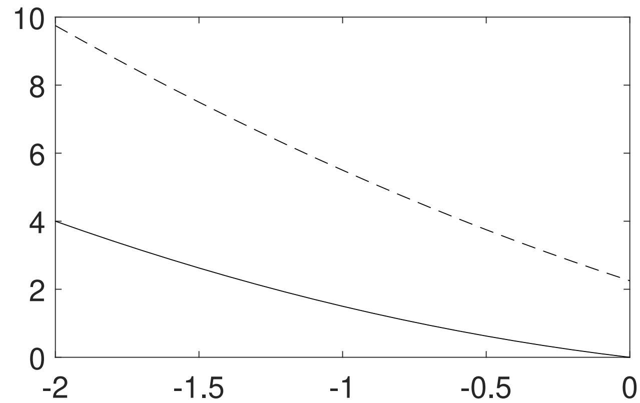

Figure 1: Comparison between the mean-square stability regions in the (x, y)-plane obtained from the recurrence relation (3.3) (below the dashed line) and that arising from (2.7) (below the continuous line), with $\vartheta = 3/4$ and $z = -2$.

Proof: Equation (3.2) is equivalent to

$$(1 - \vartheta x)Y_{n+1} - \vartheta z S_{n+1} = (\nu + V_n \sqrt{y})Y_n + (1 - \vartheta) z S_n \tag{3.4}$$

and

$$S_{n+1} - S_n = \vartheta Y_{n+1} + (1 - \vartheta)Y_n. \tag{3.5}$$

Writing (5.1) a step forward and subtracting side by side we obtain

$$\begin{aligned}(1 - \vartheta x)Y_{n+2} = (1 - \vartheta x + \nu + V_{n+1}\sqrt{y})Y_{n+1} + \vartheta z(S_{n+2} - S_{n+1})\\ - (\nu + V_n \sqrt{y})Y_n + (1 - \vartheta)z(S_{n+1} - S_n).\end{aligned}$$

Replacing the difference as in (5.2), the thesis holds true. □

Figure 1 shows, for $\vartheta = 3/4$ and $z = -2$, a comparison between the mean-square stability regions obtained from the recurrence relation (3.3) and those arising from (2.7). As visible from the figure, the recurrence relation (3.3) leads to larger stability regions. A similar behaviour occurs also for other choices of the parameters.

4. Improved stochastic ϑ-method

As we have highlighted in the previous section, the recurrence relation (3.3) provides better stability properties with respect to (2.7). Our goal is now revising the ϑ-method (1.2) in order to develop a family of methods showing (3.3) as recurrence relation when applied to the convolution test equation (2.2). To achieve the purpose, we propose a novel quadrature rule for the approximation of the deterministic integral in (1.1). Indeed, we evaluate (1.1) in t_n and split the deterministic integral as follows

$$X_n = X_0 + \sum_{i=0}^{n-2} \int_{t_i}^{t_{i+1}} a(t_n, s, X_s)ds + \int_{t_{n-1}}^{t_n} a(t_n, s, X_s)ds + \int_{t_0}^{t_n} b(t_n, s, X_s)dW_s,$$

$t \in [0, T]$. While the ϑ-method relies on the quadrature rule

$$\int_{t_i}^{t_{i+1}} a(t_n, s, X_s) ds \approx h \left(\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1 - \vartheta) a(t_n, t_i, Y_i) \right), \tag{4.1}$$

the revised method depends on the following rule

$$\int_{t_i}^{t_{i+1}} a(t_n, s, X_s) ds \approx h \left(\vartheta a(t_n, t_{i+1} - \vartheta h, Y_{i+1}) + (1 - \vartheta) a(t_n, t_i + (1 - \vartheta) h, Y_i) \right). \tag{4.2}$$

The following result analyzes the accuracy of this novel quadrature rule.

