Advances on collocation based numerical methods for Ordinary Differential Equations and Volterra Integral Equations

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Abstract. We present a survey on collocation based numerical methods for the numerical integration of Ordinary Differential Equations (ODEs) and Volterra Integral Equations (VIEs), starting from the classical collocation methods, to arrive to the most important modifications appeared in the literature, also considering the multistep case and the usage of basis of functions other than polynomials.

Keywords: Collocation, two-step collocation, Runge–Kutta methods, two-step Runge–Kutta methods, mixed collocation.

1 Introduction

Collocation is a widely applied and powerful technique in the construction of numerical methods for ODEs and VIEs. As it is well known, a collocation method is based on the idea of approximating the exact solution of a given functional equation with a suitable approximant belonging to a chosen finite dimensional space, usually a piecewise algebraic polynomial, which exactly satisfies the equation on a certain subset of the integration interval (i.e. the set of the so-called *collocation points*).

This technique, when applied to problems based on functional equations, allows the derivation of methods having many desirable properties. In fact, collocation methods provide an approximation over the entire integration interval to the solution of the equation. Moreover, the collocation function can be expressed as a linear combination of functions ad hoc for the problem we are integrating, in order to better reproduce the qualitative behaviour of the solution.

The systematic study of collocation methods for initial value problems in ODEs, VIEs, and Volterra integro-differential equations (VIDEs) has its origin, respectively, in the late '60, the early '70 and the early '80s. The idea of multistep collocation was first introduced by Lie and Norsett in [60], and further extended and investigated by several authors [13, 25, 28–30, 32–35, 37, 43, 59, 63].

Multistep collocation methods depend on more parameters than classical ones, without any significant increase in the computational cost, by regarding them as special case of multistep Runge-Kutta methods: therefore, there are much more degrees of freedom to be spent in order to obtain strong stability properties and an higher order and stage order of convergence. As a direct consequence the effective order of multistep collocation methods is generally higher with respect to one stage collocation methods with the same number of stages. Moreover, as they generally have high stage order, they do not suffer from the order reduction phenomenon (see [12, 46]), which occurs in the integration of stiff systems.

The purpose of this paper is to present a review of recently introduced families of collocation and modified collocation methods for ODEs and VIEs. In particular we aim to present the main results obtained in the context of multistep collocation and almost collocation methods, i.e. methods obtained by relaxing some collocation and/or interpolation conditions in order to obtain desirable stability properties.

The paper is organized as follows: Section 2 reviews the main results concerning classical one-step and multistep collocation methods for ODEs and their recent extensions and modifications; Section 3 is dedicated to collocation methods for second order initial value problems and also collocation methods based on functional basis other than polynomials; in Section 4 we consider the evolution of the collocation technique for Volterra integral equations.

2 Collocation based methods for first order ODEs

In this section we focus our attention on the hystorical background and more recent results concerning the collocation technique, its modifications and extensions for the derivation of highly stable continuous methods for the numerical solution of initial value problems based on ordinary differential equations

$$\begin{cases} y'(x) = f(x, y(x)), \quad x \in [x_0, X], \\ y(x_0) = y_0 \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

with $f : [x_0, X] \times \mathbb{R}^d \to \mathbb{R}^d$. It is assumed that the function f is sufficiently smooth, in such a way that the problem (2.1) is well-posed.

2.1 Classical one-step collocation methods

Let us suppose that the integration interval $[x_0, X]$ is discretized in an uniform grid $\{x_h : x_0 < x_1 < ... < x_N = X\}$. Classical collocation methods (see [7, 11, 12, 44, 45, 57, 80]) are determined by means of a continuous approximant, generally an algebraic polynomial $P(x)$, satisfying some opportune conditions: in order to advance from x_n to x_{n+1}, the polynomial $P(x)$ interpolates the numerical solution in x_n, and exactly satisfies the ODE (2.1) - i.e. *co-locates* - in the set of points $\{x_n + c_i h, \ i = 1, 2, \ldots m\}$, where $c_1, c_2, \ldots, c_m$ are m real numbers (named *collocation nodes*), that is

$$\begin{cases} P(x_n) = y_n, \\ P'(x_n + c_i h) = f(x_n + c_i h, P(x_n + c_i h)), \quad i = 1, 2, ..., m. \end{cases} \tag{2.2}$$

The solution in x_{n+1} can then be computed from the function evaluation

$$y_{n+1} = P(x_{n+1}). \tag{2.3}$$

The classical framework in which collocation methods must be placed is certainly constituted by implicit Runge-Kutta methods (IRK). In fact, Guillou and Soule in [43] and Wright in [80] independently proved that one step collocation methods form a subset of implicit Runge-Kutta methods

$$y_{n+1} = y_n + h \sum_{i=1}^{m} b_i f(x_n + c_i h, Y_i) \tag{2.4}$$

$$Y_i = y_n + h \sum_{j=1}^{m} a_{ij} f(x_n + c_j h, Y_j), \qquad i = 1, 2, ..., m, \tag{2.5}$$

where

$$a_{ij} = \int_0^{c_i} L_j(s) ds, \qquad b_j = \int_0^1 L_j(s) ds, \qquad i, j = 1, 2, ..., m \tag{2.6}$$

and $L_j(s)$, $j = 1, ..., m$, are fundamental Lagrange polynomials. The maximum attainable order of such methods is at most $2m$, and it is obtained by using Gaussian collocation points [45, 57]. Anyway, unfortunately, the order $2m$ is gained only at the mesh points: the uniform order of convergence over the entire integration interval is only m. As a consequence, they suffer from order reduction showing effective order equal to m (see [11,12,45,46]).

Butcher (see [11] and references therein) gave an interesting characterization of collocation methods in terms of easy algebraic conditions, and analogous results are also reported in [45, 57]. This characterization , together with many other several results regarding the main properties of collocation methods, comes out as natural consequence of an interesting interpretation of collocation methods in terms of quadrature formulae. In fact, if $f(x, y(x)) = f(x)$, equations (2.4)-(2.5) can be respectively interpreted as quadrature formulae for $\int_{x_n}^{x_n+h} f(x)dx$ and $\int_{x_n}^{x_n+c_ih} f(x)dx$, for i=1,2,...,m. We next consider the following linear systems

$$A(q) \quad : \quad \sum_{j=1}^{m} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \qquad k = 1, 2, ..., q, \; i = 1, 2, ..., m, \tag{2.7}$$

$$B(p) \quad : \quad \sum_{i=1}^{m} b_i c_i^{k-1} = \frac{1}{k}, \qquad k = 1, 2, ..., p. \tag{2.8}$$

Next, the following result holds (see [44, 57]):

Theorem 2.1 *If the condition $B(p)$ holds for some $p \geq m$, then the collocation method (2.2) has order p.*

As a consequence, a collocation method has the same order of the underlying quadrature formula (see [44], p. 28). Finally, the following result characterizing classical collocation methods arises (see [11, 44, 45, 57]).

Theorem 2.2 *An implicit m-stage Runge-Kutta method, satisfying $B(m)$ and having distinct collocation abscissae, is a collocation method if and only if conditions $A(m)$ holds.*

The most used collocation methods are those based on the zeros of some orthogonal polynomials, that is Gauss, Radau, Lobatto [11, 12, 45, 46, 57], having respectively order of convergence $2m$, $2m - 1$, $2m - 2$, where m is the number of collocation points (or the number of stages, regarding the collocation method as an implicit Runge-Kutta). Concerning their stability properties, it is known that Runge-Kutta methods based on Gaussian collocation points are A-stable, while the ones based on Radau IIA points are L-stable and, moreover, they are also both algebraically stable (see [12, 46, 51] and references therein contained); Runge-Kutta methods based on Lobatto IIIA collocation points, instead, are A-stable but they are not algebraically stable (see [11, 44, 45, 57]).

2.2 Perturbed collocation

As remarked by Hairer and Wanner in [46], only some IRK methods are of collocation type, i.e. Gauss, Radau IIA, and Lobatto IIIA methods. An extension of the collocation idea, the so-called *perturbed collocation* is due to Norsett and Wanner (see [65, 66]), which applies to all IRK methods.

We denote by Π_m the linear space of polynomials of degree at most m and consider the polynomial $N_j \in \Pi_m$ defined by

$$N_j(x) = \frac{1}{j!}\sum_{i=0}^{m}(p_{ij} - \delta_{ij})x^i, \quad j = 1, 2, ..., m,$$

where d_{ij} is the usual Kronecker delta. We next define the *perturbation operator* $P_{x_0,h} : \Pi_m \rightarrow \Pi_m$ by

$$(P_{x_0,h}u)(x) = u(x) + \sum_{j=1}^{n} N_j\left(\frac{x - x_0}{h}\right)u^{(j)}(x_0)h^j.$$

Next, the following definition is given (see [65, 66]).

