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Can we relax the security definition in a meaningful way?

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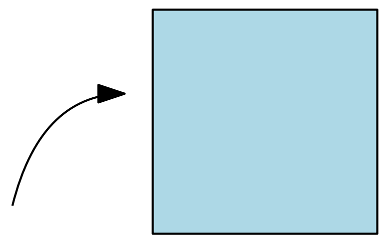

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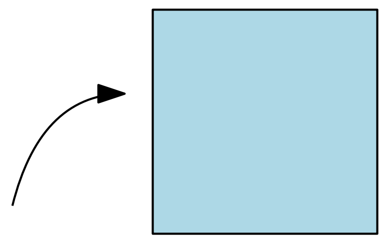

Do we need to be concerned?

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We relax perfect secrecy in two ways:

- We allow secrecy to fail with some tiny probability
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Our starting point is the following (equivalent) definition of perfect secrecy:

Definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *with message space* $\mathcal{M}$ *is* **perfectly indistinguishable** *if for every* $\mathcal{A}$ *it holds:*

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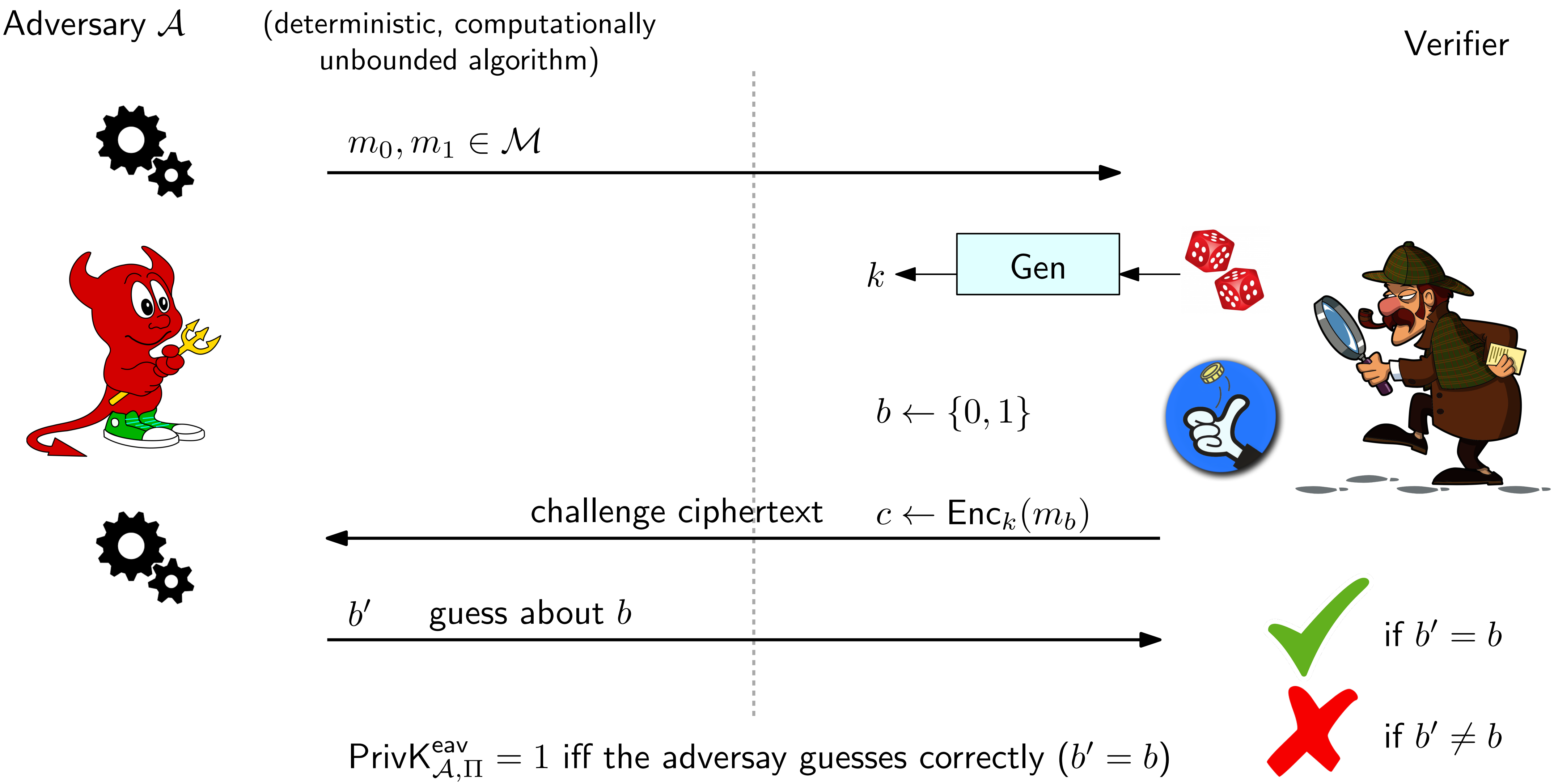
Two possible approaches:

- **Concrete**
- **Asymptotic**

Reminder: Perfect indistinguishability



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Candidate definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *is* (t, ε)*-indistinguishable if for every attacker* $\mathcal{A}$ *running in time at most* t*, it holds that:*

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Observation: $(\infty, 0)$-indistinguishability is equivalent to perfect indistinguishability

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Does not lead to a clean theory

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  Measure probabilities and running times as a function of n

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"$\eta(n)$ approaches 0 faster than the inverses of <u>all</u> polynomials in n"

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As a special case, the product of two negligible functions is negligible

Negligible and polynomially bounded functions

Which of the following functions are polynomially bounded? Which are negligible?

$n^2 + 4n - 2$	n^{100}	$n^3 + \cos(n)$	$n!$
$\frac{1}{n^{10}} + 2^{-n/2}$	2^n	3^{-n}	$\sqrt[3]{n} + \frac{1}{n}$
$n^{-n} \cdot (n^5 + n^2)$	$2^{\sqrt{n}}$	$\sqrt{n}$	$42 - \frac{1}{1+\log n}$
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$\frac{1}{n^{10}} + 2^{-n/2}$	2^n	3^{-n}	$\sqrt[3]{n} + \frac{1}{n}$
$n^{-n} \cdot (n^5 + n^2)$	$2^{\sqrt{n}}$	$\sqrt{n}$	$42 - \frac{1}{1+\log n}$
$4^{\sqrt{\log n}}$	n^{-5}	$2^{-\log n \cdot \log\log n}$	$(1 + \frac{1}{n})^n$

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If $M = \{0,1\}^{\ell(n)}$ then $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is a **fixed-length** private-key encryption scheme (for messages of length $\ell(n)$)

The adversarial indistinguishability experiment, revisited

Adversary $\mathcal{A}$ — probabilistic polynomial-time algorithm with input 1^n

Verifier

$m_0, m_1 \in \mathcal{M}$

$(|m_0| = |m_1|)$

$k \leftarrow$ Gen $\leftarrow$

$b \leftarrow \{0,1\}$

challenge ciphertext $c \leftarrow \mathsf{Enc}_k(m_b)$

b' guess about b

if $b' = b$

if $b' \neq b$

$\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1$ iff the adversay guesses correctly $(b' = b)$

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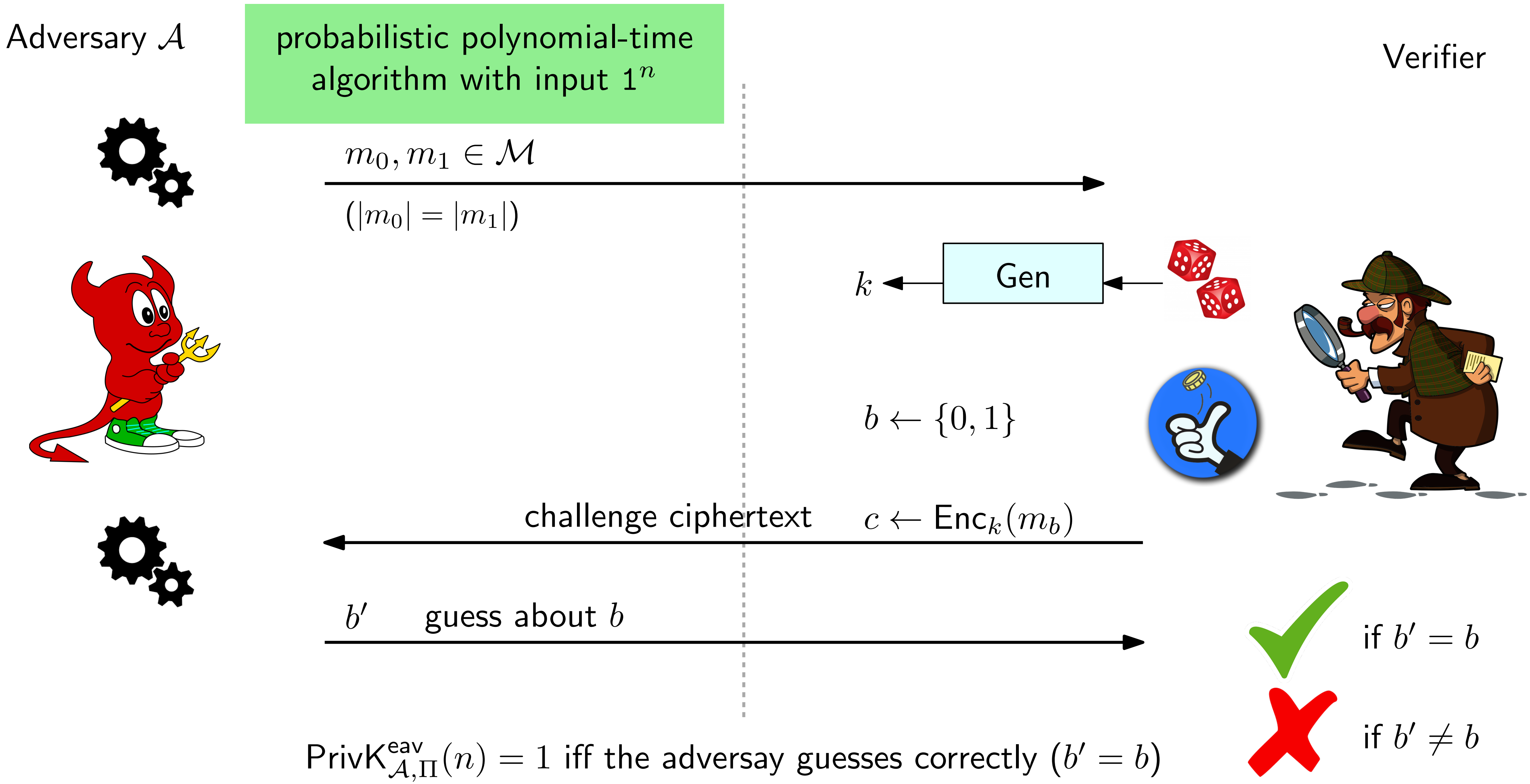

