Two-step Runge-Kutta methods for stochastic differential equations

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Abstract

We introduce a theory of two-step Runge-Kutta (TSRK) methods for stochastic differential equations, arising from the perturbation of the corresponding TSRK methods for deterministic problems. We present a proof of convergence and study the mean-square stability properties. Numerical experiments confirming the theoretical results are provided.

Keywords: Stochastic differential equations, stochastic two-step Runge-Kutta methods, mean-square stability analysis.
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1. Introduction

Numerics for stochastic differential equations (SDEs) (see [20, 21, 29]) has attracted the interest of many researchers, because of the great number of applications in biology, chemistry, epidemiology, economics and finance. In particular, we follow here the idea of building the stochastic analogue of a certain numerical method for ordinary differential equations (ODEs), following the lines drawn by several papers dedicated to stochastic multistep [2, 4, 8, 30, 33] and Runge Kutta methods [5–7, 9–11, 16, 31, 32].

The specific aim of this paper is to introduce and analyze the stochastic analogue of two-step Runge-Kutta (TSRK) methods for deterministic ODEs, introduced by Jackiewicz et al. in [25, 26, 28] (also see [24] and references therein) with purpose to heighten the usual accuracy and stability barriers of classical Runge-Kutta methods. For a given Hadamard well-posed Cauchy problem

$$\begin{cases} y' = f(y), & x \in [0, T] \\ y(0) = y_0 \end{cases}$$

and with respect to the uniform grid

$$\mathcal{I}_h = \{0 = t_0 < t_1 < t_2 < \cdots < t_N = T, \quad N = T/h\}, \tag{1}$$

TSRK method takes the form

$$
\begin{aligned}
y_{i+1} &= (1-\theta)y_i + \theta y_{i-1} + h\sum_{j=1}^{m}(v_j f(Y_{i-1}^j) + w_j f(Y_{i-1}^j)),\\
Y_{i-1}^j &= y_{i-1} + h\sum_{s=1}^{m} a_{js} f(Y_{i-1}^s), \quad j = 1,...,m,\\
Y_i^j &= y_i + h\sum_{s=1}^{m} a_{js} f(Y_i^s), \quad j = 1,...,m,
\end{aligned}
\tag{2}
$$

for $i = 1, 2, ..., N-1$. y_i approximates the solution $y(x_i)$ and θ, v_j, w_j and a_{js} are the coefficients, which characterize the method. These methods represent a middle ground between Runge-Kutta and two-step methods and provide our building blocks for analog methods for SDEs, as described in the remainder. The paper is organized as follows: in Section 2 we present the structure of the method and the study of the convergence. In Section 3, we provide a study of mean-square stability and Section 4 is dedicated to numerical experiments. Some conclusions are given in Section 5.

2. Method formulation and convergence analysis

We consider the Ito scalar stochastic differential equation with multiplicative noise

$$
\mathrm{d}x(t) = a(t, x(t))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}W(t), \tag{3}
$$

for $t \in [0, T]$, with initial conditions $x(0) = x_0$. The functions $a, \sigma : \mathbb{R} \to \mathbb{R}$ are supposed smooth enough to guarantee existence and uniqueness to the solution of (3) (see [20] and references therein). In correspondence of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $W\colon [0, T]\times\Omega \to \mathbb{R}$ is a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths on $(\Omega, \mathcal{F}, \mathbb{P})$.

With reference to the grid (1), given $\overline{x}_0 = x_0$ and computed the missing starting value $\overline{x}_1$ by a suitable one-step method, inspired by the notation introduced in [31], we design explicit stochastic TSRK method of the following form

$$
\begin{aligned}
\overline{x}_{i+1} &= (1-\theta)\overline{x}_i + \theta\overline{x}_{i-1} + h\sum_{j=0}^{m}\left(p_j K_j^i + r_j K_j^{i-1}\right)\\
&\quad + \Delta W_i \sum_{j=0}^{m} q_j G_j^i + \Delta W_{i-1}\sum_{j=0}^{m} s_j G_j^{i-1},
\end{aligned}
\tag{4}
$$

where

$$\begin{aligned}
K_0^i &= a(t_i + \alpha_0 h, \overline{x}_i), \quad G_0^i = \sigma(t_i + \alpha_0 h, \overline{x}_i), \\
x_i^{(1)} &= \overline{x}_i + \beta_{10} K_0^i h + \gamma_{10} G_0 \Delta W_i \\
K_1^i &= a(t_i + \alpha_0 h, x_i^{(1)}), \quad G_0^i = \sigma(t_i + \alpha_0 h, x_i^{(1)}), \\
&\vdots \\
x_i^{(m)} &= \overline{x}_i + \sum_{k=0}^{m-1} \beta_{mk} K_k h + \sum_{k=0}^{m-1} \gamma_{mk} G_k \Delta W_i, \\
K_m^i &= a(t_i + \alpha_m h, x_i^{(m)}), \quad G_m = \sigma(t_i + \alpha_m h, x_i^{(m)}),
\end{aligned}$$

for $i = 3, ..., N$. The coefficients of the method are then collected in the following Butcher tableau

$$\begin{array}{c|c|c} \alpha & B & \Gamma \\ \hline \theta & p^\intercal & q^\intercal \\ \hline & r^\intercal & s^\intercal \end{array} = \begin{array}{c|cccc|cccc}
\alpha_0 & \beta_{10} & & & & \gamma_{10} & & & \\
\alpha_1 & \beta_{20} & \beta_{20} & & & \gamma_{20} & \gamma_{21} & & \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\
\alpha_{m-1} & \beta_{m0} & \beta_{m1} & \cdots & \beta_{m,m-1} & \gamma_{m0} & \gamma_{m1} & \cdots & \gamma_{m,m-1} \\ \hline
\theta & p_1 & p_2 & \cdots & p_m & q_1 & q_2 & \cdots & q_m \\ \hline
 & r_1 & r_2 & \cdots & r_m & s_1 & s_2 & \cdots & s_m
\end{array} .$$