Definition 2.1 *Let $c_1, \ldots, c_m$ be given distinct collocation points. Then the corresponding perturbed collocation method is defined by*

$$\begin{aligned}
u(x_0) &= y_0, \quad u \in \Pi_m, \\
u'(x_0 + c_i h) &= f(x_0 + c_i h, (Pu)(x_0 + c_i h)), \quad i = 1, 2, ..., m, \\
y_1 &= u(x_0 + h).
\end{aligned}$$

As the authors remark in [66], if all N_j's are identically zero, then P is the identical map and the definition coincides with classical collocation. In the same paper the authors provide the equivalence result between the family of perturbed collocation methods and Runge-Kutta methods (see [66]). The interest of this results, as again is stated in [66], is that the properties of collocation methods, especially in terms of order, linear and nonlinear stability, can be derived in a reasonable short, natural and very elegant way, while it is known that, in general, these properties are very difficult to handle and investigate outside collocation.

2.3 Discontinuous collocation

In the literature, perturbed collocation has been considered as a modification of the classical collocation technique, in such a way that much more Runge-Kutta methods could be regarded as perturbed collocation based methods, rather than classically collocation based. There are other possible extensions of the collocation idea, which apply to wider classes of Runge-Kutta methods, such as the so-called *discontinuous collocation* (see [44]).

Definition 2.2 *Let $c_2, \ldots, c_{m-1}$ be distinct real numbers (usually between 0 and 1), and let b_1, b_m be two arbitrary real numbers. The corresponding* *discontinuous method is then defined via a polynomial of degree* $m-2$ *satisfying*

$$\begin{aligned} u(x_0) &= y_0 - hb_1(\dot{u}(x_0) - f(x_0, u(x_0)), \\ \dot{u}(x_0 + c_i h) &= f(x_0 + c_i h, u(x_0 + c_i h)), \quad i = 2, \ldots, m-1, \\ y_1 &= u(x_1) - hb_s(\dot{u}(x_1) - f(x_1, u(x_1)). \end{aligned}$$

Discontinuous collocation methods fall inside a large class of implicit Runge-Kutta methods, as stated by the following result (see [44]).

Theorem 2.3 *The discontinuous collocation method given in Definition 2.2 is equivalent to an* m*-stage Runge-Kutta method with coefficients determined by* $c_1 = 0$*,* $c_m = 1$ *and*

$$a_{i1} = b_1, \qquad a_{im} = 0, \qquad i = 1, \ldots, m,$$

while the other coefficients result as solutions of the linear systems $A(m-2)$ *and* $B(m-2)$ *defined in (2.7) and (2.8).*

As a consequence of this result, if $b_1 = 0$ and $b_m = 0$, then the discontinuous collocation method in Definition 2.2 is equivalent to the $(m-2)$-collocation method based on $c_2, \ldots, c_{m-1}$. An interesting example of implicit Runge-Kutta method which is not collocation based but is of discontinuous collocation type is the Lobatto IIIB method (see [11, 44, 45, 57]), which plays an important rule in the context of geometric numerical integration, together with Lobatto IIIA method (see [44], p. 33). They are both nonsymplectic methods (see Theorem 4.3 in [44]) but, considered as a pair, the resulting method is symplectic. This is a nice example of methods which possess very strong properties, but are difficult to investigate as discrete scheme (they cannot be studied as collocation methods, because they are not both collocation based); however, re-casted as discontinuous collocation based methods, their analysis is reasonably simplified and very elegant [44].

2.4 Multistep collocation

The successive results which appeared in literature (see [23, 43, 46, 59, 60]) have been devoted to the construction of multistep collocation methods. Guillou and Soulé introduced multistep collocation methods [43], by adding interpolation conditions in the previous k step points, so that the collocation polynomial is defined by

$$\begin{cases} P(x_{n-i}) = y_{n-i} \quad i = 0, 1, ..., k-1, \\ P'(x_n + c_j h) = f(x_n + c_j h, P(x_n + c_j h)) \quad j = 1, ..., m. \end{cases} \tag{2.9}$$

The numerical solution is given, as usual,

$$y_{n+1} = P(x_{n+1}). \tag{2.10}$$

Hairer-Wanner [46] and Lie-Norsett [60] derived different strategies to obtain multistep collocation methods. In [46] the Hermite problem with incomplete data (2.9) is solved by means of the introduction of a generalized Lagrange basis

$$\{\varphi_i(s),\ \psi_j(s),\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m\}$$

and, correspondingly, the collocation polynomial is expressed as linear combination of this set of functions, i.e.

$$P(x_n + sh) = \sum_{i=1}^{k} \varphi_i(s) y_{n-k+i} + h \sum_{i=1}^{s} \psi_i(s) P'(x_n + c_i h),$$

where $s = \frac{x - x_n}{h}$. Therefore, the problem (2.9) is transformed in the problem of deriving $\{\varphi_i,\ \psi_j,\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m\}$ in such a way that the corresponding polynomial $P(s)$ satisfies the conditions (2.9).

Lie-Norsett in [60] completely characterized multistep collocation methods, giving the expressions of the coefficients of collocation based multistep Runge-Kutta methods in closed form, as stated by the following

Theorem 2.4 *The multistep collocation method (2.9)-(2.10) is equivalent to a multistep Runge-Kutta method*

$$\begin{aligned} Y_j &= \sum_{i=0}^{k-1} \varphi_i(c_j) y_{n+k-1-i} \\ &+ h \sum_{i=1}^{m} \psi_i(c_j) f(x_{n+k-1} + c_i h, Y_i), \qquad j = 1, ..., m, \end{aligned}$$

$$y_{n+k} = \sum_{i=0}^{k-1} \varphi_i(1) y_{n+k-1-i} + h \sum_{i=1}^{m} \psi_i(1) f(x_{n+k-1} + c_i h, Y_i),$$

where the expression of the polynomials $\varphi_i(s)$, $\psi_i(s)$ are provided in Lemma 1 of [60]. $\square$

Lie and Norsett in [60] also provided a complete study of the order of the resulting methods, stating order conditions by means of the study of variational matrices, and showing that the maximum attainable order of a k-step m-stage collocation method is $2m + k - 1$. They also proved that there exist $\binom{m+k-1}{k-1}$ nodes that allow superconvergence and, in analogy with Runge-Kutta methods, they are named *multistep Gaussian* collocation points. As Hairer-Wanner stated in [46], these methods are not stiffly stable and, therefore, they are not suited for stiff problems: in order to obtain better stability properties, they derived methods of highest order $2m+k-2$, imposing $c_m = 1$ and deriving the other collocation abscissa in a suited way to achieve this highest order and named the corresponding methods of "Radau"-type, studied their stability properties, deriving also many A-stable methods.

2.5 Two-step collocation and almost collocation methods

In more recent times, our strenghts have been devoted to extend the multistep collocation technique to the class of two-step Runge-Kutta methods (TSRK)

$$\begin{cases} y_{n+1} = \theta y_{n-1} + \tilde{\theta} y_n + h \displaystyle\sum_{j=1}^{m} \Big(v_j f(Y_j^{[n-1]}) + w_j f(Y_j^{[n]}) \Big), \\ Y_i^{[n]} = u_i y_{n-1} + \tilde{u}_i y_n + h \displaystyle\sum_{j=1}^{m} \Big(a_{ij} f(Y_j^{[n-1]}) + b_{ij} f(Y_j^{[n]}) \Big), \end{cases} \tag{2.11}$$

introduced by Jackiewicz and Tracogna [53] and further investigated by several authors (see [51] and references therein contained). Two-step Runge-Kutta methods (2.11) differ from the multistep Runge-Kutta methods above described, because they also depend on the stage derivatives at two consecutive step points: as a consequence, "we gain extra degrees of freedom associated with a two-step scheme without the need for extra function evaluations" (see [53]), because the function evaluations $f(Y_j^{[n-1]})$ are completely inherited from the previous step. Therefore, the computational cost of these formulae only depends on the structure of the matrix B. Different approaches to the construction of continuous TSRK methods outside collocation are presented in [1], [3] and [54].

The continuous approximant

$$\begin{cases} P(x_n + sh) = \varphi_0(s) y_{n-1} + \varphi_1(s) y_n \\ \quad + h \displaystyle\sum_{j=1}^{m} \Big(\chi_j(s) f(P(x_{n-1} + c_j h)) + \psi_j(s) f(P(x_n + c_j h)) \Big), \\ y_{n+1} = P(x_{n+1}), \end{cases} \tag{2.12}$$

expressed as linear combination of the basis functions

$$\{\varphi_0(s),\ \varphi_1(s),\ \chi_j(s),\ \psi_j(s),\ j = 1, 2, \ldots, m\},$$

is an algebraic polynomial which is derived in order to satisfy some interpolation and collocation conditions, i.e.

$$\begin{array}{l} P(x_{n-1}) = y_{n-1}, \quad P(x_n) = y_n, \\ P'(x_{n-1}+c_ih) = f(x_{n-1}+c_ih, P(x_{n-1}+c_ih)),\ i = 1,2,...,m, \\ P'(x_n+c_ih) = f(x_n+c_ih, P(x_n+c_ih)), \quad i = 1,2,...,m. \end{array} \tag{2.13}$$

As a first attempt, we have generalized in [33], [34] the techniques introduced by Guillou-Soulé [43], Hairer–Wanner [46] and Lie–Norsett [60], adapting and extending this technique to TSRK methods. Using the techniques introduced in [60], we have derived in [34] the coefficients of (2.12) in closed form: the corresponding results are reported in the following theorem (see [34]).