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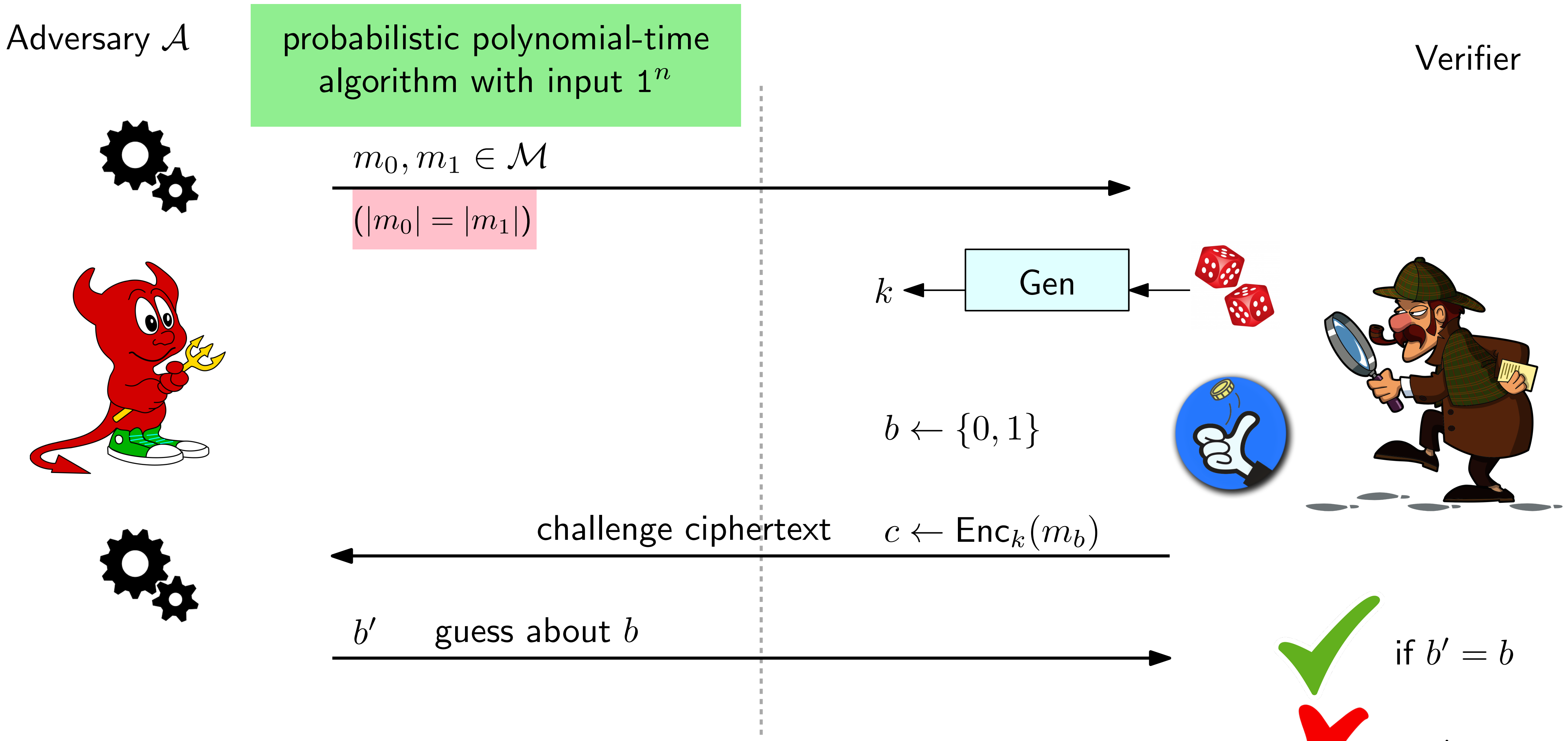