Theorem 4.1. *Let $f(t, y)$ and $y(t)$ be sufficiently smooth functions. Then,*

$$\begin{aligned} \int_{t_i}^{t_{i+1}} f(s, y(s)) ds = {} & h \left(\vartheta f(t_{i+1} - \vartheta h, y(t_{i+1})) + (1 - \vartheta) f(t_i + (1 - \vartheta) h, y(t_i)) \right) \\ & + E(h), \end{aligned}$$

with $E(h) = h^2 \left((\frac{1}{2} - \vartheta) \frac{d}{dt} f(t, y(t))|_{t=t_i} + (2\vartheta - 1) \frac{\partial f}{\partial t}(t_i, y(t_i)) \right) + O(h^3)$.

Proof: Let $g(t)$ be such that $g'(t) = f(t, y(t))$. Then,

$$\begin{aligned} \int_{t_i}^{t_{i+1}} f(s, y(s)) &= g(t_{i+1}) - g(t_i) = h g'(t_i) + \frac{h^2}{2} g''(t_i) + O(h^3) \\ &= h f(t_i, y(t_i)) + \frac{h^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t))|_{t=t_i} + O(h^3). \end{aligned}$$

Expanding $f(t_{i+1} - \vartheta h, y(t_{i+1}))$ and $f(t_i + (1 - \vartheta) h, y(t_i))$ around $(t_i, y(t_i))$ we obtain

$$\begin{aligned} f(t_{i+1} - \vartheta h, y(t_{i+1})) &= f(t_i, y(t_i)) + (1 - \vartheta) h \frac{\partial f}{\partial t}(t_i, y(t_i)) \\ &\quad + h y'(t_i) \frac{\partial f}{\partial y}(t_i, y(t_i)) + O(h^2) \\ &= f(t_i, y(t_i)) + h \frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t))|_{t=t_i} - \vartheta h \frac{\partial f}{\partial t}(t_i, y(t_i)) \\ &\quad + O(h^2) \end{aligned}$$

and

$$f(t_i + (1 - \vartheta) h, y(t_i)) = f(t_i, y(t_i)) + (1 - \vartheta) h \frac{\partial f}{\partial t} f(t_i, y(t_i)) + O(h^2).$$

Then, the thesis holds. □

Assuming $Y_0 = X_0$, the revised method, denoted as *improved stochastic ϑ-method*, takes the form

$$\begin{aligned} Y_n = Y_0 + h \sum_{i=0}^{n-1} & \left(\vartheta a(t_n, t_{i+1} - \vartheta h, Y_{i+1}) + (1 - \vartheta) a(t_n, t_i + (1 - \vartheta h), Y_i) \right) \\ + & \sqrt{h} \sum_{i=0}^{n-1} b(t_n, t_i, Y_i) V_i. \end{aligned} \tag{4.3}$$

Remark 4.1. *Under hypothesis of sufficient regularity for the functions a and b, from above Theorem 4.1 and Theorem 2.1 in [14], it follows that the improved stochastic ϑ-method* (4.3) *preserves the order of convergence of the ϑ-method* (1.2)*. Indeed, it is convergent of order 1/2, i.e. there exist a constant C such that*

$$\mathbb{E}|X_n - Y_n| \leq C h^{\frac{1}{2}}.$$

We observe that the recurrence relation of the improved method (4.3) with respect to the basic test equation (2.1) is the same as the ϑ-method (1.2). Indeed, the quadrature formulae (4.1) and (4.2) coincide when $a(t, s, y) = \lambda y$, with constant λ.

As regards the convolution test equation (2.2), the following result provides the expression of the recurrence relation for the improved stochastic ϑ-method (4.3).

Theorem 4.2. *Let $x = h\lambda$, $y = h\mu^2$ and $z = h^2\sigma$. The recurrence relation for the improved stochastic ϑ-method* (4.3) *applied to the convolution test equation* (2.2) *assumes the form* (3.3)*.*

Proof: Applying the improved ϑ-method to the convolution test equation (2.2) and collecting the coefficients of Y_n leads to

$$\begin{aligned}(1 - \vartheta x - \vartheta^2 z)Y_n = Y_0 &+ \sum_{i=0}^{n-2} \left[\vartheta x + \vartheta z(n - i - 1) + \vartheta^2 z\right] Y_{i+1} \\ &+ \sum_{i=0}^{n-1} \left[(1 - \vartheta)x + (1 - \vartheta)z(n - i) - (1 - \vartheta)^2 z + \sqrt{y} V_i\right] Y_i.\end{aligned} \tag{4.4}$$