Theorem 2.5 *The method (2.12) is equivalent to a TSRK method in the form*

$$\begin{aligned} Y_j^{[n]} = \phi_0(c_j)y_{n-1} \quad &+ \quad \phi_1(c_j)y_n + h\sum_{i=1}^{m}[\chi_j(c_i)f(x_{n-1}+c_ih, Y_i^{[n-1]}) \\ &+ \quad \psi_j(c_i)f(x_n+c_ih, Y_i^{[n]})], \quad j = 1,\ldots,m, \end{aligned}$$

$$\begin{aligned} y_{n+1} = \phi_0(1)y_{n-1} \quad &+ \quad \phi_1(1)y_n + h\sum_{j=1}^{m}[\chi_j(1)f(x_{n-1}+c_jh, Y_j^{[n-1]}) \\ &+ \quad \psi_j(1)f(x_n+c_jh, Y_j^{[n]})], \end{aligned}$$

where

$$\psi_j(s) = \int_0^s l_j(\tau)d\tau - \frac{\int_{-1}^0 l_j(\tau)d\tau}{\int_{-1}^0 M(\tau)d\tau}\int_0^s M(\tau)d\tau, \quad j = 1,\ldots,m,$$

$$\chi_j(s) = \int_0^s \tilde{l}_j(\tau)d\tau - \frac{\int_{-1}^0 \tilde{l}_j(\tau)d\tau}{\int_{-1}^0 M(\tau)d\tau}\int_0^s M(\tau)d\tau, \quad j = 1,\ldots,m,$$

$$\phi_0(s) = -\frac{\int_0^s M(\tau)d\tau}{\int_{-1}^0 M(\tau)d\tau},$$

$$\phi_1(s) = 1 + \frac{\int_0^s M(\tau)d\tau}{\int_{-1}^0 M(\tau)d\tau}.$$

with

$$l_i(s) = \prod_{j=1, j\neq i}^{2m} \frac{s-d_j}{d_i-d_j}, \qquad M(s) = \prod_{j=1}^{2m}(s-d_j), \qquad \begin{cases} d_i = c_i \\ d_{m+i} = c_i - 1, \end{cases}$$

$$i = 1, 2, ..., m,$$

$$\tilde{l}_j(s) = \prod_{i=1, i\neq j}^{2m} \frac{s-e_i}{e_j-e_i}, \qquad \begin{cases} e_i = c_i - 1 \\ e_{m+i} = c_i, \end{cases} \qquad i = 1, 2, ..., m. \qquad \square$$

We proved in [34] that the resulting methods have uniform order $2m+1$ but such an high order enforces these methods to have bounded stability regions only. For this reason, in order to derive highly stable methods (i.e. A-stable and L-stable), we have introduced in [24,25,37] the class of *almost collocation methods*, which are obtained in such a way that only some of the above interpolation and collocation conditions are satisfied. Relaxing the above conditions, we obtain more degrees of freedom, which have been used in order to derive many A-stable and L-stable methods of order $m+r$, $r = 0,\ 1, ...,\ m$. Therefore the highest attainable order is $2m$ which, in principle, can seem a loss in comparison with standard Runge-Kutta methods. As a matter of fact, this is not true: in fact, Runge-Kutta-Gauss methods have order $2m$ in the grid points, while the stage order is equal to m, therefore they suffer from order reduction in the integration of stiff problems (see [11, 12, 46]), i.e. the effective order of convergence in presence of stiffness is only m. Our methods, instead, do not suffer from order reduction, i.e. the effective order of convergence in the integration of stiff problems is $2m$, because they have high stage order. In [37] we have studied the existence of such methods, derived continuous order conditions, provided characterization results and studied their stability properties. A complete analysis of m-stage two-step continuous methods, with $m = 1, 2, 3, 4$, has been provided in [38], while the analysis of the implementation issues for two-step collocation methods has been provided in [39]. The construction of algebraically stable two-step collocation methods is object of a current research project.

3 Collocation methods for second order ODEs of special type

We now concentrate our attention on the hystorical evolution of the collocation technique for the numerical solution of initial value problems based on second order ordinary differential equations with periodic and oscillating solution

$$\begin{cases} y'(x) = f(x, y(x)), \quad x \in [x_0, X], \\ y'(x_0) = y'_0 \in \mathbb{R}^d, \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

where $f : [x_0, X] \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be a is sufficiently smooth function, in order to ensure the existence and the uniqueness of the solution.

3.1 Direct and indirect collocation methods

In the context of collocation methods for second order equations, two possibilities have been taken into account in the literature, i.e. methods based on *indirect* or *direct collocation* [77]. Indirect collocation methods are generated by applying a collocation based Runge-Kutta method to the first order representation of (3.1), which has doubled dimension. If

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

is the Butcher array of a collocation Runge-Kutta method, the tableau of the corresponding indirect collocation method is

$$\begin{array}{c|c} c & A^2 \\ \hline & A^T b \\ & b^T \end{array}$$

which results in a Runge-Kutta-Nyström method [45]. The theory of indirect collocation methods completely parallels the well–known theory of collocation methods for first order equations (see [77]) and, therefore, the properties of a collocation method are totally inherited by the corresponding indirect collocation method. Thus, the maximum attainable order is $2m$, where m is the number of stages, and it is achieved by Gauss-type methods, which are also A-stable, while L-stability is achieved by Radau IIA-type methods, of order $2m - 1$.

In the case of direct collocation methods, the collocation polynomial is derived directly for the second order problem. Van der Houwen et al. in [77] studied the order, stage order of direct collocation methods and also provided their stability analysis, extending the results of Kramarz [56]. Concerning order and stage order, the following result holds (see [77]):

Theorem 3.1 *Direct and indirect collocation methods with the same collocation nodes have the same order. The stage order of direct collocation* *methods is one higher whenever*

$$\int_0^1 \prod_{i=1}^m (s-c_i)ds = 0. \qquad \square$$

Therefore, while indirect and direct collocation methods have the same order, their stage order is different and, in particular, direct methods have higher stage order. However, they are not competitive in terms of stability. Van der Houwen et al. in [77] clearly state that "From a practical point of view, direct collocation methods based on Gauss, Radau and Lobatto collocation points are of limited value, because the rather small stability or periodicity boundaries make them unsuitable for stiff problems. The A-stable indirect analogues are clearly more suitable for integrating stiff problems".

Moreover, Coleman [18] proved that no P-stable one step symmetric collocation methods exist. P-stability (see Lambert-Watson paper [58]) is a very relevant property for the numerical treatment of a second order system whose theoretical solution is periodic with a moderate frequency and a high frequency oscillation of small amplitude superimposed. This phenomenon is known in literature as *periodic stiffness* [75], which can be reasonably faced using P-stable methods, exactly as A-stable methods are suitable for stiff problems. In other terms, P-stability ensures that the choice of the stepsize is independent from the values of the frequencies, but it only depends on the desired accuracy [22, 70].

In [58], the authors proved that P-stable linear multistep methods

$$\sum_{j=0}^p \alpha_j y_{n+j} = h^2 \sum_{j=0}^p \beta_j f_{n+j}.$$

can achieve maximum order 2. In the context of Runge–Kutta–Nyström methods

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + h^2 \sum_{i=1}^m \bar{b}_i f(x_n + c_i h, Y_i), \\ y'_{n+1} &= y'_n + h \sum_{i=1}^m b_i f(x_n + c_i h, Y_i), \\ Y_i &= y_n + c_i h y'_n + h^2 \sum_{j=1}^m a_{ij} f(x_n + c_j h, Y_j), \quad i = 1, 2, ..., m, \end{aligned}$$

many A-stable and P-stable methods exist, but the ones falling in the subclass of collocation methods, whose coefficients (see [45]) are of the form

$$
\begin{aligned}
a_{ij} &= \int_0^{c_i} L_j(s)ds, \\
b_i &= \int_0^{1} L_i(s)ds, \\
\bar{b}_i &= \int_0^{1} (1-s)L_i(s)ds,
\end{aligned}
$$

have only bounded stability intervals and are not P-stable [70].