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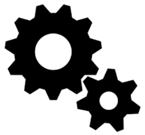

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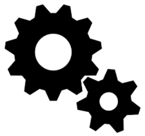

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Computational indistinguishability (asymptotic)

Definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *has indistinguishable encryptions in the presence of an eavesdropper (is* **EAV-secure***) if, for every probabilistic polynomial-time adversary* $\mathcal{A}$*, there is a negligible function* ε *such that:*

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Observation: perfect indistinguishability implies EAV-security

Example 1

Consider a scheme where:

- $\mathsf{Gen}(1^n)$ returns a key chosen uniformly at random in $\{0,1\}^n$
- The best possible adversary $\mathcal{A}$ performs a brute-force search over the key space
- If the running time of the adversary is $t(n)$ then:

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Is this scheme EAV-secure? Yes!

For all polynomial running times $t(n)$, all functions in $O\left(\frac{t(n)}{2^n}\right)$ are negligible

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A increase in computing power resulted in a more difficult attack!

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How large do we need to choose n?

n	48	64	128	256	512	1024
running time	2.5 months	6 months	4 years	32 years	255 years	2041 years
probability of success	1 in 256	$\approx$ 1 in 17 mil	$\approx$ 3 in 10^{26}	$\approx$ 3 in 10^{65}	$\approx$ 1 in 10^{142}	$\approx$ 2 in 10^{296}

Computational secrecy (asymptotic)

We relax perfect secrecy in two ways:

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In general, encryption does not hide the plaintext length

- This is captured in the indistinguishably experiment by requiring $|m_0| = |m_1|$

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In Google maps, the map tiles are compressed and (essentially) static. The size of the ciphertext can be used to determine the viewed location

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Pragmatic approach: assume that PRGs exist (and hope for the best) 🤞

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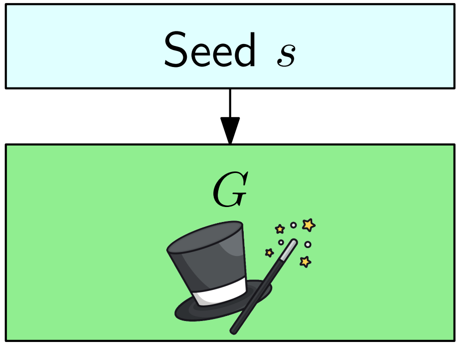
A pseudorandom number generator is a deterministic polynomial-time algorithm that takes a binary string s (**seed**) chosen uniformly at random...



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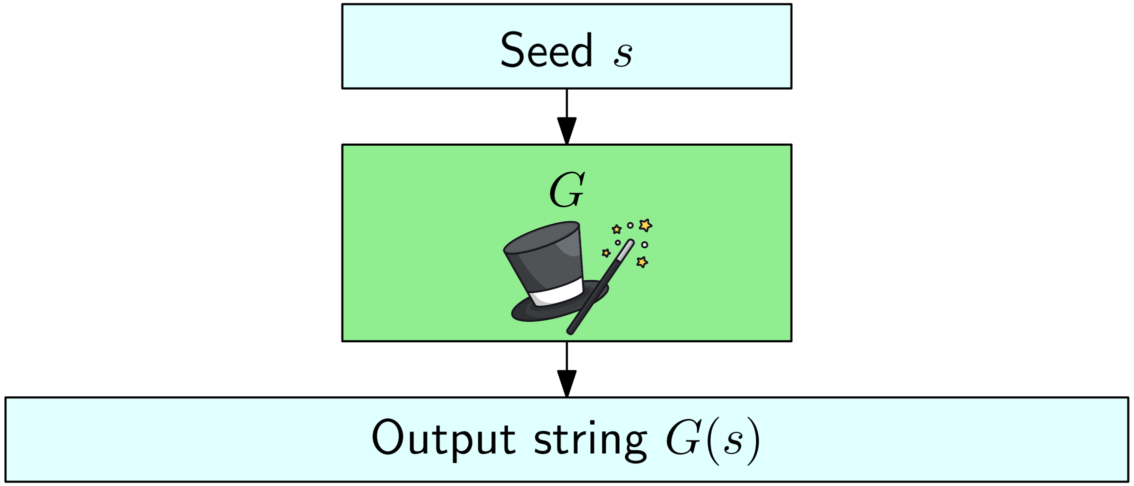