Writing above formula one step forward and subtracting (4.4), we obtain

$$(1 - \vartheta x - \vartheta^2 z)Y_{n+1} = \left[\tilde{\nu} + \sqrt{y} V_n + (1 - \vartheta)z\right] Y_n + \sum_{i=0}^{n-1} [\vartheta z Y_{i+1} + (1 - \vartheta z)Y_i].$$

Writing the analogous formula for Y_{n+2} and subtracting side by side leads to the thesis. □

The following lemma highlights a useful property to derive the stability matrix of the improved ϑ-method (4.3).

Lemma 4.1. *For the improved ϑ-method* (4.3) *applied to the test equation* (2.2)*, the following equality holds true:*

$$(1 - \vartheta x - \vartheta^2 z)\mathbb{E}[\sqrt{y} V_n Y_n Y_{n+1}] = y\mathbb{E}[Y_n^2]. \tag{4.5}$$

Proof: From the recurrence relation (3.3) referred to the point t_{n+1}

$$(1 - \vartheta x - \vartheta^2 z)Y_{n+1} = (\tilde{\mu} + \sqrt{y} V_n)Y_n - (\tilde{\nu} + \sqrt{y} V_{n-1})Y_{n-1},$$

multiplying both sides by $\sqrt{y} V_n Y_n$ and passing to the expectations, we obtain

$$\begin{aligned}(1 - \vartheta x - \vartheta^2 z)\mathbb{E}(\sqrt{y} V_n Y_n Y_{n+1}) &= \mathbb{E}(V_n)\mathbb{E}\left[\tilde{\mu} \sqrt{y} Y_n^2 - \tilde{\nu} \sqrt{y} Y_n Y_{n-1} - y V_{n-1} Y_n Y_{n-1}\right] \\ &\quad + \mathbb{E}(V_n^2)\mathbb{E}[y Y_n^2]\end{aligned}$$

and the thesis immediately follows. □

Theorem 4.3. *The improved stochastic ϑ-method* (4.3) *is mean-square stable with respect to the convolution test equation* (2.2) *if* $\rho(\tilde{K}) < 1$, *with*

$$\tilde{K} = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{y}{(1-\vartheta x-\vartheta^2 z)^2} & -\dfrac{\tilde{\nu}}{1-\vartheta x-\vartheta^2 z} & \dfrac{\tilde{\mu}}{1-\vartheta x-\vartheta^2 z} \\ \dfrac{(\tilde{\nu}^2+y)(1-\vartheta x-\vartheta^2 z)-2\tilde{\mu}y}{(1-\vartheta x-\vartheta^2 z)^3} & -\dfrac{2\tilde{\nu}\tilde{\mu}}{(1-\vartheta x-\vartheta^2 z)^2} & \dfrac{\tilde{\mu}^2+y}{(1-\vartheta x-\vartheta^2 z)^2} \end{bmatrix}. \tag{4.6}$$

Proof: We obtain the thesis by proving that

$$\begin{bmatrix} \mathbb{E}[Y_{n+1}^2] \\ \mathbb{E}[Y_{n+1}Y_{n+2}] \\ \mathbb{E}[Y_{n+2}^2] \end{bmatrix} = \tilde{K} \begin{bmatrix} \mathbb{E}[Y_n^2] \\ \mathbb{E}[Y_nY_{n+1}] \\ \mathbb{E}[Y_{n+1}^2] \end{bmatrix},$$

with $\tilde{K}$ defined in (4.6). Multiplying (3.3) by Y_{n+1} and passing to the expectations leads to

$$(1-\vartheta x-\vartheta^2 z)\mathbb{E}[Y_{n+1}Y_{n+2}] = \tilde{\mu}\mathbb{E}[Y_{n+1}^2] - \tilde{\nu}\mathbb{E}[Y_nY_{n+1}] - \mathbb{E}[\sqrt{y}V_nY_nY_{n+1}].$$

By applying (4.5), we obtain

$$(1-\vartheta x-\vartheta^2 z)\mathbb{E}[Y_{n+1}Y_{n+2}] = -\frac{y}{1-\vartheta x-\vartheta^2 z}\mathbb{E}[Y_n^2] - \tilde{\nu}\mathbb{E}[Y_nY_{n+1}] + \tilde{\mu}\mathbb{E}[Y_{n+1}^2],$$

that provides the second row of the matrix $\tilde{K}$.

