Even Subsequence

- **Problem**

Given a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$, where $a_i \in \mathbb{N}_0^+$, return the number of contiguous non-empty subsequences $B =$ of A whose elements sum to an even number.

3	4	6	5	8

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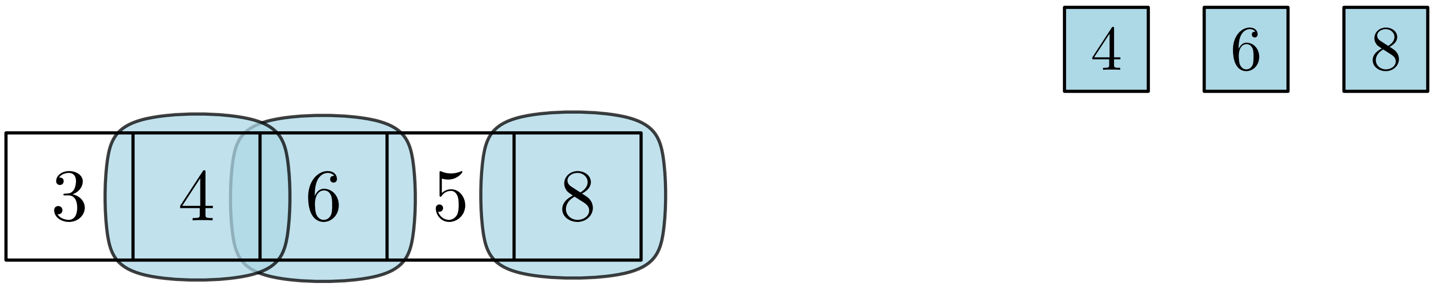

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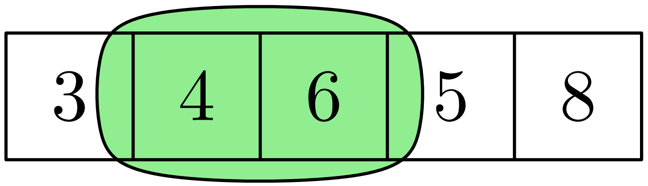

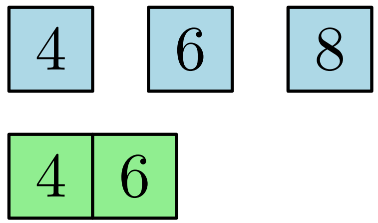

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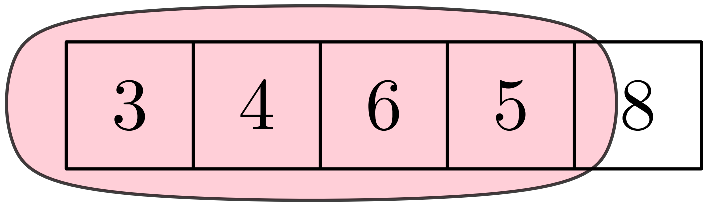

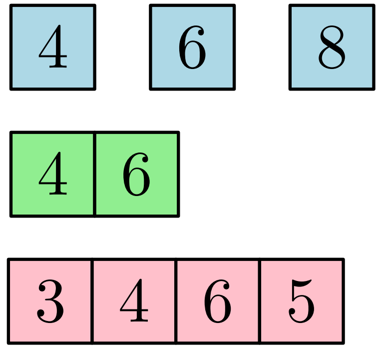

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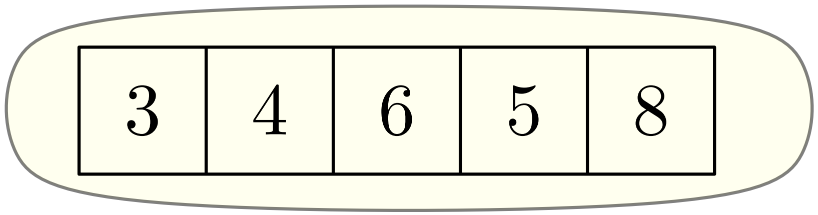