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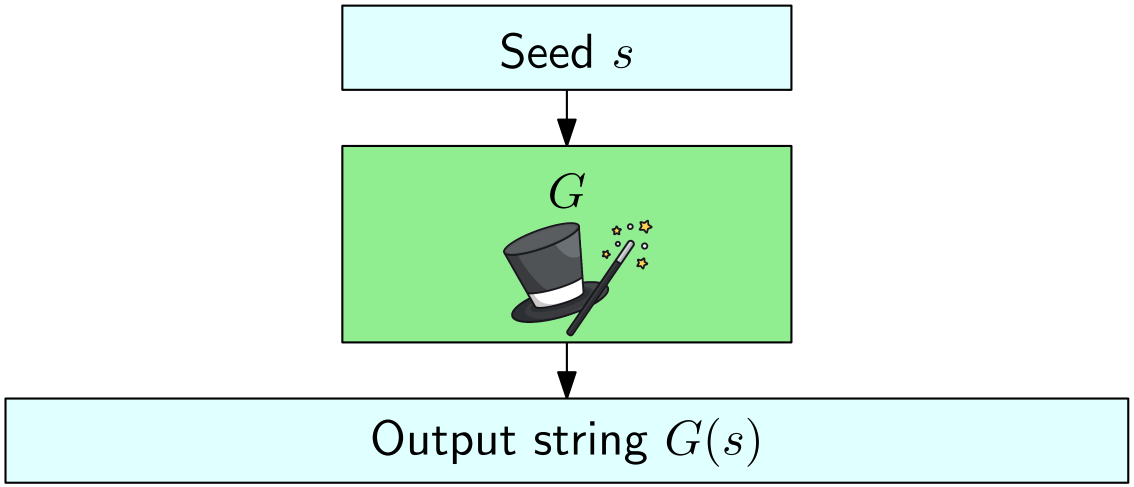

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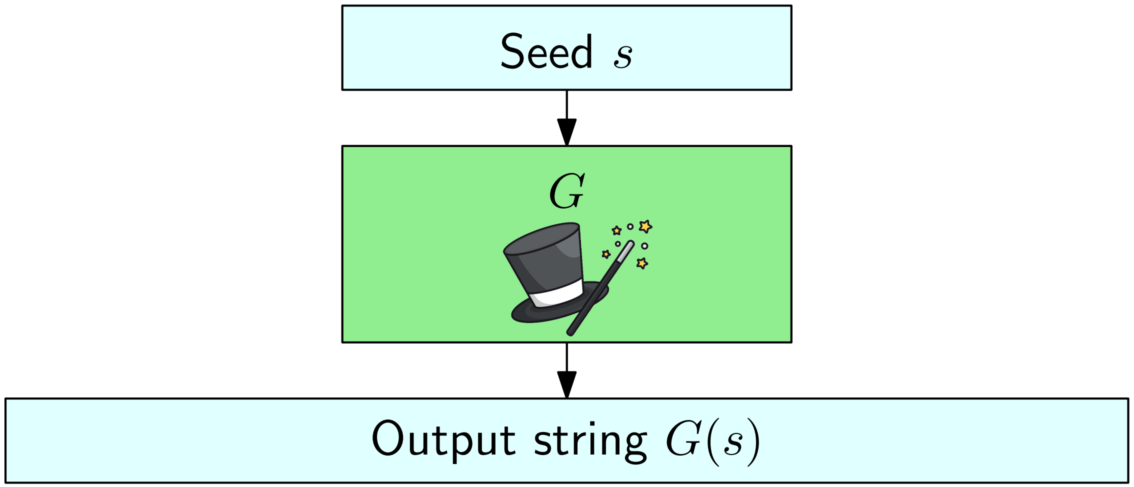

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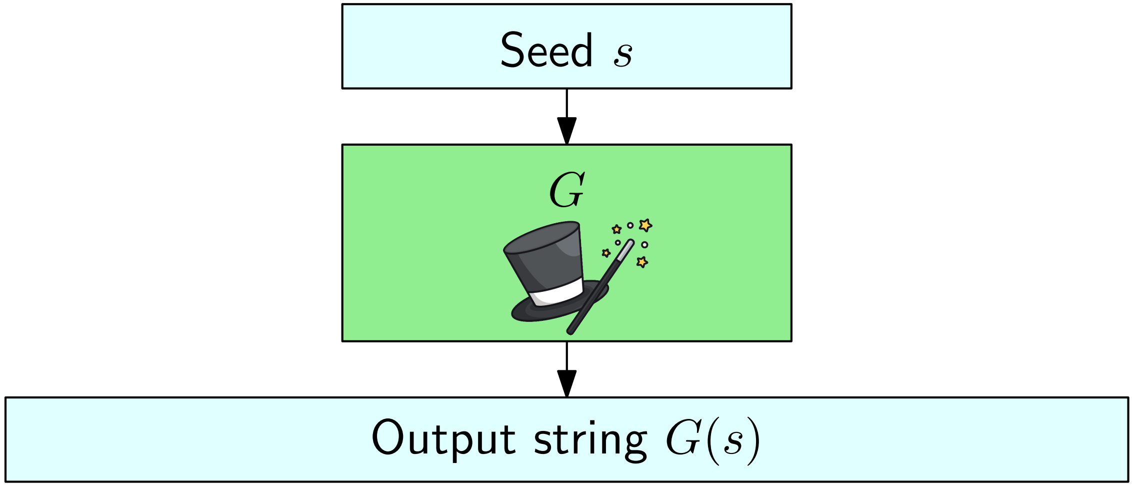