In order to guarantee the convergence of the underlying deterministic TSRK method (see [25]), we set

$$-1 < \theta \leq 1, \quad \text{and} \quad \sum_{j=0}^{m} (p_j + r_j) = 1 + \theta. \tag{5}$$

The analysis of the mean-square convergence for the stochastic method (4) is presented in the following result.

Theorem 2.1. *Consider the scalar Ito SDE* (3) *and suppose that the functions*

$$a, \sigma, \frac{\partial a}{\partial x}, \frac{\partial a}{\partial t}, \frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial t}, \frac{\partial^2 \sigma}{\partial x^2}, \frac{\partial^2 \sigma}{\partial t^2}, \frac{\partial^2 \sigma}{\partial t \partial x}$$

are bounded. Then, the approximation $\overline{x}_t$, $t \in [0, T]$ given by the TSRK method (4)- (5) *converges in mean-square sense to the solution y_t of the equation*

$$dy = \left[a(t, y) + \lambda \frac{\partial \sigma}{\partial x}(t, y) \sigma(t, y) \right] dt + \sigma(t, y) dw \tag{6}$$

where

$$\lambda = \sum_{j=1}^{m} q_j \sum_{k=0}^{j-1} \gamma_{jk}, \quad m \geq 1 \tag{7}$$

The presentation of the proof to this result benefits of the following remarks.

Remark 2.1. *Since*

$$\begin{aligned}\overline{x}_{i+1} - \overline{x}_i = \theta(\overline{x}_{i-1} - \overline{x}_i) + h\sum_{j=0}^{m}\left(p_j K_j^i + r_j K_j^{i-1}\right)\\ + \Delta W_i \sum_{j=0}^{m} q_j G_j^i + \Delta W_{i-1}\sum_{j=0}^{m} s_j G_j^{i-1},\end{aligned}$$

under the hypothesis of boundedness in the statement of Theorem (2.1) *and since* $\theta < 1$, *we can say that*

$$\begin{aligned}|\overline{x}_{i+1} - \overline{x}_i| &\le \theta|\overline{x}_{i-1} - \overline{x}_i| + C_1 h + C_2\Delta W_i + C_3\Delta W_{i-1}\\ &< \cdots < D_1 h + \sum_{k=0}^{i} C_k \Delta W_k,\end{aligned} \tag{8}$$

with $D_1, C_0, \ldots, C_i \in \mathbb{R}$, *supposing that the missing value* $\overline{x}_1$ *is computed by a starting method satisfying* $|\overline{x}_1 - \overline{x}_0| = O(h)$. *As consequence, we get*

$$\mathbb{E}|\overline{x}_{i+1} - \overline{x}_i| = D_1 h.$$

Remark 2.2. *Following the approach of [31] for proving the convergence of stochastic explicit Runge-Kutta methods, we consider the two-step Maruyama method*

$$\begin{aligned}y_{i+1} =&(1-\theta)y_i + \theta y_{i-1} + h\beta_0(\overline{a}_{i-1} + \lambda\frac{\partial\overline{\sigma}_{i-1}}{\partial x}\overline{\sigma}_{i-1}) + h\beta_1(\overline{a}_i + \lambda\frac{\partial\overline{\sigma}_i}{\partial x}\overline{\sigma}_i)\\ &+ \theta\overline{\sigma}_{i-1}\Delta W_{i-1} + \overline{\sigma}_i dw_i,\end{aligned} \tag{9}$$

where $\overline{a}_i = a(t_i, \overline{y}_i)$, $\overline{\sigma}_i = \sigma(t_i, \overline{y}_i)$, $\partial\overline{\sigma}_i/\partial x = \partial\sigma/\partial x(t_i, \overline{y}_i)$, $\sum_{j=0}^{m} p_j = \beta_0$, $\sum_{j=0}^{m} r_j = \beta_1$. *With this choice of the coefficients, the method* (9) *is convergent, see [30]. By the triangle inequality and Hölder continuity, it is sufficient to prove that*

$$\max_i \mathbb{E}(\overline{x}_i - y_i)^2 \longrightarrow 0, \quad \textit{for} \;\; h \longrightarrow 0$$

Remark 2.3. *As noted also in [31], the hypotesis of boundedness is not too strong in computation.*

We are now ready to prove the result.