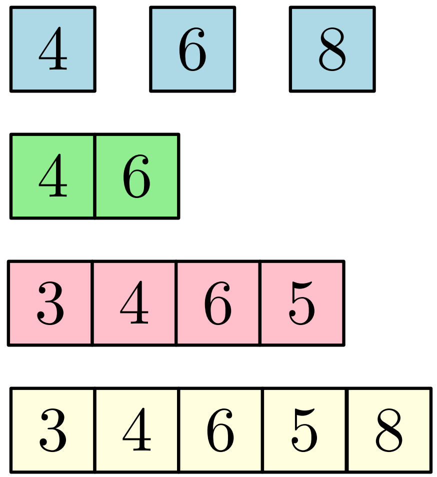

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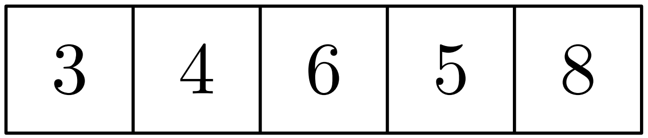
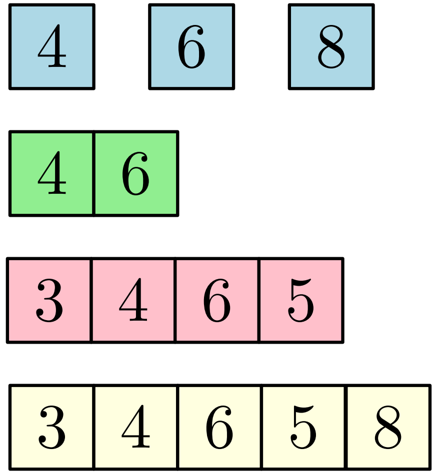
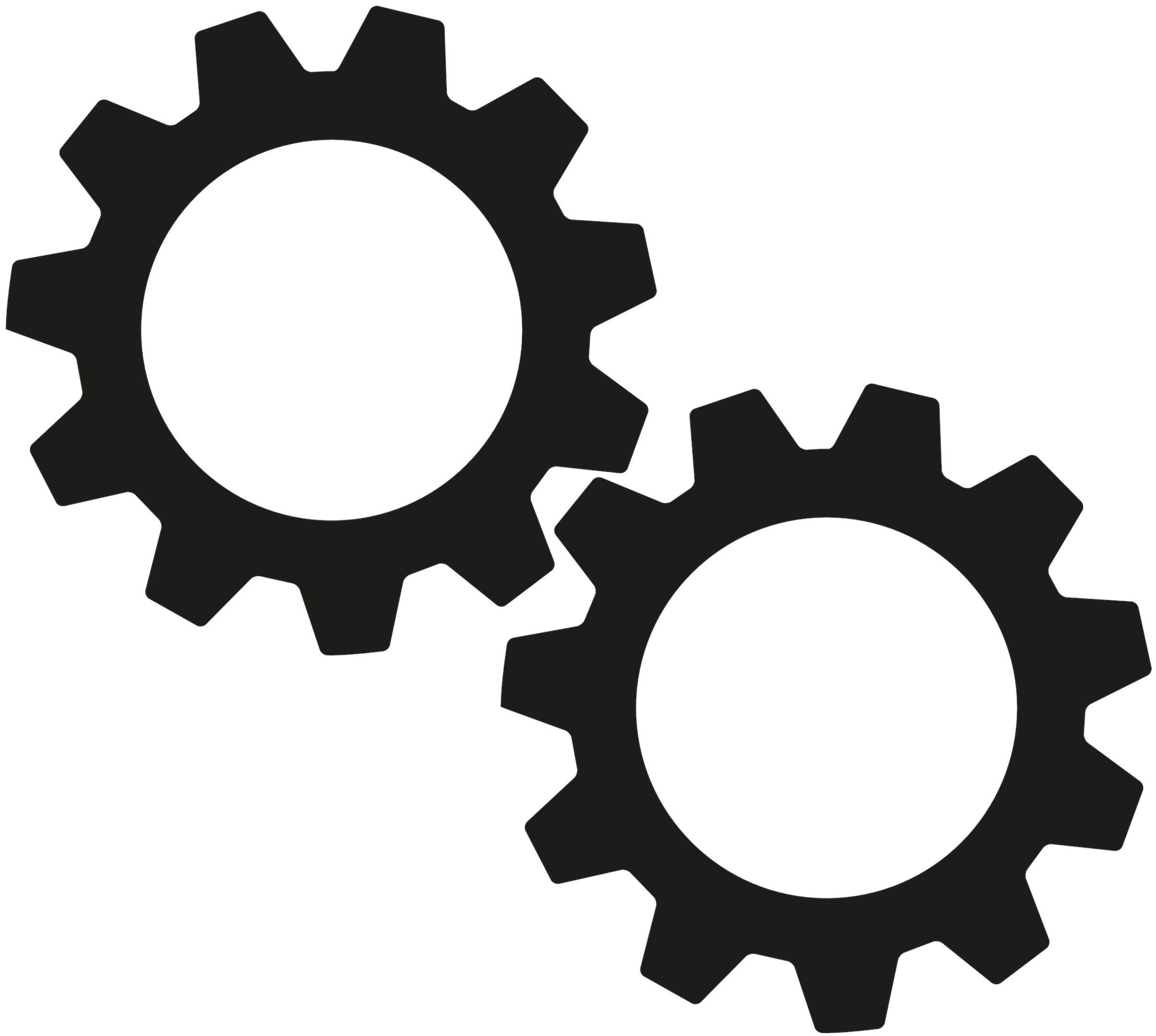