By squaring (3.3) and passing to the expectations, we obtain

$$\begin{aligned}(1-\vartheta x-\vartheta^2 z)^2\mathbb{E}[Y_{n+2}^2] = (\tilde{\mu}^2+y)\mathbb{E}[Y_{n+1}^2] + (\tilde{\nu}^2+y)\mathbb{E}[Y_n^2] - 2\tilde{\mu}\tilde{\nu}\mathbb{E}[Y_nY_{n+1}] \\ - 2\tilde{\mu}\mathbb{E}[\sqrt{y}V_nY_nY_{n+1}].\end{aligned}$$

By applying (4.5), we obtain

$$\begin{aligned}(1-\vartheta x-\vartheta^2 z)^2\mathbb{E}[Y_{n+2}^2] = (\tilde{\nu}^2+y-\frac{2\tilde{\mu}y}{1-\vartheta x-\vartheta^2 z})\mathbb{E}[Y_n^2] - 2\tilde{\mu}\tilde{\nu}\mathbb{E}[Y_nY_{n+1}] \\ + (\tilde{\mu}^2+y)\mathbb{E}[Y_{n+1}^2],\end{aligned}$$

that provides the third row of the matrix $\tilde{K}$. □

Figures 2 and 3 show the mean-square stability regions with respect to the convolution test equation (2.2) in the (x, y)-plane of the ϑ-method (1.2) and the improved ϑ-method (4.3), for different values of ϑ and z. We observe that, the more ϑ and $|z|$ increase, the more the stability region of the improved ϑ-method (4.3) is larger, as it is particularly visible in Figures 4 and 5.

The evidence obtained by applying both methods (1.2) and (4.3) is displayed in Figure 6, confirming the theoretical results. The figure shows the mean-square of the numerical solution over 1000 realizations of problem (2.2), with $\lambda = -2$, $\mu = 2\sqrt{3}$, $\sigma = -8$. In correspondence of $h = 1/2$, we obtain the point $(-1, 6, -2)$ belonging to the stability region of (4.3) and outside that of (1.2), for $\vartheta = 1$.

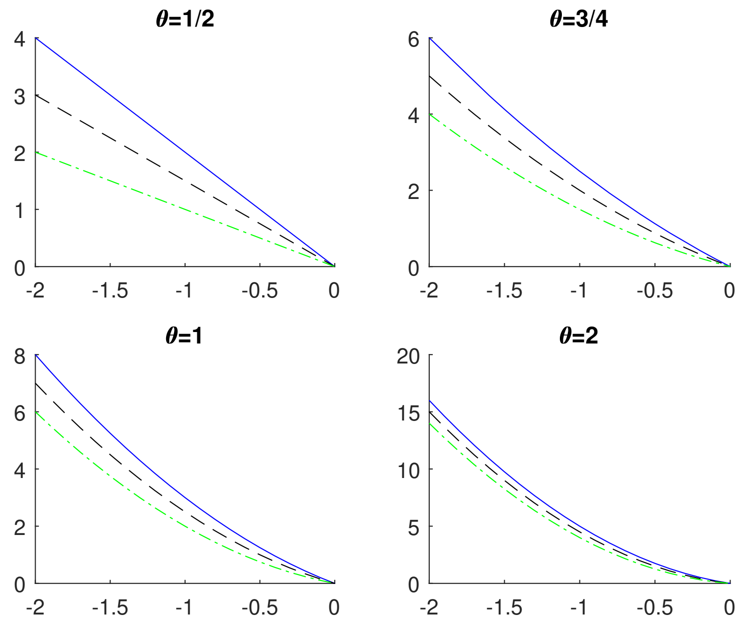

Figure 2: Mean-square stability regions with respect to the convolution test equation (2.2) in the (x, y)-plane of the ϑ-method (1.2) for different values of ϑ and $z = 0$ (solid line), $z = -1$ (dashed line) and $z = -2$ (dashed-dotted line).