Proof 2.1. *Setting* $a_i = a(t_i, \overline{x}_i)$, $\sigma_i = \sigma(t_i, \overline{x}_i)$ *and* $\overline{\Delta W_i} = |\Delta W_i|$, *we consider the Ito-Taylor expansions of* $K_j h$ *and* $G_l \Delta W$, *we have*

$$K_0 h = a_i h + \frac{\partial a}{\partial t}(\xi_0)\alpha_0 h^2 = a_i h + O(h^2),$$

$$G_0\Delta W_i = \sigma_i\Delta W_i + \frac{\partial\sigma}{\partial t}(\eta_0)\alpha_0 h\Delta W_i = \sigma\Delta W_i + O(h\overline{\Delta W_i}),$$

with $\xi_0, \eta_0\ [t_i, t_i + \alpha_0 h]$. *Then,*

$$\begin{aligned} x_i^{(1)} - \overline{x}_i &= \beta_{10}(a_i h + O(h^2)) + \gamma_{10}(\sigma_i \Delta W_i + O(h\overline{\Delta W_i}) \\ &= \beta_{10} a_i h + \gamma_{10}\sigma_i \Delta W_i + O(h\overline{\Delta W_i}) + O(h^2)), \end{aligned}$$

$$\begin{aligned} K_1^i\, h &= a_i h + \frac{\partial a}{\partial t}(\xi_1)\alpha_1 h^2 + \frac{\partial a}{\partial x}(\xi_1) h (x_i^{(1)} - \overline{x}_i) \\ &= a_i h + O(h\overline{\Delta W_i}) + O(h^2)), \end{aligned}$$

and

$$\begin{aligned} G_1 \Delta W_i &= \sigma_i \Delta W_i + \frac{\partial \sigma_i}{\partial t}\alpha_1 h \Delta W_i + \frac{\partial \sigma_i}{\partial x}\Delta W_i (x_i^{(1)} - \overline{x}_i) + \frac{1}{2}\frac{\partial^2 \sigma}{\partial t^2}(\eta_1)(\alpha_1 h)^2 \Delta W_i \\ &+ \frac{\partial^2 \sigma}{\partial t \partial x}(\eta_1)\alpha_1 h \Delta W_i (x_i^{(1)} - \overline{x}_i) + \frac{1}{2}\frac{\partial^2 \sigma}{\partial^2 x}(\eta_1)\Delta W_i (x_i^{(1)} - \overline{x}_i)^2 \\ &= \sigma_i \Delta W_i + \gamma_{10}\frac{\partial \sigma_i}{\partial x}\sigma_i (\Delta W_i)^2 + O(h\overline{\Delta W_i}) + O(\overline{\Delta W_i}^3). \end{aligned}$$

At the step $j \geq 1$*, we have*

$$x_i^{(j)} - \overline{x}_i = \sum_{k=0}^{j-1} \beta_{jk} a_i h + \sum_{k=0}^{j-1} \gamma_{jk}\sigma_i \Delta W_i + O(\Delta W_i^2) + O(h\overline{\Delta W_i}) + O(\overline{\Delta W_i}^3)$$

and

$$K_j h = a_i h + O(h\overline{\Delta W_i}) + O(h^2),$$

$$G_j \Delta W_i = \sigma_i \Delta W_i + \sum_{k=0}^{j-1} \gamma_{jk}\frac{\partial \sigma_i}{\partial x}\sigma_i (\Delta W_i)^2 + O(h\overline{\Delta W_i}) + O(\overline{\Delta W_i}^3).$$

Moreover,

$$\begin{aligned} x_{i+1} &= (1-\theta)x_i + \theta x_{i-1} + h\sum_{j=0}^{m} p_j a_i + h\sum_{j=0}^{m} r_j a_{i-1} + \Delta W_i \sum_{j=0}^{m} q_j \sigma_i \\ &+ \sum_{j=1}^{m} q_j \sum_{k=0}^{j-1} y_{jk}\frac{\partial \sigma_i}{\partial x}\sigma_i \Delta W_i^2 + \sum_{j=1}^{m} q_j \sigma_{i-1}\Delta W_{i-1} \\ &+ \sum_{j=1}^{m} q_j \sum_{k=0}^{j-1} y_{jk}\frac{\partial \sigma_{i-1}}{\partial x}\sigma_{i-1}\Delta W_{i-1}^2 \\ &+ O(h\overline{\Delta W_i}) + O(h\overline{\Delta W_{i-1}}) + O(h^2) + O(\overline{\Delta W_i}^3) + O(\overline{\Delta W_{i-1}}^3). \end{aligned}$$

Side-by-side subtraction with (9) *yields*

$$
\begin{aligned}
x_{i+1} - y_{i+1} &= x_i - y_i + \theta(x_i - x_{i-1} + y_i - y_{i-1}) + h\beta_1(a_i - \overline{a}_i) \\
&\quad + h\beta_0(a_{i-1} - \overline{a}_{i-1}) + \lambda\frac{\partial\sigma_i}{\partial x}\sigma_i(\Delta W_i^2 - h\beta_1) \\
&\quad + \lambda\beta_1 h\left(\frac{\partial\sigma_i}{\partial x}\sigma_i - \frac{\partial\overline{\sigma}_i}{\partial x}\overline{\sigma}_i\right) + \frac{\partial\sigma_i}{\partial x}\sigma_i(\mu\Delta W_{i-1}^2 - \lambda h\beta_0) \\
&\quad + \lambda\beta_0 h\left(\frac{\partial\sigma_{i-1}}{\partial x}\sigma_{i-1} - \frac{\partial\overline{\sigma}_{i-1}}{\partial x}\overline{\sigma}_{i-1}\right) + \left(\sum_{j=0}^{m} q_j\sigma_i - \overline{\sigma}_i\right)\Delta W_i \\
&\quad + \left(\sum_{j=0}^{m} r_j\sigma_{i-1} - \theta\overline{\sigma}_{i-1}\right)\Delta W_{i-1} + O(h\overline{\Delta W_i}) + O(h\overline{\Delta W_{i-1}}) + O(h^2) \\
&\quad + O(\overline{\Delta W_{i-1}}^3) + O(\overline{\Delta W_i}^3).
\end{aligned}
$$