Answer: 6



Naive Solution

- For each pair of integers (i, j) with $i \leq j$
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```
unsigned int result = 0;
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    for(unsigned int j=i; j<A.size(); j++)
        if(std::accumulate(A.begin()+i, A.begin()+j+1, 0)
                                                        %2 == 0)
            result++;

return result;
```

Naive Solution

- For each pair of integers (i, j) with $i \leq j$ $O(n^2)$
  - Check whether $x \leftarrow \sum_{k=i}^{j} a_k$ is even $O(n)$

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Can we do better?

Partial Sums Technique

Compute the partial sums $P_k = \sum_{i=1}^{k} a_k$ in one iteration $O(n)$

$$P_0 = 0 \qquad\qquad P_k = P_{k-1} + a_k$$

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A		3	4	6	5	8
P	0	3	7	13	18	26

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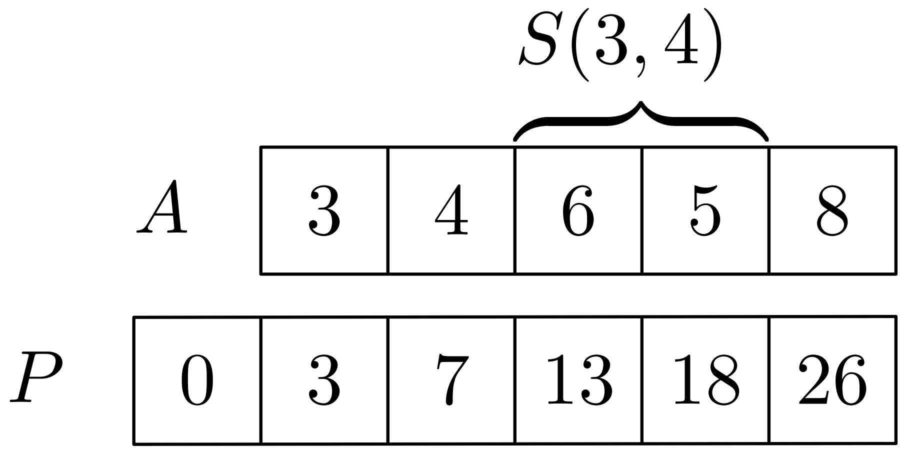

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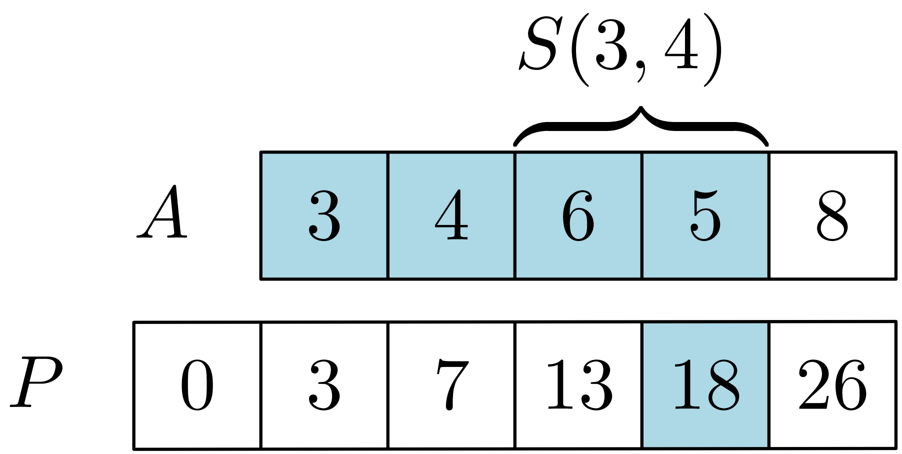