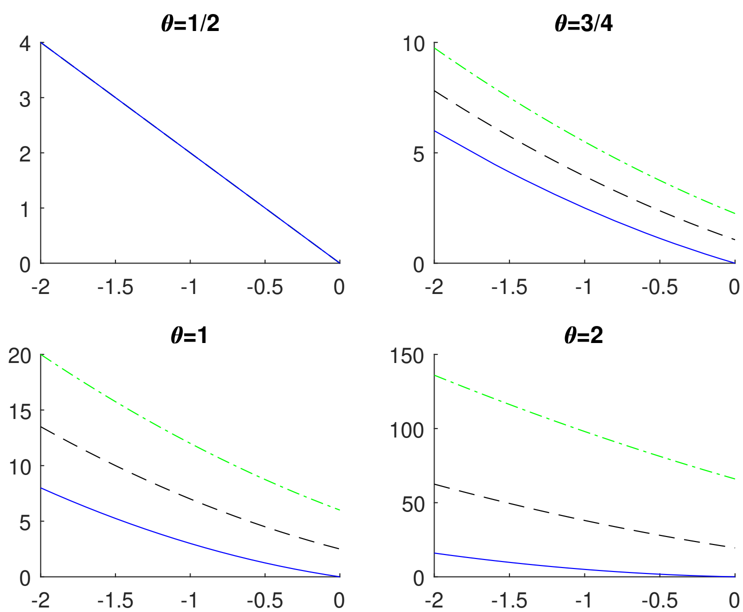

Figure 3: Mean-square stability regions with respect to the convolution test equation (2.2) in the (x, y)-plane of the improved ϑ-method (4.3) for different values of ϑ and $z = 0$ (solid line), $z = -1$ (dashed line) and $z = -2$ (dashed-dotted line).

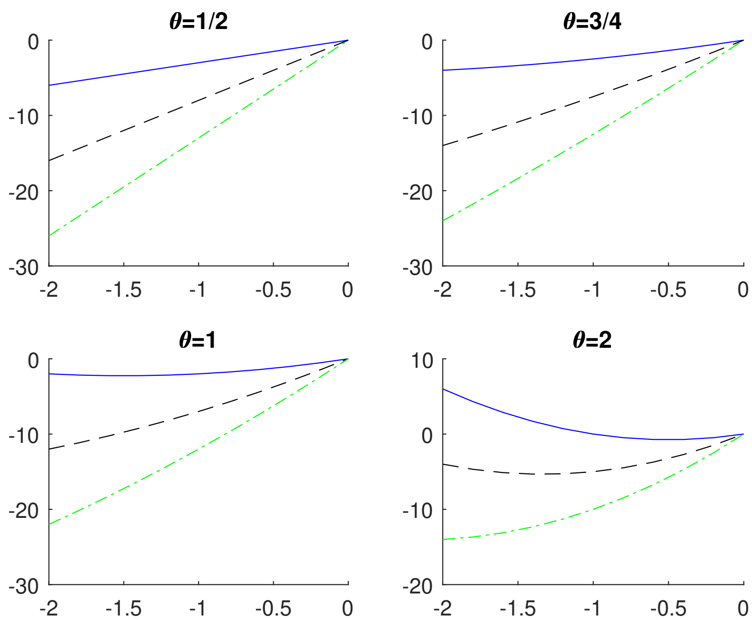

Figure 4: Mean-square stability regions with respect to the convolution test equation (2.2) in the (x, y)-plane of the ϑ-method (1.2) for different values of ϑ and $z = -10$ (solid line), $z = -20$ (dashed line) and $z = -30$ (dashed-dotted line).