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These questions are meaningless...

- Randomness is captured by probability distributions
- Uniformity is a property of distributions (not binary strings)
- The uniform distribution over a set X assigns probability $\frac{1}{|X|}$ to every element in X

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These questions are meaningless...

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- The uniform distribution over a set X assigns probability $\frac{1}{|X|}$ to every element in X

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Which of the following binary strings is random?

Which of the following binary strings is uniform?

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...and that x is "pseudorandom" if it is the output of a PRG

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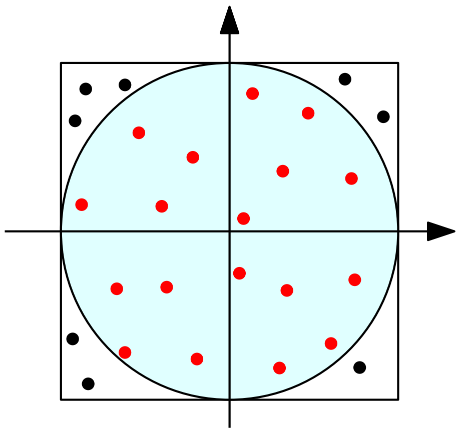

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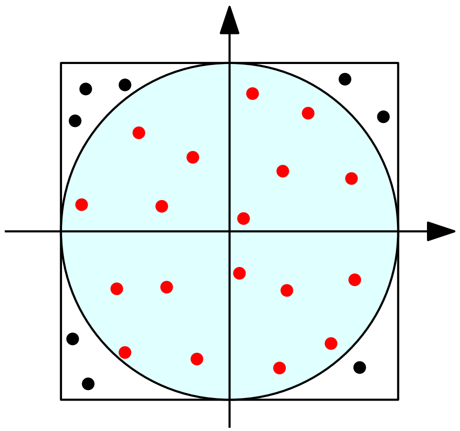

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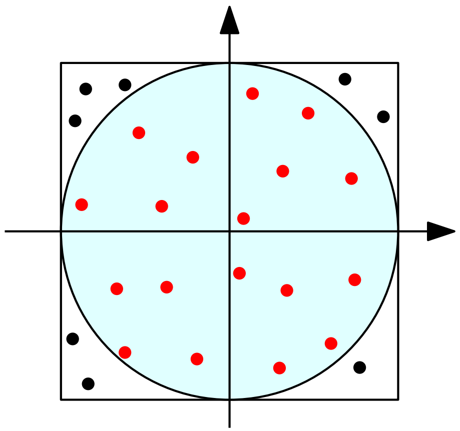

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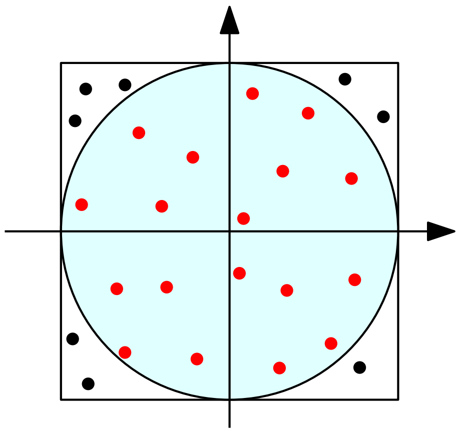

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Is this even possible?