3.2 Two-step Runge-Kutta-Nyström methods

We have observed in the previous paragraph that P-stability is a desirable property that only few methods in the context of linear multistep methods and Runge-Kutta-Nyström methods possess. In order to create a good balance between high order and strong stability properties, further steps in the literature have been devoted to the development of multistep Runge-Kutta-Nystrom methods for second order problems. Much of this work has been done by Paternoster (see [64, 70–74]). In particular, the author proved that no P-stable methods can be found in the class of indirect collocation TSRK methods, while it was possible to find P-stable methods in the context of *two-step Runge-Kutta-Nyström methods*

$$
\begin{aligned}
Y_j^{[n-1]} &= y_{n-1} + c_j h y'_{n-1} + h^2 \sum_{k=1}^{m} a_{jk} f(x_{n-1} + c_k h, Y_k^{[n-1]}), \quad j = 1, 2, ..., m, \\
Y_j^{[n]} &= y_n + c_j h y'_n + h^2 \sum_{k=1}^{m} a_{jk} f(x_n + c_k h, Y_k^{[n]}), \quad j = 1, 2, ..., m, \\
y_{n+1} &= (1-\theta) y_n + \theta y_{n-1} + h \sum_{j=1}^{m} (v_j y'_{n-1} + w_j y'_n) \\
&+ h^2 \sum_{j=1}^{m} \bar{v}_j f(x_{n-1} + c_j h, Y_j^{[n-1]}) + \bar{w}_j f(x_n + c_j h, Y_j^{[n]}), \\
y'_{n+1} &= (1-\theta) y'_n + \theta y'_{n-1} + h \sum_{j=1}^{m} (v_j f(x_{n-1} + c_j h, Y_j^{[n-1]}) \\
&+ w_j f(x_n + c_j h, Y_j^{[n]})),
\end{aligned}
$$

which represent the extension to second order problems of the two-step Runge-Kutta methods introduced in [52] for first order problems.

3.3 Collocation based two-step hybrid methods

In the numerical integration of Second Order Inital Value Problems through collocation, many possibilities can be taken into account: for example, Runge–Kutta–Nyström methods provide an approximation to the solution and its first derivative at each step point. However, as Henrici observed in [47], "If one is not particularly interested in the values of the first derivatives, it seems unnatural to introduce them artificially". For this reason, other types of methods have been taken into account in the literature, i.e. methods which provide an approximation to the solution without computing any approximation to the first derivative: these formulae are denoted in literature as *hybrid methods*. Coleman introduced in [20] the following class of two-step hybrid methods for second order equations:

$$
\begin{aligned}
Y_i^{[n]} &= u_i y_{n-1} + (1-u_i)y_n + h^2 \sum_{j=1}^{m} a_{ij} f(x_n + c_j h, Y_j^{[n]}), \quad (3.2)\\
& \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, m,\\
y_{n+1} &= \theta y_{n-1} + (1-\theta)y_n + h^2 \sum_{j=1}^{m} w_j f(x_n + c_j h, Y_j^{[n]}). \quad (3.3)
\end{aligned}
$$

This class of methods has been further investigated in [17, 36, 41, 78, 79]. In more recent times, we derived in [35] collocation based methods belonging to the class of Coleman hybrid methods (3.2)-(3.3), extending the technique introduced by Hairer and Wanner in [46] for first order problems. The collocation polynomial takes the form

$$
P(x_n + sh) = \varphi_1(s) y_{n-1} + \varphi_2(s) y_n + h^2 \sum_{j=1}^{m} \chi_j(s) P''(x_n + c_j h), \quad (3.4)
$$

where $s = \frac{x - x_n}{h} \in [0, 1]$, and unknown basis functions

$$
\{\varphi_1(s),\ \varphi_2(s),\ \chi_j(s),\ j = 1,\ 2,\ \dots,\ m\},
$$

are derived imposing the following $m+2$ conditions

$$
\begin{aligned}
P(x_{n-1}) &= y_{n-1}, \qquad P(x_n) = y_n,\\
P''(x_n + c_j h) &= f(x_n + c_j h, P(x_n + c_j h)), \quad j = 1, \dots, m.
\end{aligned}
$$

After computing the basis functions as solutions of $m+2$ linear systems (see [63]), the resulting class of methods takes the following form

$$
\begin{aligned}
Y_i^{[n]} &= \varphi_1(c_i)y_{n-1} + \varphi_2(c_i)y_n + h^2 \sum_{j=1}^{m} \chi_j(c_i)P''(x_n + c_j h), \quad (3.5) \\
y_{n+1} &= \varphi_1(1)y_{n-1} + \varphi_2(1)y_n + h^2 \sum_{j=1}^{m} \chi_j(1)P''(x_n + c_j h). \quad (3.6)
\end{aligned}
$$

In [35] we have provided the study of stability and periodicity properties and derived continuous order conditions for (3.6)-(3.5), which are object of the following result.

Theorem 3.2
Assume that the function f is sufficiently smooth. The collocation method associated to (3.4) has uniform order p if the following conditions are satisfied:

$$
\begin{aligned}
&1 - \varphi_1(s) - \varphi_2(s) = 0, \qquad s + \varphi_1(s) = 0, \\
&s^k + (-1)^{k+1}\varphi_1(s) - k(k-1)\sum_{j=1}^{m} \chi_j(s) c_j^{k-2} = 0.
\end{aligned}
$$

$k = 2, \ldots, p, \quad s \in [0, 1].$

Theorem 3.2 allows us to prove that every two–step collocation method associated to (3.4), has order $p = m$ on the whole integration interval, and this is result is in keeping with [20].

3.4 Mixed collocation methods

The development of classical collocation methods (i.e. methods based on algebraic polynomials), even if it is not the most suitable choice for second order problems that do not possess solutions with polynomial behaviour, is the first necessary step in order to construct collocation methods whose collocation function is expressed as linear combination of different functions, e.g. trigonometric polynomials, mixed or exponential basis (see, for instance, [21, 50]), which can better follow the qualitative behaviour of the solution. It is indeed more realistic to choose basis functions which are not polynomials.

Many authors have considered in literature different functional basis, instead of the polynomial one, e.g. [8,19,22,36,40,42,50,55,67,69,71,73,76]. In particular we mention here the work by Coleman and Duxbury [21], where the authors introduced mixed collocation methods applied to the Runge-Kutta-Nyström scheme, where the collocation function is expressed as linear combination of trigonometric functions and powers, in order to provide better approximations for oscillatory solutions. The methods are derived in order to exactly integrate the armonic oscillator

$$y'' = -k^2 y,$$

where k is a constant, a feature which is not achievable by algebraic polynomial collocation. The term *mixed interpolation* appeared for the first time in [40] to describe interpolation by a linear combination of a sine and cosine of a given frequency, and powers of the relevant variable, and later used by Brunner et al. in [8] in the context of Volterra integral equations. The solution on the generic integration interval $[x_n, x_{n+1}]$ is approximated by the collocating function

$$u(x_n + sh) = a\cos\theta s + b\sin\theta s + \sum_{i=0}^{m-1} \Gamma_i s^i, \tag{3.7}$$

which satisfies the following collocation and interpolation conditions

$$\begin{aligned} u(x_n) &= y_n, \qquad u'(x_n) = y'_n, \\ u''(x_n + c_j h) &= f(x_n + c_j h, u(x_n + c_j h)), \quad j = 1, \ldots, m. \end{aligned}$$

Integrating (3.7) twice, we obtain the Runge-Kutta-Nystrom formulation of the methods, i.e.

$$\begin{aligned} u'(x_n + sh) &= y'_n + h\sum_{i=1}^{m} \alpha_i(s) f_{n+c_i}, \\ u(x_n + sh) &= y_n + shy'_n + h^2 \sum_{i=1}^{m} \beta_i(s) f_{n+c_i}, \end{aligned}$$

where

$$\alpha_i(s) = \int_0^s L_i(\tau)d\tau, \qquad \beta_i(s) = \int_0^s (s-\tau)L_i(\tau)d\tau.$$

Outside collocation, many authors derived methods having frequency dependent parameters (see, for instance, [50, 55, 68, 76] and references therein contained). The linear stability analysis of these methods is carried out in [22]. In [36] also a method with parameters depending on two frequency is presented, and the modification in the stability analysis is performed, leading to a three dimensional region.

4 Collocation methods for VIEs

Piecewise polynomial collocation methods for VIEs introduce a number of aspects not present when solving ODEs. In this section we will present the main results in the context of collocation and almost collocation methods for VIEs of the form

$$y(x) = g(x) + \int_0^x k(x, \tau, y(\tau))d\tau \qquad x \in I := [0, X], \tag{4.1}$$

where $k \in C(D \times \mathbb{R})$, with $D := \{(x, \tau) : 0 \leq \tau \leq x \leq X\}$, and $g \in C(I)$, also underlying connections and differences with the case of ODEs.