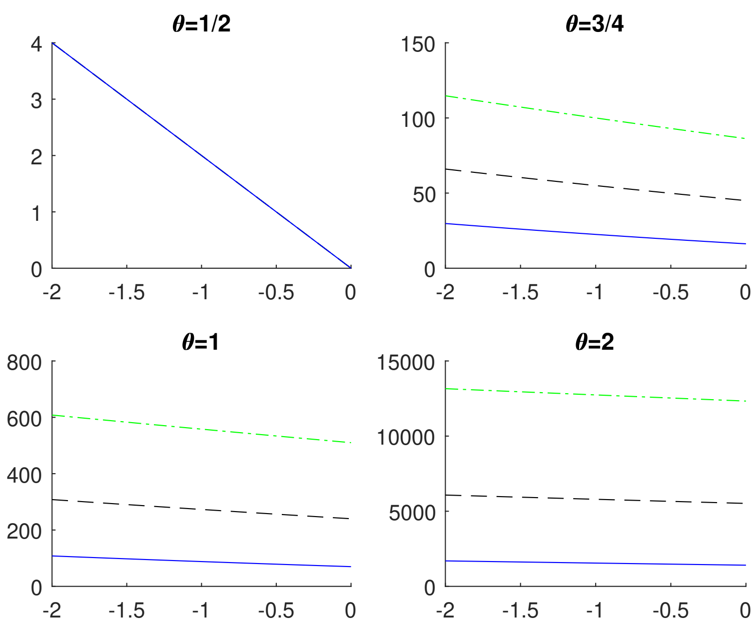

Figure 5: Mean-square stability regions with respect to the convolution test equation (2.2) in the (x, y)-plane of the improved ϑ-method (4.3) for different values of ϑ and $z = -10$ (solid line), $z = -20$ (dashed line) and $z = -30$ (dashed-dotted line).

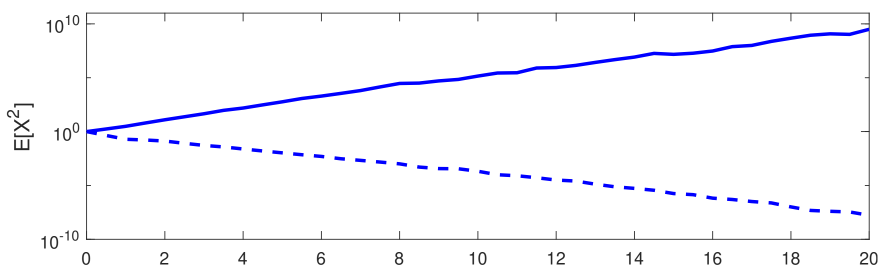

Figure 6: Mean-square of the numerical solution of problem (2.2), with $\lambda = -2$, $\mu = 2\sqrt{3}$, $\sigma = -8$ and stepsize $h = 1/2$, obtained by applying the ϑ-method (solid line) and the improved ϑ-method (dashed line) for $\vartheta = 1$.

5. Numerical evidence on nonlinear problems

This section is focused on providing the numerical evidence originating from the application of the improved methods introduced in Section 4 on a selection of nonlinear problems. Specifically, we consider

- the nonlinear SVIE [25]:

$$X_t = \frac{1}{10} - \frac{1}{800}\int_0^t \tanh(X_s)\,\mathrm{sech}^2(X_s)ds + \frac{1}{20}\int_0^t \mathrm{sech}(X_s)dW_s, \tag{5.1}$$

  for $t \in [0, 0.55]$, whose exact solution is

$$X(t) = \mathrm{arcsinh}\left(\frac{1}{20}W_t + \sinh\left(\frac{1}{10}\right)\right);$$

- the nonlinear problem [35, 38, 39]

$$X_t = 1 + \int_0^t e^{-(t-s)}\sin(X_s)ds + \int_0^t e^{-(t-s)}\cos(X_s)dW_s, \tag{5.2}$$

  for $t \in [0, 1]$.