Thanks to the boundedness condition, and exploiting (8)*, we get from* (10)

$$
\begin{aligned}
|x_{i+1} - y_{i+1}| &< |x_i - y_i| + Ch + \sum_{k=0}^{i} C_k\Delta W_k + F\Delta W_i^2 + O(h\overline{\Delta W_i}) \\
&\quad + O(h\overline{\Delta W_{i-1}}) + O(h^2) + O(\overline{\Delta W_i}^3) + O(\overline{\Delta W_{i-1}}^3).
\end{aligned}
\tag{10}
$$

Squaring (10) *and exploiting the inequality*

$$
(a_1 + a_2 + ... + a_n)^2 \leq n(a_1^2 + a_2^2 + ... + a_n^2), \quad \forall a_1, a_2, ..., a_n \in \mathbb{R},
$$

we get

$$
\begin{aligned}
|x_{i+1} - y_{i+1}|^2 &< |x_i - y_i|^2 + C^2h^2 + \left(\sum_{k=0}^{i} C_k\Delta W_k\right)^2 + F^2\Delta W_i^4 O(h^2\overline{\Delta W_i}^2) \\
&\quad + O(h^2\overline{\Delta W_{i-1}}^2) + O(h^4) + O(\overline{\Delta W_i^6}) + O(\overline{\Delta W_{i-1}^6}) \\
&< |x_i - y_i|^2 + C^2h^2 + (i+1)\left(\sum_{k=0}^{i} C_k^2\Delta W_k^2\right) + F^2\Delta W_i^4 \\
&\quad + O(h^2\overline{\Delta W_i}^2) + O(h^2\overline{\Delta W_{i-1}}^2) + O(h^4) + O(\overline{\Delta W_i}^6) + O(\overline{\Delta W_{i-1}}^6),
\end{aligned}
$$

where $\mu = \sum_{j=1}^{m} s_j \sum_{k=0}^{j-1} y_{jk}$*. Taking the expected value, we get*

$$
\mathbb{E}|x_{i+1} - y_{i+1}|^2 < \mathbb{E}|x_i - y_i|^2 + Ch^2 + (i+1)h\sum_{k=0}^{i} C_k^2 + F^2 3h^2 + O(h^3) + O(h^4).
$$

3. Mean-square stability analysis

In this section, we provide a study of the mean-square stability properties of method (4). Let us consider the scalar test equation [22, 23]

$$
\mathrm{d}x = \lambda x\mathrm{d}t + \mu x\mathrm{d}W(t) \tag{11}
$$

and suppose that it is mean-square stable, i.e.

$$\lim_{t\longrightarrow\infty} \mathbb{E}|x^2(t)| = 0 \Longleftrightarrow \mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0. \tag{12}$$

We aim to provide conditions on the stepsize h, such that the numerical solution given by the TSRK method (4) reproduces numerically the property (12), i.e.

$$\lim_{n\longrightarrow\infty} \mathbb{E}|x_n^2| = 0.$$

Let us denote by X_i the vector of the stages at the i-th step

$$X_i = \left[x_i^{(0)} \;\; x_i^{(1)} \;\; \dots \;\; x_i^{(m)}\right]^{\intercal}.$$

Applying our method to (11), we get

$$X_i = x_i e + \alpha B X_i + \eta_i \Gamma X_i, \tag{13}$$

where

$$B = (\beta_{ij}), \quad \Gamma = (\gamma_{ij}),$$

e the unit $n-$dimensional vector, $\alpha = h\lambda$ and $\eta_i = \mu\Delta W_i$. As a consequence,

$$X_i = (\mathbb{I} - \alpha B - \eta_i\Gamma)^{-1}\, x_i e \tag{14}$$

and

$$x_{i+2} = (1-\theta)x_{i+1} + \theta x_i + \alpha\left(p^T X_{i+1} + r^T X_i\right) + \eta_{i+1} q^T X_{i+1} + \eta_i s^T X_i. \tag{15}$$

Setting

$$\Lambda_i = (\mathbb{I} - \alpha B - \eta_i\Gamma)^{-1}$$

we get the recurrence relation

$$x_{i+2} = A_{i+1} x_{i+1} + C_i x_i \tag{16}$$

where

$$A_{i+1} = (1-\theta) + \left(\alpha p^T + \eta_{i+1} q^T\right)\Lambda_{i+1}\underline{e},$$
$$C_{i+1} = \theta + \left(\alpha r^T + \eta_i s^T\right)\Lambda_i e.$$

Squaring (16) and taking the expected value yields

$$\mathbb{E}|x_{i+2}^2| = \mathbb{E}|A_{i+1}^2|\mathbb{E}|x_{i+1}^2| + \mathbb{E}|C_i^2|\mathbb{E}|x_i^2| + 2\mathbb{E}|A_{i+1}|\mathbb{E}|C_i\; x_i\; x_{i+1}| \tag{17}$$