4.1 Classical one–step collocation methods

Let us discretize the interval I by introducing a uniform mesh

$$I_h = \{x_n := nh, n = 0, ..., N,\ h \geq 0, Nh = X\}.$$

The equation (4.1) can be rewritten, by relating it to this mesh, as

$$y(x) = F_n(x) + \Phi_n(x) \qquad x \in [x_n, x_{n+1}],$$

where $F_n(x) := g(x) + \int_0^{x_n} k(x, \tau, y(\tau))d\tau$ and $\Phi_n(x) := \int_{x_n}^{x} k(x, \tau, y(\tau))d\tau$ represent respectively the *lag term* and the *increment function*. Let us fix m collocation parameters $0 \leq c_1 < ... < c_m \leq 1$ and denote by $x_{nj} = x_n + c_j h$ the collocation points. The collocation polynomial, restricted to the interval $[x_n, x_{n+1}]$, is of the form:

$$u_n(x_n + sh) = \sum_{j=1}^{m} L_j(s)U_{nj} \quad s \in [0, 1] \quad n = 0, ..., N-1 \tag{4.2}$$

where $L_j(s)$ is the $j-th$ Lagrange fundamental polynomial with respect to the collocation parameters and $U_{nj} := u_n(x_{nj})$. *Exact* collocation methods are obtained by imposing that the collocation polynomial (4.2) exactly satisfies the VIE (4.1) in the collocation points x_{ni} and by computing $y_{n+1} = u_n(x_{n+1})$:

$$\begin{cases} U_{ni} = F_{ni} + \Phi_{ni} \\ y_{n+1} = \sum_{j=1}^{m} L_j(1)U_{nj} \end{cases}, \tag{4.3}$$

where

$$F_{ni} = g(x_{ni}) + h\sum_{\nu=0}^{n-1} \int_0^1 k(x_{ni}, x_\nu + sh, u_\nu(x_\nu + sh))ds \quad i = 1, ..., m \tag{4.4}$$

$$\Phi_{ni} = h\int_0^{c_i} k(x_{ni}, x_n + sh, u_n(x_n + sh))ds \quad i = 1, ..., m. \tag{4.5}$$

Note that the first equation in (4.3) represents a system of m nonlinear equations in the m unknowns U_{ni}. We obtain an approximation $u(x)$ of the solution $y(x)$ of the integral equation (4.1) in $[0, X]$, by considering

$$u(x)|_{(x_n,x_{n+1}]} = u_n(x) \tag{4.6}$$

where $u_n(x)$ given by (4.2).

We recall that, in contrast with what happens in the case of ODEs, generally $u(x)$ is not continuous in the mesh points, as

$$u(x) \in S^{(-1)}_{m-1}(I_h), \tag{4.7}$$

where

$$S^{(d)}_{\mu}(I_h) = \left\{ v \in C^d(I) : v|_{(x_n,x_{n+1}]} \in \Pi_\mu \ (0 \leq n \leq N-1) \right\}.$$

Here, Π_μ denotes the space of (real) polynomials of degree not exceeding μ. A complete analysis of collocation methods for linear and nonlinear Volterra integral and integro–differential equations, with smooth and weakly singular kernels is given in [7]. In particular, as shown in [7, 10], the classical one–step collocation methods for a second-kind VIE do no longer exhibit $O(h^{2m})$ superconvergence at the mesh points if collocation is at the Gauss points, in fact they have uniform order m for any choice of the collocation parameters and local superconvergence order in the mesh points of $2m-2$ (m Lobatto points or $m-1$ Gauss points with $c_m = 1$) or $2m-1$ (m Radau II points). The optimal order is recovered only in the iterated collocation solution.

We observe that, differently from the case of ODEs, the collocation equations are in general not yet in a form amenable to numerical computation, due to the presence of the memory term given by the Volterra integral operator. Thus, another discretisation step, based on quadrature formulas $\bar{F}_{ni} \simeq F_{ni}$ and $\bar{\Phi}_{ni} \simeq \Phi_{ni}$ for approximating the lag term (4.4) and the increment function (4.5), is necessary to obtain the fully discretised collocation scheme, thus leading to *Discretized* collocation methods. Such methods preserve, under suitable hypothesis on the quadrature formulas, the same order of the exact collocation methods [10].

The connection between collocation and implicit Runge–Kutta methods for VIEs (the so called VRK methods) is not immediate: a collocation method for VIEs is equivalent to a VRK method if and only if $c_m = 1$ (see Theorem 5.2.2 in [10]). Some other continuous extensions of Runge–Kutta methods for VIEs, which do not necessarily lead to collocation methods, have been introduced in [5].

Many efforts have been made in the literature with the aim of obtaining fast collocation and more general Runge–Kutta methods for the numerical solution of VIEs. It is known that the numerical treatment of VIEs is very expensive from computational point of view because of presence of the "lag-term", which contains the entire history of the phenomenon. To this cost, it has also to be added the one due to the "increment term" which leads, for implicit methods (generally possessing the best stability properties), to the resolution of a system of nonlinear equations at each step of integration. In order to reduce the computational effort in the lag–term computation, fast collocation and Runge–Kutta methods have been constructed for convolution VIEs of Hammerstein type, see [14, 27, 61, 62].

The stability analysis of collocation and Runge–Kutta methods for VIEs can be found in [4, 10, 15, 31] and the related bibliography. In particular a collocation method for VIEs is A–stable if the corresponding method for ODEs is A–stable.

4.2 Multistep collocation

Multistep collocation and Runge–Kutta methods for VIEs, have been introduced in order to bring down the computational cost related to the resolution of non-linear systems for the computation of the increment term. As a matter of fact such methods, showing a dependence on stages and steps in more consecutive grid points, permit to raise the order of convergence of the classical methods, without inflating the computational cost or, equivalently, having the same order at a lower computational cost.

A first analysis of multistep collocation methods for VIEs appeared in [29,30], where the methods are obtained by introducing in the collocation polynomial the dependence from r previous time steps; namely we seek for a collocation polynomial, whose restriction to the interval $[x_n, x_{n+1}]$ takes the form

$$P_n(x_n+sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) Y_{nj}, \quad s \in [0,1], \quad n = 0, ..., N-1, \tag{4.8}$$

where

$$Y_{nj} := P_n(x_{nj}) \tag{4.9}$$

and $\varphi_k(s)$, $\psi_j(s)$ are polynomials of degree $m+r-1$ to be determined by imposing the interpolation conditions at the points x_{n-k}, that is $u_n(x_{n-k}) = y_{n-k}$, and by satisfying (4.9). It is proved in [26, 28] that, assuming $c_i \neq c_j$

and $c_1 \neq 0$, the polynomials $\varphi_k(s)$, $\psi_j(s)$ have the form:

$$\begin{aligned} \varphi_k(s) &= \prod_{i=1}^{m} \tfrac{s-c_i}{-k-c_i} \cdot \prod_{\substack{i=0\\ i\neq k}}^{r-1} \tfrac{s+i}{-k+i}, \\ \psi_j(s) &= \prod_{i=0}^{r-1} \tfrac{s+i}{c_j+i} \cdot \prod_{\substack{i=1\\ i\neq j}}^{m} \tfrac{s-c_i}{c_j-c_i}. \end{aligned} \tag{4.10}$$

The discretized multistep collocation method is then obtained by imposing the collocation conditions, i.e. that the collocation polynomial (4.8) exactly satisfies the VIE (4.1) at the collocation points x_{ni}, and by computing $y_{n+1} = P_n(x_{n+1})$:

$$\left\{ \begin{array}{c} Y_{ni} = \bar{F}_{ni} + \bar{\Phi}_{ni} \\ y_{n+1} = \sum\limits_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum\limits_{j=1}^{m} \psi_j(1) Y_{nj} \end{array} \right. . \tag{4.11}$$

The lag–term and increment–term approximations

$$\bar{F}_{ni} = g(x_{ni}) + h \sum_{\nu=0}^{n-1} \sum_{l=0}^{\mu_1} b_l k(x_{ni}, x_\nu + \xi_l h, P_\nu(x_\nu + \xi_l h)) \quad i = 1, ..., m \tag{4.12}$$

$$\bar{\Phi}_{ni} = h \sum_{l=0}^{\mu_0} w_{il} k(x_{ni}, x_n + d_{il} h, P_n(x_n + d_{il} h)) \quad i = 1, ..., m \tag{4.13}$$

are obtained by using quadrature formulas of the form

$$(\xi_l, b_l)_{l=1}^{\mu_1}, \quad (d_{il}, w_{il})_{l=1}^{\mu_0}, \; i = 1, ..., m, \tag{4.14}$$

where the quadrature nodes ξ_l and d_{il} satisfy $0 \leq \xi_1 < ... < \xi_{\mu_1} \leq 1$ and $0 \leq d_{i1} < ... < d_{i\mu_0} \leq 1$, μ_0 and μ_1 are positive integers and w_{il}, b_l are suitable weights.