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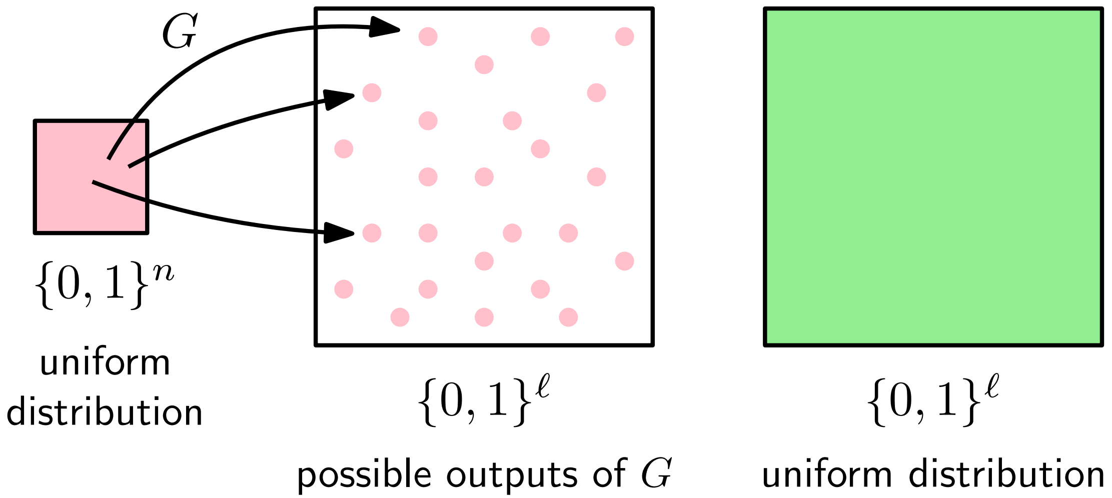
- At least half of the ℓ-bit strings (actually a $\frac{2^{\ell-n}-1}{2^{\ell-n}}$-fraction) can never be output by G !



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G will never pass the following statistical test (for some n):

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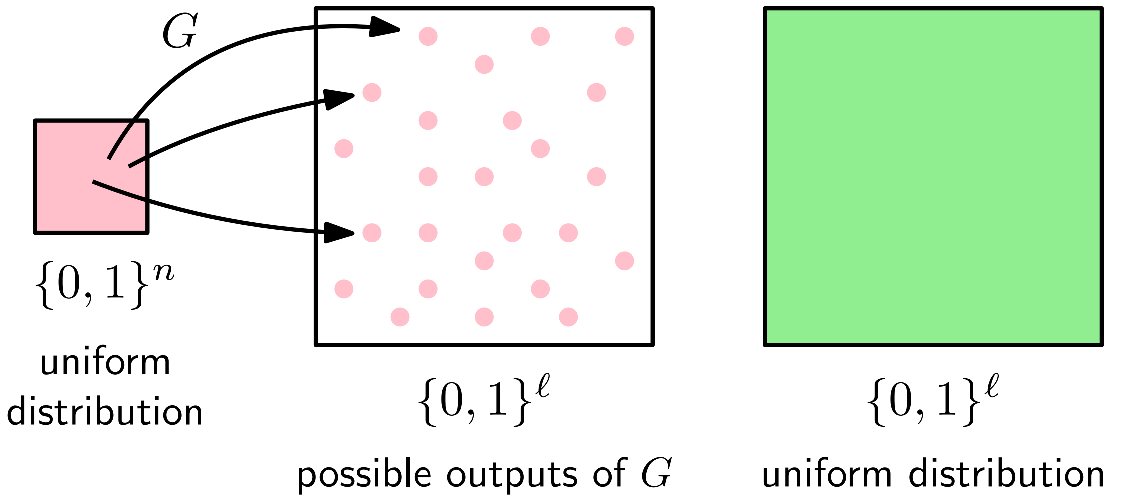

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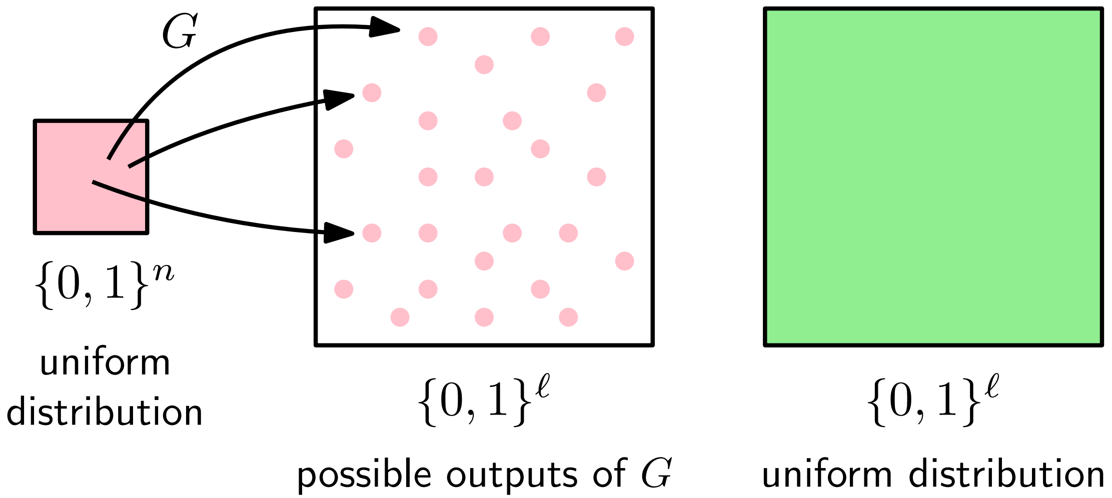