We observe that

$$\begin{aligned}\mathbb{E}|C_i\; x_i\; x_{i+1}| &= \mathbb{E}|C_i A_i x_i^2| + \mathbb{E}|C_i C_{i-1}\; x_i\; x_{i-1}| \\ &= \mathbb{E}|C_i A_i|\mathbb{E}|x_i^2| + \mathbb{E}|C_i|\mathbb{E}|C_{i-1}\; x_i\; x_{i-1}|\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathbb{E}|C_i|\mathbb{E}|x_i\; x_{i+1}| &= \mathbb{E}|C_i|\mathbb{E}|A_i x_i^2| + \mathbb{E}|C_i|\mathbb{E}|C_{i-1}\; x_i\; x_{i-1}| \\ &= \mathbb{E}|C_i|\mathbb{E}|A_i|\mathbb{E}|x_i^2| + \mathbb{E}|C_i|\mathbb{E}|C_{i-1}\; x_i\; x_{i-1}|.\end{aligned} \tag{19}$$

Thanks to (18) and (19), (17) becomes

$$\begin{aligned}
\mathbb{E}|x_{i+2}^2| =&\mathbb{E}|A_{i+1}^2|\mathbb{E}|x_{i+1}^2| + \mathbb{E}|C_i^2|\mathbb{E}|x_i^2| + 2\mathbb{E}|A_{i+1}|\mathbb{E}|C_i|\mathbb{E}|x_i\ x_{i+1}| \\
&+ 2\mathbb{E}|A_{i+1}|\left(\mathbb{E}|C_iA_i|\mathbb{E}|x_i^2| - \mathbb{E}|C_i|\mathbb{E}|A_i|\mathbb{E}|x_i^2|\right) \\
=&\mathbb{E}|A_{i+1}^2|\mathbb{E}|x_{i+1}^2| + \Big[\mathbb{E}|C_i^2| + 2\mathbb{E}|A_{i+1}|\cdot \\
&(\mathbb{E}|C_iA_i| - \mathbb{E}|C_i|\mathbb{E}|A_i|)\Big]\mathbb{E}|x_i^2| + 2\mathbb{E}|A_{i+1}|\mathbb{E}|C_i|\mathbb{E}|x_i\ x_{i+1}|
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\mathbb{E}|x_{i+2}\ x_{i+1}| &= \mathbb{E}|A_{i+1}x_{i+1}^2| + \mathbb{E}|C_ix_ix_{i+1}| \\
&= \mathbb{E}|A_{i+1}|\mathbb{E}|x_{i+1}^2| + (\mathbb{E}|A_iC_i| - \mathbb{E}|A_i|\mathbb{E}|C_i|)\,\mathbb{E}|x_i^2| + \mathbb{E}|C_i|\mathbb{E}|x_ix_{i+1}|.
\end{aligned} \tag{21}$$

Thanks to (20) and (21), we get

$$\begin{bmatrix} \mathbb{E}|x_{i+2}^2| \\ \mathbb{E}|x_{i+2}\ x_{i+1}| \\ \mathbb{E}|x_{i+1}^2| \end{bmatrix} = M \begin{bmatrix} \mathbb{E}|x_{i+1}^2| \\ \mathbb{E}|x_i\ x_{i+1}| \\ \mathbb{E}|x_i^2| \end{bmatrix},$$

where the stability matrix M is given by

$$M = \begin{bmatrix} \mathbb{E}|A_{i+1}^2| & 2\mathbb{E}|A_{i+1}|\mathbb{E}|C_i| & \mathbb{E}|C_i^2| + 2\mathbb{E}|A_{i+1}|\,(\mathbb{E}|C_iA_i| - \mathbb{E}|C_i|\mathbb{E}|A_i|) \\ \mathbb{E}|A_{i+1}| & \mathbb{E}|A_iC_i| - \mathbb{E}|A_i|\mathbb{E}|C_i| & \mathbb{E}|C_i| \\ 1 & 0 & 0 \end{bmatrix}. \tag{22}$$

Remark 3.1. *For any method of the form* (4)*, it is always possible to have an explicit form of M as function of h. Therefore, for any stepsize h, it is always possible to establish if the method is mean-square stable, checking if*

$$\rho(M) < 1. \tag{23}$$

In the following sections, we study the mean square stability of two classes of methods.

3.1. Two-stage methods

We consider a general two stage method of the form (4) (i.e. $m = 2$), characterized by the matrices

$$B = \begin{bmatrix} 0 & 0 \\ b_1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ g_1 & 0 \end{bmatrix}$$

and by the vectors of coefficients $p = [p_1\ \ p_2]^\intercal$, $r = [r_1\ \ r_2]^\intercal$, $q = [q_1\ \ q_2]^\intercal$ and $s = [s_1\ \ s_2]^\intercal$. In the remainder, we set $\alpha = h\lambda$, $\gamma = h\mu^2$ and

$$u = [1 \quad \alpha b_1 + 1]\,, \quad \eta = \theta + \alpha\ r \cdot u, \quad \xi = \alpha\ p \cdot u + 1 - \theta,$$

$$\psi = s \cdot u + \alpha g_1 r_2, \quad \chi = q \cdot u + \alpha\ g_1 p_2, \quad \varsigma = g_1 q_2,$$

$$\upsilon = g_1 s_2, \quad \kappa = \left(3\upsilon^2 + 2\upsilon\varsigma\xi + 2\upsilon^2\varsigma^2\right)\gamma^2 + \left(\psi^2 + 2\sigma\eta + 2\xi\psi\chi + 2\varsigma\psi\chi\right)\gamma + \eta^2.$$