The discretized multistep collocation method (4.8)-(4.11) provides a continuous approximation $P(x)$ of the solution $y(x)$ of the integral equation (4.1) in $[0, X]$, by considering

$$P(x)|_{(x_n, x_{n+1}]} = P_n(x) \tag{4.15}$$

where $P_n(x)$ is given by (4.8). We note that usually the polynomial constructed in the collocation methods for VIEs doesn't interpolate the numerical solution in the previous step points, resulting a discontinuous approximation of the solution (4.7). In this multistep extension, the collocation polynomial is instead a continuous approximation to the solution, i.e. $u(x) \in S_{m-1}^{(0)}(I_h)$

The discretized multistep collocation method (4.8)-(4.11) can be regarded as a multistep Runge–Kutta method for VIEs:

$$\begin{cases} Y_{ni} = \bar{F}_n(x_{ni}) + h \sum\limits_{l=1}^{\mu_0} w_{il} k\left(x_n + e_{il}h,\ x_n + d_{il}h,\ \sum\limits_{k=0}^{r-1} \gamma_{ilk} y_{n-k} + \sum\limits_{j=1}^{m} \beta_{ilj} Y_{nj}\right) \\ y_{n+1} = \sum\limits_{k=0}^{r-1} \theta_k y_{n-k} + \sum\limits_{j=1}^{m} \lambda_j Y_{nj} \end{cases}, \tag{4.16}$$

where

$$\bar{F}_n(x) = g(x) + h \sum_{\nu=0}^{n-1} \sum_{l=1}^{\mu_1} b_l k\left(x,\ x_\nu + \xi_l h,\ \sum_{k=0}^{r-1} \delta_{lk} y_{\nu-k} + \sum_{j=1}^{m} \eta_{lj} Y_{\nu,j}\right) \tag{4.17}$$

and

$$\begin{aligned} &e_{il} = c_i,\ \gamma_{ilk} = \varphi_k(d_{il}),\ \beta_{ilj} = \psi_j(d_{il}),\\ &\theta_k = \varphi_k(1),\ \lambda_j = \psi_j(1),\\ &\delta_{lk} = \varphi_k(\xi_l),\ \eta_{lj} = \psi_j(\xi_l). \end{aligned}$$

The reason of interest of the multistep collocation methods lies in the fact that they increase the order of convergence of collocation methods without increasing the computational cost, except for the cost due to the starting procedure. As a matter of fact, in advancing from x_n to x_{n+1}, we make use of the approximations y_{n-k}, $k = 0, ..., r-1$, which have already been evaluated at the previous steps. This permits to increase the order, by maintaining in (4.11) the same dimension m of the nonlinear system (4.3).

The r–steps m–points collocation methods have uniform order $m + r$, and order of local superconvergence $2m+r-1$. The knowledge of the collocation polynomial, which provides a continuous approximation of uniform order of the solution, will allow a cheap variable stepsize implementation. Indeed, when the stepsize changes, the new approximation values can be computed by simply evaluating the collocation polynomial, without running into problems of order reduction, as a consequence of the uniform order.

4.3 Two–step collocation and almost collocation methods

Unfortunately multistep methods of the form (4.8)-(4.11) do not lead to a good balance between high order and strong stability properties, infact, altough methods with unbounded stability regions exist, no A–stable methods have been found. Therefore in [30] a modification in the technique has been introduced, thus obtaining two-step *almost* collocation methods, also for systems of VIEs, by relaxing some of the collocation conditions and by introducing some previous stage values, in order to further increase the order and to have free parameters in the method, to be used to get A-stability.

The methods are defined by

$$\begin{cases} P(x_n + sh) = \varphi_0(s)y_{n-1} + \varphi_1(s)y_n + \sum_{j=1}^{m} \chi_j(s)P(x_{n-1,j}) \\ \quad + \sum_{j=1}^{m} \psi_j(s)\big(\bar{F}_{nj} + \bar{\Phi}_{nj}\big)\Big), \\ y_{n+1} = P(x_{n+1}), \end{cases} \tag{4.18}$$

$s \in (0,1]$, $n = 1, 2, \ldots, N-1$.

If the polynomials $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$ and $\psi_j(s)$, $j = 1, 2, ..., m$ satisfy the interpolation conditions

$$\begin{array}{llll} \varphi_0(0) = 0, & \varphi_1(0) = 1, & \chi_j(0) = 0, & \psi_j(0) = 0, \\ \varphi_0(-1) = 1, & \varphi_1(-1) = 0, & \chi_j(-1) = 0, & \psi_j(-1) = 0, \end{array}$$

and the collocation conditions

$$\begin{array}{llll} \varphi_0(c_i) = 0, & \varphi_1(c_i) = 0, & \chi_j(c_i) = 0, & \psi_j(c_i) = \delta_{ij}, \\ \varphi_0(c_i - 1) = 0, & \varphi_1(c_i - 1) = 0, & \chi_j(c_i - 1) = \delta_{ij}, & \psi_j(c_i - 1) = 0, \end{array}$$

$i = 1, 2, ..., m$, then we obtain order $p = 2m + 1$.

In our search for A-stable methods we will have been mainly concerned with methods of order $p = 2m - r$, where $r = 1$ or $r = 2$ is the number of relaxed conditions. Namely we for $p = 2m + 1 - r$, $r = 1, 2$, have chosen $\varphi_0(s)$ as a polynomial of degree $\leq 2m+1-r$, which satisfies the collocation conditions

$$\varphi_0(c_i) = 0, \quad i = 1, 2, \ldots, m. \tag{4.19}$$

This leads to the polynomial $\varphi_0(s)$ of the form

$$\varphi_0(s) = (q_0 + q_1 s + ... + q_{m+1-r} s^{m+1-r}) \prod_{i=1}^{m} (s - c_i), \tag{4.20}$$

where $q_0, q_1, ..., q_{m+1-r}$ are free parameters. Moreover, for $p = 2m - 1$ we have chosen $\varphi_1(s)$ as a polynomial of degree $\leq 2m - 1$ which satisfies the collocation conditions

$$\varphi_1(c_i) = 0, \quad i = 1, 2, \ldots, m. \tag{4.21}$$

This leads to the polynomial $\varphi_1(s)$ of the form

$$\varphi_1(s) = (p_0 + p_1 s + ... + p_{m-1} s^{m-1}) \prod_{i=1}^{m} (s - c_i), \tag{4.22}$$

where $p_0, p_1, ..., p_{m-1}$ are free parameters.

The methods have uniform order of convergence $p = 2m - r$, and are therefore suitable for an efficient variable stepsize implementation. Moreover methods which are A-stable with respect to the basic test equation and have unbounded stability regions with respect to the convolution test equation have been provided.

4.4 Mixed collocation

In the case of VIEs with periodic highly oscillatory solutions, traditional methods may be inefficient, as they may require the use of a small stepsize in order to follow accurately the oscillations of high frequency. As in the case of ODEs "ad hoc" numerical methods have been constructed, incorporating the a priori knowledge of the behaviour of the solution, in order to use wider stepsizes with respect to classical methods and simultaneously to simulate with high accuracy the oscillations of the solution.

A first work on the numerical solution of VIEs with periodic solution is [6], where numerical methods were constructed by means of mixed interpolation. Recently, mixed collocation methods have been introduced in [8, 9] for VIEs and VIDEs. In particular in [8], mixed collocation methods have been introduced for linear convolution VIEs of the form

$$y(x) = g(x) + \int_{-\infty}^{x} k(x - \tau)y(\tau)d\tau, \quad x \in [0, X], \tag{4.23}$$

with

$$y(x) = \psi(x), \quad x \in [-\infty, 0],$$

where $k \in L^1(0, \infty)$, g is a continuous periodic function and ψ is a given bounded and continuous function. The collocation polynomial is taken in the form

$$P_n(x_n + sh) = \sum_{k=0}^{m} B_k(s)Y_{n,k}$$

where the $B_k(s)$ are combinations of trigonometric functions and algebraic polynomials given in [8]. The numerical method is of the form

$$\left\{ \begin{array}{c} Y_{ni} = \bar{F}_{ni} + \bar{\Phi}_{ni} \\ y_{n+1} = \sum_{k=0}^{m} B_k(1)Y_{n,k} \end{array} \right. , \tag{4.24}$$

where the lag-term and increment term approximations are given by

$$\bar{F}_{ni} = g(x_{ni}) + \int_{-\infty}^{0} k(x_{ni} - \tau)\psi(\tau)d\tau + h\sum_{\nu=0}^{n-1}\sum_{l=0}^{m} w_l(1)k(x_{nj} - x_{\nu,l}P_\nu(x_{\nu,l})$$

$$\bar{\Phi}_{ni} = hc_i \sum_{l=0}^{m} w_l(1)k(x_{ni} - x_n - hc_ic_l)\left(\sum_{k=0}^{m} B_k(c_ic_l)Y_{n,k}\right)$$

with

$$w_l(s) = \int_0^s B_l(\tau)d\tau.$$

With some suitable choices for collocation parameters such methods accurately integrates systems for which the period of oscillation of the solution is known. In the paper [16] the authors introduce a family linear methods, namely Direct Quadrature (DQ) methods, specially tuned on the specific feature of the problem, based on the exponential fitting [49, 50], which is extremely flexible when periodic functions are treated. Such formulae are based on a three-term quadrature formula, that is of the same form as the usual Simpson rule, but specially tuned on integrands of the form $k(s)y(s)$ where k and y are of type

$$k(x) = e^{\alpha x}, \qquad y(x) = a + b\cos(\omega x) + c\sin(\omega x), \tag{4.25}$$

where $\alpha, \omega, a, b, c \in I\!R$. The coefficients of the new quadrature rule depend on the parameters of the integrand, i.e. α and ω. It has been shown as the use of exponentially fitted based three-point quadrature rules produces a definite improvement in the accuracy when compared with the results from the classical Simpson rule, and that the magnitude of the gain depends on how good is the knowledge of the true frequencies. The results also indicate that, as a rule, if the input accuracy is up to 10 percent, then the accuracy gain in the output is substantial.