The results, contained in Figures 7 and 8 confirm the effectiveness of improved θ-methods when applied to nonlinear problems. In particular, the expected order of convergence 1/2 is confirmed. In the case of Problem (5.1), the error is computed as the expected gap between the exact solution on the numerical solution in the endpoint of the integration interval, i.e. $\mathbb{E}\,|X(T) - X_N|$. As regards Problem (5.2), the unknown exact solution is replaced by a reference solution computed with a small enough stepsizes.

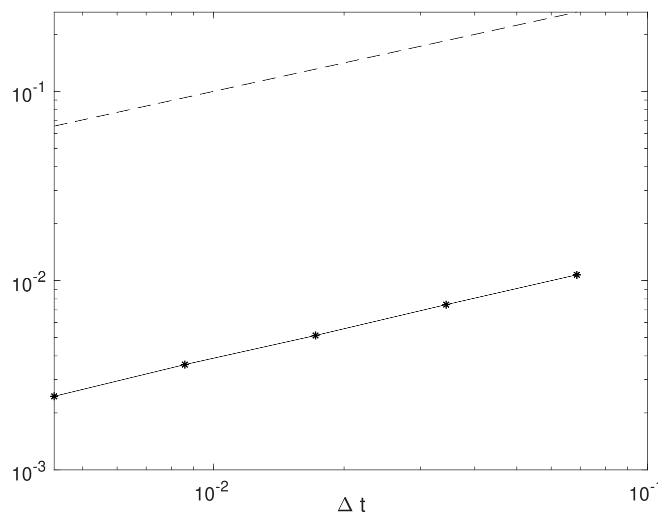

Figure 7: Sampled expected value of the error in the endpoint of the integration interval (solid line), for Problem (5.1). The dashed line gives the slope for order 1/2. The sampling is computed over 1000 trajectories.

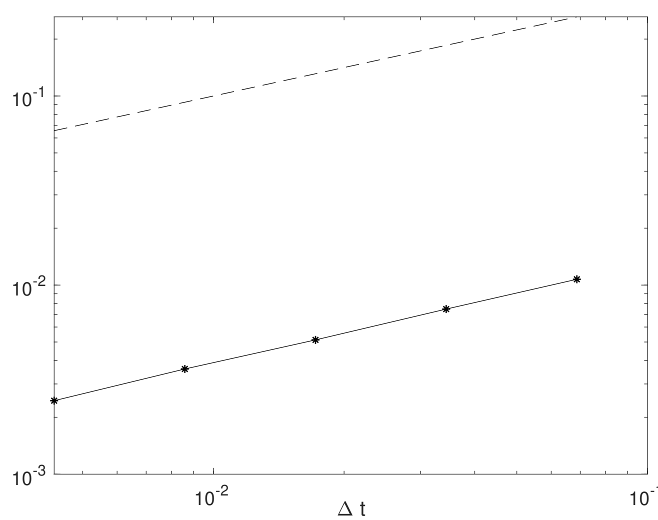

Figure 8: Sampled expected value of the error in the endpoint of the integration interval (solid line), for Problem (5.2). The dashed line gives the slope for order 1/2. The sampling is computed over 1000 trajectories.

6. Conclusions

We have introduced the family of improved ϑ-methods (4.3) for the numerical solution of SVIEs (1.1). The improvement lies in achieving better stability properties with respect to the convolution test problem (2.2), namely the improved method applied to (2.2) provides the same recurrence relation of the ϑ-method for SDEs, applied to the equivalent system of SDEs (2.4). The improved method is obtained by the quadrature formula (4.2) for the approximation of the deterministic integral in (1.1). A confirmation of the improvement in the stability properties has been provided by depicting some selected stability regions. Further developments of the research regard the generalization of the idea to higher order methods relying on more accurate quadrature formulae, as well as on its application to other stochastic evolutionary operators with memory, such as the stochastic perturbation of fractional differential equations [7, 11]. Moreover, following the lines drawn in the context of SDEs, a further investigation will also be focused on the analysis of nonlinear stability properties [4], also in perspective of a long-term analysis of nonlinear stochastic oscillatory problems [8, 13, 17, 18] and stochastic Hamiltonian problems [12].

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