Then, the corresponding stability matrix has the following form

$$M = \begin{bmatrix} \varsigma^2\gamma^2 + \left(\chi^2 + 2\varsigma\xi\right)\gamma + \xi^2 & 2(\upsilon\varsigma\gamma^2 + (\varsigma\eta + \upsilon\xi)\gamma + \xi\eta) & \kappa \\ \varsigma\gamma + \xi & 2\upsilon\varsigma\gamma^2 + \psi\chi\gamma & \upsilon\gamma + \eta \\ 1 & 0 & 0 \end{bmatrix}.$$

3.2. Three-stage methods

Let us focus on the general class threestage methods (i.e. $m = 3$), characterized by the matrices

$$B = \begin{bmatrix} 0 & 0 & 0 \\ b_1 & 0 & 0 \\ 0 & b_2 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 \\ g_1 & 0 & 0 \\ 0 & g_2 & 0 \end{bmatrix}$$

and the vectors of coefficients $p = [p_1 \ \ p_2 \ \ p_3]$, $r = [r_1 \ \ r_2 \ \ r_3]$, $q = [q_1 \ \ q_2 \ \ q_3]$ and $s = [s_1 \ \ s_2 \ \ s_3]$. We set

$$v = [g_1 \quad g_2 + \alpha(b_1 g_2 + b_2 g_1) \quad \alpha g_1 g_2] \quad u = \left[1 \quad \alpha b_1 + 1 \quad b_1 b_2 \alpha^2 + b_2\alpha + 1\right],$$

$$l = [g_1 \quad g_2 + \alpha(b_1 g_2 + b_2 g_1)], \quad \eta = \alpha p \cdot u + 1 - \theta, \quad \sigma = [q_2 \ q_3 \ p_3] \cdot v,$$

$$\phi = [s_2 \ s_3 \ r_3] \cdot v, \quad \chi = \alpha\, r \cdot u + \theta, \quad \kappa = s \cdot u + \alpha[r_2 \ r_3] \cdot l,$$

$$\nu = s \cdot v + \alpha[r_2 \ r_3] \cdot l, \quad \zeta = g_1 g_2 s_3, \quad \xi = g_1 g_2 q_3, \quad \delta = q \cdot u + \alpha[p_2 \ p_3] \cdot l.$$

The entries of the stability matrix are then given by

$$\begin{aligned} M_{11} &= 15\xi^2\gamma^3 + 3(\sigma^2 + 2\xi\delta)\gamma^2 + (2\sigma\eta + \delta)\gamma + \eta^2, \\ M_{12} &= 2(-\theta^2\sigma\phi\gamma^2 + (\theta\phi\eta + \theta^2\phi + \theta^2\sigma - \theta\chi)\gamma + \eta\chi - \eta\theta + \theta\chi - \theta^2), \\ M_{13} &= 15\zeta^2\gamma^3 + (\psi^2 + 2\zeta\kappa)\gamma^2 + \nu^2\gamma + 2\psi\chi + 2(\eta + \theta - \theta\sigma\gamma)M_{22}, \\ M_{21} &= \eta + \theta - \theta\sigma\gamma, \\ M_{22} &= 15\xi\gamma^3 + 3(\sigma\phi + \zeta\delta + \xi\kappa)\gamma^2 + (\phi\eta + \kappa\delta + \sigma\chi - \theta\phi\eta - \theta^2\phi - \theta^2\sigma + \theta\chi)\gamma \\ &\quad - \eta\chi + \eta\theta - \theta\chi + \theta^2\chi\eta, \\ M_{23} &= \chi - \theta + \theta\phi\gamma. \end{aligned}$$

4. Numerical Experiments

In this section we present some numerical experiments confirming the theoretical expectations in terms of convergence and stability properties.

4.1. Numerical evidence for two-stage methods

We first construct an example of two-stage method starting from the second order Heun method, represented by the following Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}.$$

h	err
0.5	0.3819
0.25	0.0096
0.1250	0.0018
0.0625	7.3934×10^{-4}
0.0313	2.8874×10^{-4}
0.0156	1.9080×10^{-4}
0.0078	1.1539×10^{-4}

Table 1: Mean-square error at the endpoint $T = 1$, obtained by method (24) for different values of the stepsize h.

We set

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Gamma = B$$

and choose

$$r = \frac{11}{16}[1 \quad 1]^{\intercal}, \quad p = \frac{1}{16}[1 \quad 1]^{\intercal}, \quad q = r, \quad s = p, \quad \theta = 1/2.$$

The corresponding Butcher tableau is given by

$$\begin{array}{c|c|c} \alpha & B & \Gamma \\ \hline \theta & p^{\intercal} & q^{\intercal} \\ \hline & r^{\intercal} & s^{\intercal} \end{array} = \begin{array}{c|cc|cc} 0 & & & & \\ 1 & 1 & & 1 & \\ \hline \frac{1}{2} & \frac{1}{16} & \frac{1}{16} & \frac{11}{16} & \frac{11}{16} \\ \hline & \frac{11}{16} & \frac{11}{16} & \frac{1}{16} & \frac{1}{16} \end{array}. \tag{24}$$

The underlying deterministic TSRK has (at least) order one, since (5) is satisfied. To check the properties of this method, we consider the linear equation (11), with $\lambda = -3$ and $\mu = 1/2$, and plot with a solid magenta line the solution

$$x(t) = x_0 \exp\left(\left(\eta - \frac{1}{2}\mu^2\right) t + \mu W(t)\right), \tag{25}$$

where $x_0 = 1$ and $\eta = \lambda + \kappa\mu^2$, with κ computed according to formula (7).