5 Conclusions and future perspectives

In this paper we have described, at the best of our knowledge, some of the collocation methods appeared in the literature for ODEs and VIEs. Some interesting properties of collocation-based methods are, in our opinion, still to be exploited. For instance, the knowledge of the collocation function on the whole interval of integration might allow cheap and realiable error estimators, to be used in a variable stepsize-variable order environment, also for problems with delay. Therefore, although collocation technique is an old idea in Numerical Analysis, we strongly believe that it will constitute building blocks for the development of modern software for an efficient and accurate integration of evolutionary problems.

References

[1] Bartoszewski, Z. and Jackiewicz, Z., Derivation of continuous explicit two-step Runge-Kutta methods of order three, *J. Comput. Appl. Math.* **205** (2007), 764–776.

[2] Bartoszewski, Z. and Jackiewicz, Z., Construction of highly stable parallel two-step Runge-Kutta methods for delay differential equations, *J. Comput. Appl. Math.* **220**, 1-2 (2008), 257–270.

[3] Bartoszewski, Z., Podhaisky, H., and Weiner, R., Construction of stiffly accurate two-step Runge-Kutta methods of order three and their continuous extensions using Nordsieck representation, Report No. 01, Martin-Luther-Universitt Halle-Wittenberg, Fachbereich Mathematik und Informatik (2007).

[4] Bellen, A., Jackiewicz, Z., Vermiglio, R., and Zennaro, M., Stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind, *IMA J. Numer. Anal.* **10**, 1 (1990), 103–118.

[5] Bellen, A., Jackiewicz, Z., Vermiglio, R., and Zennaro, M., Natural continuous extensions of Runge-Kutta methods for Volterra integral equations of the second kind and their applications, *Math. Comp.* **52**, 185 (1989), 49–63.

[6] Bother, P., De Meyer, H., and Vanden Berghe, G., Numerical solution of Volterra equations based on mixed interpolation, *Comput. Math. Appl.* **27** (1994), 1–11.

[7] Brunner, H., *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, 2004.

[8] Brunner, H., Makroglou, A., and Miller, R. K., On mixed collocation methods for Volterra integral equations with periodic solution, *Appl. Numer. Math.* **24** (1997), 115-130.

[9] Brunner, H., Makroglou, A., and Miller, R. K., Mixed interpolation collocation methods for first and second order Volterra integro-differential equations with periodic solution, *Appl. Numer. Math.* **23** (1997), 381–402.

[10] Brunner, H. and van der Houwen, P. J., *The Numerical Solution of Volterra Equations*, CWI Monographs, 3, North-Holland, Amsterdam, 1986.

[11] Butcher, J. C., *The Numerical Analysis of Ordinary Differential Equations. Runge-Kutta and General Linear Methods*, John Wiley & Sons, Chichester, New York, 1987.

[12] Butcher, J. C., *Numerical Methods for Ordinary Differential Equations*, 2nd Edition, John Wiley & Sons, Chichester, 2008.

[13] Butcher, J. C. and Tracogna, S., Order conditions for two-step Runge-Kutta methods, *Appl. Numer. Math.* **24** (1997), 351–364.

[14] Capobianco, G., Conte, D., Del Prete, I., and Russo, E., Fast Runge–Kutta Methods for nonlinear convolution systems of Volterra Integral Equations, *BIT* **47**, 2 (2007), 259–275.

[15] Capobianco, G., Conte, D., Del Prete, I., and Russo, E., Stability analysis of fast numerical methods for Volterra Integral Equations, *Electron. Trans. Numer. Anal.* **30** (2008), 305–322.

[16] Cardone, A., Ixaru, L. Gr., and Paternoster, B., Exponential fitting Direct Quadrature methods for Volterra integral equations, *submitted*.

[17] Chan, R. P. K., Leone, P., and Tsai, A., Order conditions and symmetry for two-step hybrid methods, *Int. J. Comput. Math.* **81**, 12 (2004), 1519–1536.

[18] Coleman, J. P., Rational approximations for the cosine function; P-acceptability and order, *Numer. Algorithms* **3**, 1-4 (1992), 143-158.

[19] Coleman, J. P., Mixed interpolation methods with arbitrary nodes, *J. Comput. Appl. Math.* **92** (1998), 69-83.

[20] Coleman, J. P., Order Conditions for a class of two–step methods for $y'' = f(x, y)$, *IMA J. Numer. Anal.* **23** (2003), 197–220.

[21] Coleman, J. P. and Duxbury, S.C., Mixed collocation methods for $y" = f(x, y)$, *J. Comput. Appl. Math.* **126**, 1-2 (2000), 47–75.

[22] Coleman, J. P.; Ixaru, L.Gr., P-stability and exponential-fitting methods for $y'' = f(x, y)$, *IMA J. Numer. Anal.* **16**, 2 (1996), 179–199.

[23] Cong, Nguyen huu and Mitsui, T., Collocation-based two-step Runge-Kutta methods, *Japan J. Indust. Appl. Math.* **13**, 1 (1996), 171–183.

[24] Conte, D., D'Ambrosio, R., Ferro, M., and Paternoster, B., Piecewise-polynomial approximants for solutions of Functional Equations, in press on *Collana Scientifica di Ateneo, Univ. degli Studi di Salerno.*

[25] Conte, D., D'Ambrosio, R., Ferro, M., and Paternoster, B., Practical construction of Two-Step Collocation Runge–Kutta methods for Ordinary Differential Equations, in press on *Applied and Industrial Mathematics in Italy III*.

[26] Conte, D., D'Ambrosio, R., Ferro, M., and Paternoster, B., Modified Collocation Techniques for Volterra Integral Equations, in press on *Applied and Industrial Mathematics in Italy III*.

[27] Conte, D. and Del Prete, I., Fast collocation methods for Volterra integral equations of convolution type, *J. Comput. Appl. Math.* **196**, 2 (2006), 652–663.

[28] Conte, D., Jackiewicz, Z., and Paternoster, B., Two-step almost collocation methods for Volterra integral equations, *Appl. Math. Comput.* **204** (2008), 839–853.

[29] Conte, D. and Paternoster, B., A Family of Multistep Collocation Methods for Volterra Integral Equations, in *Numerical Analysis and Applied Mathematics* ed. by T. E. Simos, G. Psihoyios, Ch. Tsitouras; *AIP Conference Proceedings* **936**, Springer, 2007, 128–131.

[30] Conte, D. and Paternoster, B., Multistep collocation methods for Volterra Integral Equations, *Appl. Numer. Math.* **59** (2009), 1721–1736.

[31] Crisci, M. R., Russo, E., and Vecchio, A., Stability results for one-step discretized collocation methods in the numerical treatment of Volterra integral equations, *Math. Comp.* **58**, 197 (1992), 119–134.

[32] D'Ambrosio, R., Two–step collocation methods for ordinary differential equations, Ph.D. thesis, University of Salerno, *in preparation*.

[33] D'Ambrosio, R., Ferro, M., and Paternoster, B., A General Family of Two Step Collocation Methods for Ordinary Differential Equations, in *Numerical Analysis and Applied Mathematics* ed. by T. E. Simos, G. Psihoyios, Ch. Tsitouras; *AIP Conference Proceedings* **936**, Springer, 2007, 45–48.

[34] D'Ambrosio, R., Ferro, M., and Paternoster, B., Collocation based two step Runge–Kutta methods for Ordinary Differential Equations, in *ICCSA 2008* ed. by O. Gervasi *et al.*; *Lecture Notes in Comput. Sci.* **5073**, Part II (2008), Springer, New York, 736–751.

[35] D'Ambrosio, R., Ferro, M., and Paternoster, B., Two-step hybrid collocation methods for $y'' = f(x, y)$, *Appl. Math. Lett.* **22** (2009), 1076–1080.

[36] D'Ambrosio, R., Ferro, M., and Paternoster, B., Trigonometrically fitted two-step hybrid methods for second order ordinary differential equations with one or two frequencies, to appear on *Math. Comput. Simulat.*

[37] D'Ambrosio, R., Ferro, M., Jackiewicz, Z., and Paternoster, B., Two-step almost collocation methods for ordinary differential equations, in press on *Numer. Algorithms* (doi: 10.1007/s11075-009-9280-5).