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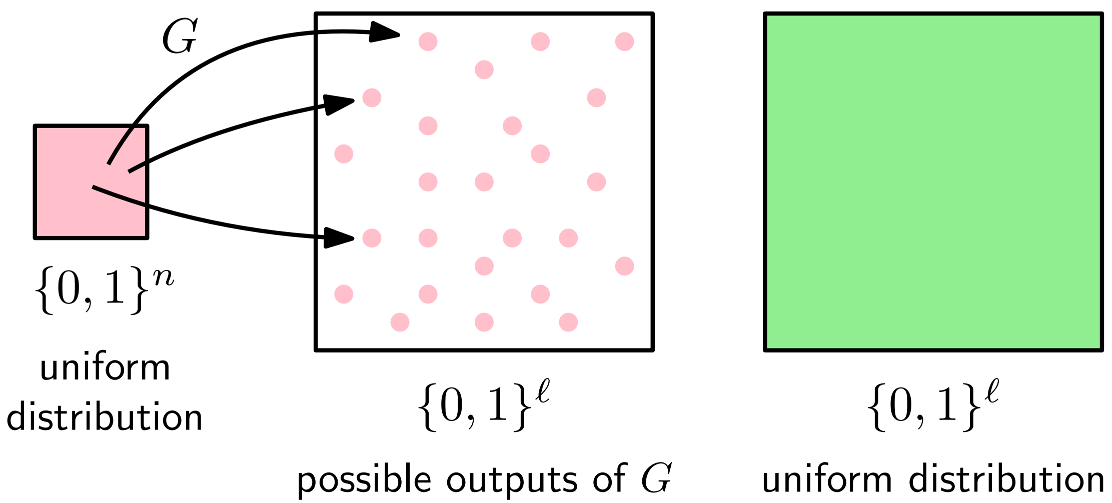
Idea: If adversaries are polynomially bounded, we only need to pass statistical tests that run in polynomial time



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Let G be a deterministic polynomial-time algorithm such that for any n and any input $s \in \{0,1\}^n$, the output $G(s)$ is a string of length $\ell(n)$

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$\left|1 - \frac{1}{2^{\ell(n)}}\right|$ is not negligible

Why are PRGs useful?

As far as polynomial-time algorithms are concerned, the output of $G(s)$ with a random seed s is indistinguishable (up to some negligible probability) from a random string r

Seed $s \in \{0,1\}^n$

$\downarrow$

G

$\downarrow$

Output string $G(s) \in \{0,1\}^{\ell(n)}$

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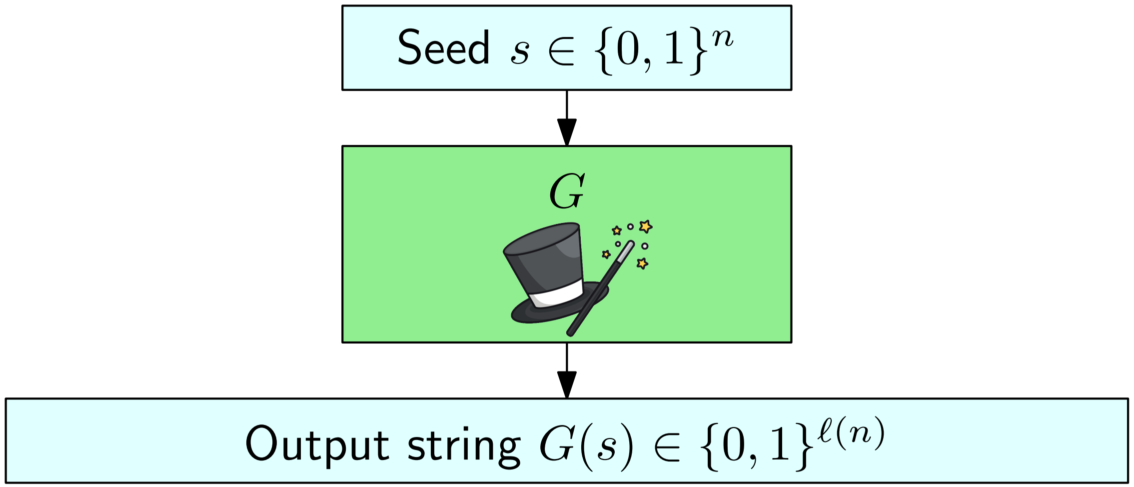

If we have a randomized polynomial-time algorithm that uses $\ell(n)$ random bits, and we replace those random bits with the output of $G(s)$, the resulting (randomized) algorithm "behaves the same" except for a negligible probability