According to Theorem 2.1 , the constructed method should converge to the solution of the equation

$$\mathrm{d}x = \eta x \mathrm{d}t + \mu x \mathrm{d}W(t), \tag{26}$$

which is given by (25). We choose various values of the stepsize and integrate the equation in the interval $[0, 1]$. Correspondingly, Table 1 shows the decay of the mean-square error at $T = 1$, computed over 1000 paths, confirming the mean-square convergence of the method.

We are able to express the stability matrix (22) as function of h. In Figure 1, we plot the spectral radius of M as function of h. In Figures 2, we represent $\mathbb{E}|X_n|^2$ for two different values of the stepsize h, i.e. $h = 0.5$ (top of the figure) and $h = 0.9$ (bottom of the figure). Since for a given h, we expect that the method is mean-square stable if the spectral radius of M is less than 1, the graphs in Figure 2, perfectly agree with such condition. In fact, only the solution on the right is stable.

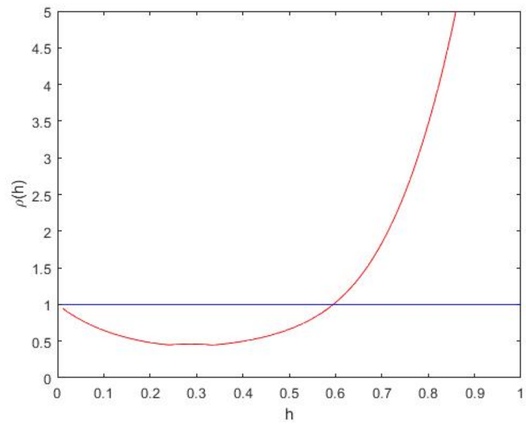

Figure 1: Behaviour of the spectral radius of the stability matrix of the method (24), as function of h.

4.2. Numerical evidence for three-stage methods

We start from the following third order Heun method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

and choose

$$r = \frac{1}{24}[1 \quad 1 \quad 1]^{\mathsf{T}}, \quad p = \frac{5}{32}[1 \quad 0 \quad 3]^{\mathsf{T}}, \quad q = r, \quad s = p, \quad \theta = 1/2.$$

Also in this case, the underlying TSRK is convergent. The corresponding Butcher tableau is given by

$$\begin{array}{c|c|c} \alpha & B & \Gamma \\ \hline \theta & p^{\mathsf{T}} & q^{\mathsf{T}} \\ \hline & r^{\mathsf{T}} & s^{\mathsf{T}} \end{array} = \begin{array}{c|ccc|ccc} 0 & & & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \frac{1}{3} & & \\ \frac{2}{3} & & \frac{2}{3} & & & \frac{2}{3} & \\ \hline \frac{1}{2} & \frac{5}{32} & \frac{5}{32} & \frac{5}{32} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\ \hline & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{5}{32} & \frac{5}{32} & \frac{5}{32} \end{array}. \tag{27}$$

Similarly to Section 4.1, the reduction of the error according to the stepsize is highlighted in Table 2. Figure 3 shows the behaviour of the spectral radius of the stability matrix of method (27). According to our analysis, in Figure 4, it is clear that taking the stepsize $h = 0.313$ (top of the picture) we have mean-square stability; on the contrary, the value $h = 0.837$ gives rise to instability (bottom of picture).

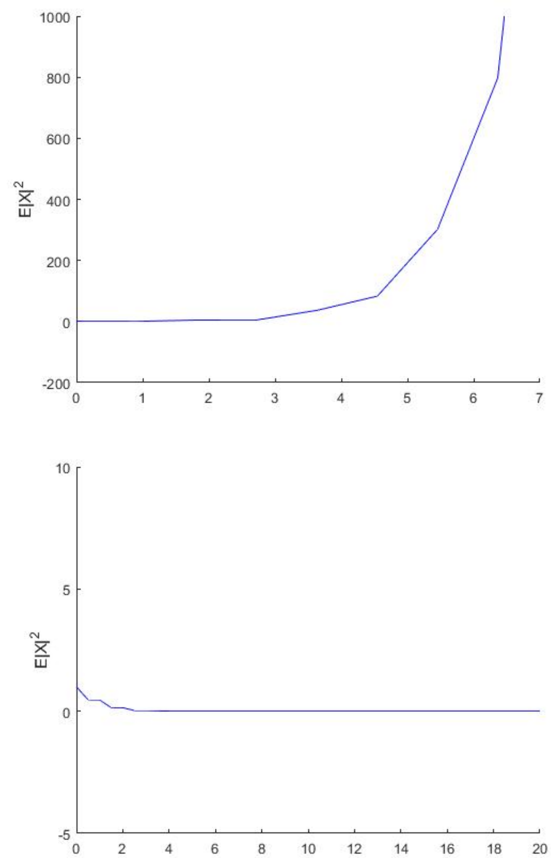

Figure 2: Behaviour of x_n^2, computed by (24) with stepsize $h = 0.9$ (top) and $h = 0.5$ (bottom), for problem (26).

4.3. Considerations about stability

Let us consider the Explicit Midpoint method, represented by the Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}.$$

We choose

$$B = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \Gamma = B$$

and $p = [0 \;\; 1]$, $r = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $q = p$ and $s = r$. We construct a TSRK method with $\theta = \frac{1}{2}$ and consider the same test equation of Sections 4.1 and 4.2 (with $\lambda = -3$ and $\mu = 0.5$). In Figure 5, we plot the behaviour of the spectral radius of the stability

h	err
0.5	0.2304
0.25	0.0243
0.1250	0.0011
0.0625	0.001
0.0313	8.0053×10^{-4}
0.0156	7.7344×10^{-4}
0.0078	5.3000×10^{-4}

Table 2: Mean-square error at the endpoint $T = 1$, obtained by method (27) for different values of the stepsize h.