[38] D'Ambrosio, R. and Jackiewicz Z., Continuous Two–Step Runge–Kutta Methods for Ordinary Differential Equations, *submitted*.

[39] D'Ambrosio, R. and Jackiewicz Z., Construction and implementation of highly stable two-step collocation methods, *in preparation*.

[40] De Meyer, H., Vanthournout, J., and Vanden Berghe, G., On a new type of mixed interpolation, *J. Comput. Appl. Math.* **30** (1990), 55-69.

[41] Franco, J. M., A class of explicit two-step hybrid methods for second-order IVPs, *J. Comput. Appl. Math.* **187**, 1 (2006), 41–57.

[42] Gautschi, W., Numerical integration of ordinary differential equations based on trigonometric polynomials, *Numer. Math.* **3** (1961), 381-397.

[43] Guillou, A. and Soulé, F. L., La résolution numérique des problèmes differentiels aux conditions par des méthodes de collocation, *RAIRO Anal. Numér. Ser. Rouge* **R-3** (1969), 17–44.

[44] Hairer, E., Lubich, C., and Wanner, G., *Geometric Numerical Integration – Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics, Springer–Verlag, Berlin, 2000.

[45] Hairer, E., Norsett, S. P., and Wanner, G., *Solving Ordinary Differential Equations I – Nonstiff Problems*, Springer Series in Computational Mathematics 8, Springer–Verlag, Berlin, 2000.

[46] Hairer, E. and Wanner, G., *Solving Ordinary Differential Equations II – Stiff and Differential–Algebraic Problems*, Springer Series in Computational Mathematics 14, Springer–Verlag, Berlin, 2002.

[47] Henrici, P., *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, 1962.

[48] Nguyen, Huu Cong, Note on the performance of direct and indirect Runge–Kutta–Nyström methods, *J. Comput. Appl. Math.* **45**, 3 (1993), 347–355.

[49] Ixaru, L.Gr., Operations on oscillatory functions, *Comput. Phys. Comm.* **105** (1997), 1–19.

[50] Ixaru, L. Gr. and Vanden Berghe, G., *Exponential Fitting*, Kluwer Academic Publishers, Dordrecht, (2004).

[51] Jackiewicz, Z., *General Linear Methods for Ordinary Differential Equations*, in press on John Wiley & Sons.

[52] Jackiewicz, Z., Renaut, R., and Feldstein, A., Two-step Runge-Kutta methods, *SIAM J. Numer. Anal.* **28**, 4 (1991), 1165–1182.

[53] Jackiewicz, Z. and Tracogna, S., A general class of two-step Runge-Kutta methods for ordinary differential equations, *SIAM J. Numer. Anal.* **32** (1995), 1390–1427.

[54] Jackiewicz, Z. and Tracogna, S., Variable stepsize continuous two-step Runge-Kutta methods for ordinary differential equations, *Numer. Algorithms* **12** (1996), 347–368.

[55] Konguetsof, A.; Simos, T. E. An exponentially-fitted and trigonometrically-fitted method for the numerical solution of periodic initial-value problems. Numerical methods in physics, chemistry, and engineering. Comput. Math. Appl. 45 (2003), no. 1-3, 547554.

[56] Kramarz, L., Stability of collocation methods for the numerical solution of $y'' = f(t, y)$, *BIT* **20**, 2 (1980), 215–222.

[57] Lambert, J. D., *Numerical methods for ordinary differential systems: The initial value problem*, John & Wiley, Chichester, 1991.

[58] Lambert, J. D. and Watson, I. A., Symmetric multistep methods for Periodic Initial Value Problems, *J. Inst. Maths. Applics.* **18** (1976), 189–202.

[59] Lie, I., The stability function for multistep collocation methods, *Numer. Math.* **57**, 8 (1990), 779–787.

[60] Lie, I. and Norsett, S. P., Superconvergence for Multistep Collocation, *Math. Comp.* **52**, 185 (1989), 65–79.

[61] Lopez-Fernandez, M., Lubich, C., and Schdle, A., Adaptive, fast, and oblivious convolution in evolution equations with memory, *SIAM J. Sci. Comp.* **30** (2008), 1015–1037.

[62] Lopez-Fernandez, M., Lubich, C., and Schdle, A., Fast and oblivious convolution quadrature, *SIAM J. Sci. Comp.* **28** (2006), 421–438.

[63] Martucci, S. and Paternoster, B., Vandermonde-type matrices in two step collocation methods for special second order Ordinary Differential Equations, in *Computational Science ICCS 2004*, ed. by M.Bubak *et al.*; *Lecture Notes in Comput. Sci.* **3039**, Part IV, Springer Verlag, 2004, 418–425.

[64] Martucci, S. and Paternoster, B., General two step collocation methods for special second order Ordinary Differential Equations, Paper Proceedings of the 17th IMACS World Congress Scientific Computation, Applied Mathematics and Simulation, Paris, July 11-15, 2005.

[65] Norsett, S. P., Collocation and perturbed collocation methods, in *Numerical analysis, Proc. 8th Biennial Conf.*, Univ. Dundee, Dundee, 1979), 119–132; *Lecture Notes in Math.* **773**, Springer, Berlin, 1980.

[66] Norsett, S. P. and Wanner, G., Perturbed collocation and Runge Kutta methods, *Numer. Math.* **38**, 2 (1981), 193–208.

[67] Paternoster, B., Runge-Kutta(-Nyström) methods for ODEs with periodic solutions based on trigonometric polynomials, *Appl. Numer. Math.* **28** (1998), 401-412.

[68] Paternoster, B., A phase–fitted collocation–based Runge–Kutta–Nyström methods, *Appl. Numer. Math.* **35**, 4 (2000), 339–355.

[69] Paternoster, B., General two-step Runge-Kutta methods based on algebraic and trigonometric polynomials, *Int. J. Appl. Math.* **6**, 4 (2001), 347–362.

[70] Paternoster, B., Two step Runge-Kutta-Nystrom methods for y=f(x,y) and P-stability, in *Computational Science ICCS 2002*, ed. by P.M.A.Sloot, C.J.K.Tan, J.J.Dongarra, A.G.Hoekstra; *Lecture Notes in Computer Science* **2331**, Part III, 459-466, Springer Verlag, Amsterdam, 2002.

[71] Paternoster, B., Two step Runge-Kutta-Nystrom methods for oscillatory problems based on mixed polynomials, *Computational Science–ICCS 2003*, ed. by P.M.A.Sloot, D.Abramson, A.V.Bogdanov, J.J.Dongarra, A.Y.Zomaya, Y.E.Gorbachev; *Lecture Notes in Computer Science* **2658**, Part II, 131-138, Springer, Berlin Heidelberg, (2003).

[72] Paternoster, B., Two step Runge-Kutta-Nystrom methods based on algebraic polynomials, *Rendiconti di Matematica e sue Applicazioni* **23**, Serie VII (2003), 277–288.

[73] Paternoster, B., Two step Runge-Kutta-Nystrom methods for oscillatory problems based on mixed polynomials, in *Computational Science – ICCS 2003* ed. by P.M.A.Sloot, D.Abramson, A.V.Bogdanov, J.J.Dongarra, A.Y.Zomaya, Y.E.Gorbachev; *Lecture Notes in Computer Science* **2658**, Part II, 131–138, Springer, Berlin Heidelberg, 2003.

[74] B.Paternoster, A general family of two step Runge-Kutta-Nyström methods for $y'' = f(x, y)$ based on algebraic polynomials, *Computational Science – ICCS 2003* ed. by V.N.Alexandrov, G.D. van Albada, P.M.A.Sloot, J.J.Dongarra; *Lecture Notes in Computer Science* **3994** Part IV, 700-.707, Springer Verlag, 2006.

[75] Petzold, L. R., Jay, L. O., and Yen, J., Numerical solution of highly oscillatory ordinary differential equations, *Acta Numer.* **6** (1997), 437483.

[76] Simos, T. E., Dissipative trigonometrically-fitted methods for linear second-order IVPs with oscillating solution, *Appl. Math. Lett.* **17**, 5 (2004), 601–607.

[77] Van den Houwen, P. J., Sommeijer, B. P., and Nguyen, Huu Cong, Stability of collocation-based Runge–Kutta–Nyström methods, *BIT* **31**, 3 (1991), 469–481.

[78] Van de Vyver, H., Phase-fitted and amplification-fitted two-step hybrid methods for $y'' = f(x, y)$, *J. Comput. Appl. Math.* **209**, 1 (2007), 33–53.

[79] Van de Vyver, H., A phase-fitted and amplification-fitted explicit two-step hybrid method for second-order periodic initial value problems, *Internat. J. Modern Phys.* **C 17**, 5 (2006), 663–675.

[80] Wright, K., Some relationships between implicit RungeKutta, collocation and Lanczos τ-methods, and their stability properties, *BIT* **10** (1970), 217–227.