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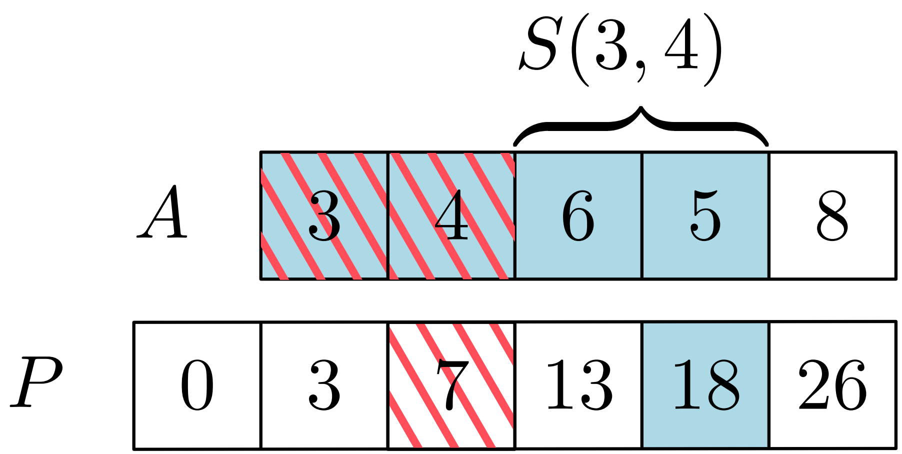

Observation: $S(i,j) = \sum_{k=i}^{j} a_k = P_j - P_{i-1}$ constant time!

A $O(n^2)$ algorithm

- Compute the vector P of prefix sums.
- For each pair of integers (i, j) with $i \leq j$
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```
std::vector<int> P(A.size()+1);
std::partial_sum(A.begin(), A.end(), P.begin()+1);

unsigned int result = 0;
for(unsigned int i=0; i<A.size(); i++)
    for(unsigned int j=i; j<A.size(); j++)
        if((P[j+1] - P[i])%2 == 0)
            result++;

return result;
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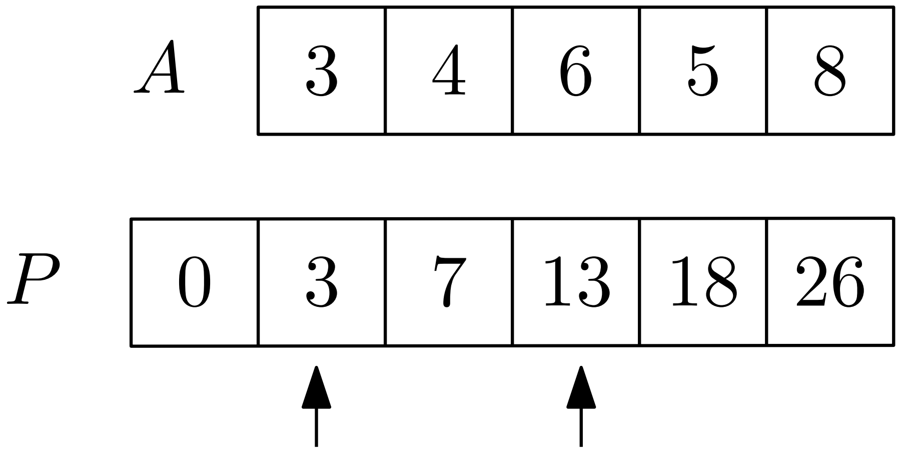

Observation 1: Each pair i, j with $\ell < j$ for which $P_j - P_i$ is even contributes 1 to the result.

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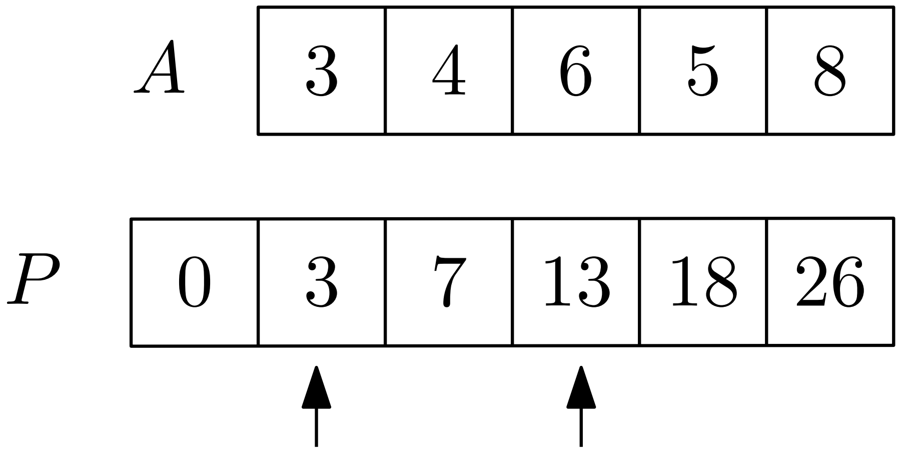

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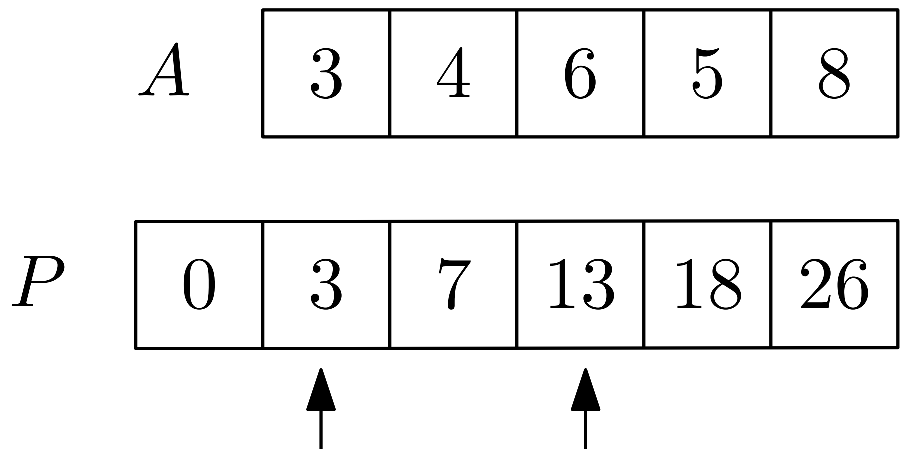

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Let E and O be the number of even and odd values is P, respectively.

Number of even pairs in P: $\binom{E}{2} = E(E-1)/2$.

Number of odd pairs in P: $\binom{O}{2} = O(O-1)/2$.

A Linear-Time Solution

- Compute E and O. $\quad O(n)$
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std::partial_sum(A.begin(), A.end(), A.begin());
int odd = std::count_if(A.begin(), A.end(),
                                    [](int x) {return x%2;});

return odd*(odd-1)/2 + (A.size()-odd)*(A.size()-odd+1)/2;
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$O(n)$ time $O(n)$ space

Observation: Each partial sum is used only once.

A Linear-Time Solution

- Compute E and O. $O(n)$
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```
unsigned int n;
std::cin >> n;

int x, odd=0, prefix_sum=0;
for(unsigned int i=0; i<n; i++)
{
    std::cin >> x;
    prefix_sum += x;
    odd += (prefix_sum%2)?1:0;
}

std::cout << odd*(odd-1)/2 + (n-odd)*(n-odd+1)/2 << "\n";
```

Streaming algorithm. $O(n)$ time $O(1)$ space