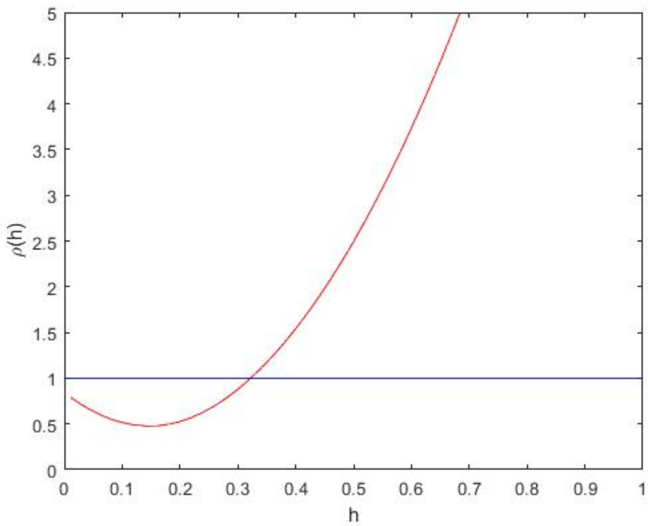

Figure 3: Behaviour of the spectral radius of the stability matrix of method (27), as function of h.

matrix M as function of h. In [22], we find the mean-square stability condition for the Euler-Maruyama method, thanks to which we are able to compute the stability interval $(0, 0.6389)$, which is clearly smaller than the stability interval of the considered TSRK. We can say that this class of new methods offers potentially more advantageous stability properties.

5. Conclusions

In this article, we present a possibility of extend to the stochastic case the family of TSRK methods, which are well-known in the deterministic ODEs context. We provide convergence and stability results, which are confirmed by the experimental evidence. We consider this work as the first step to enlarge the class of the stochastic numerical methods in a family analogous to that of General Linear Methods [24]. Furthermore,

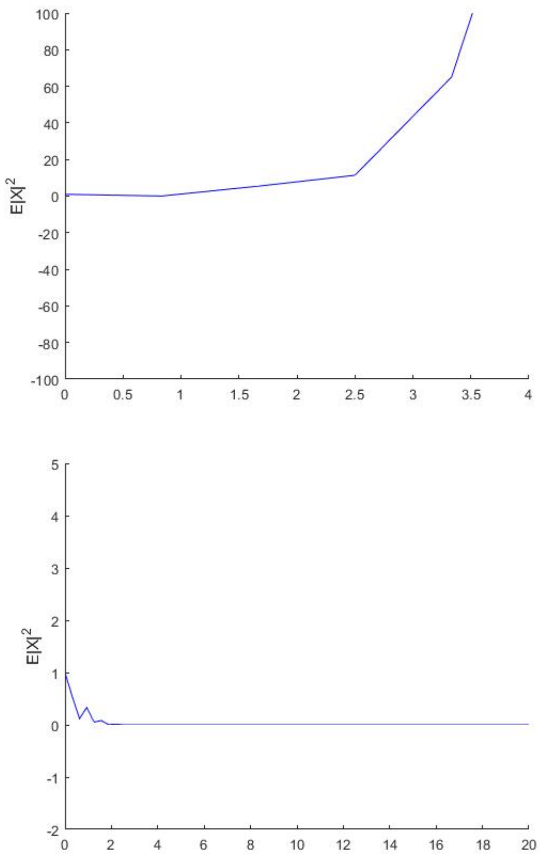

Figure 4: Behaviour of x_n^2, computed by (27) with stepsize $h = 0.837$ (top) and $h = 0.313$ (bottom), for problem (26).

future works may be devoted to different stability issues [1, 3, 17, 23] and to the investigation of properties of conservation of invariance laws [12–15, 18, 19] in a geometric integration perspective.

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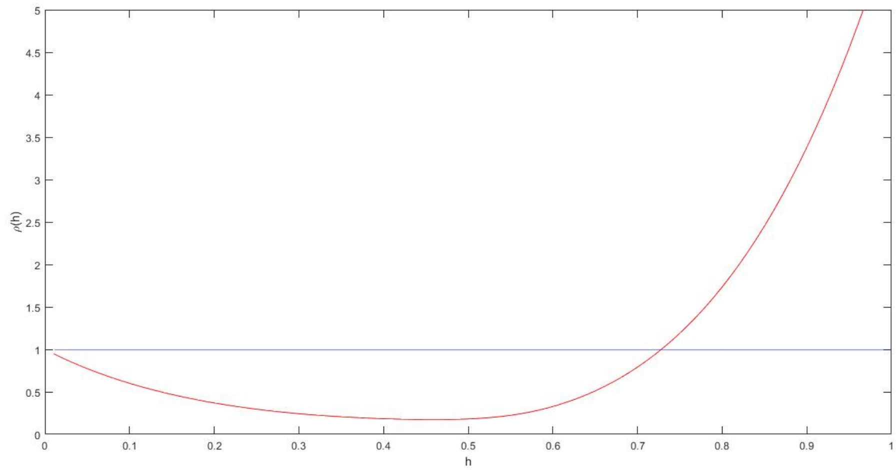

Figure 5: Behaviour of the spectral radius of the stability matrix as function of h.

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