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Hash function (inf.): a function H that maps a long input string to a short, fixed-length, output string.

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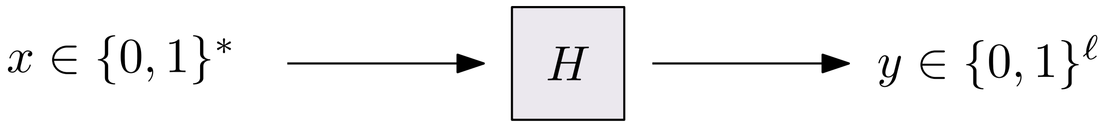

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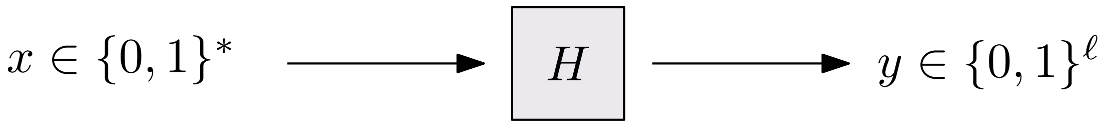

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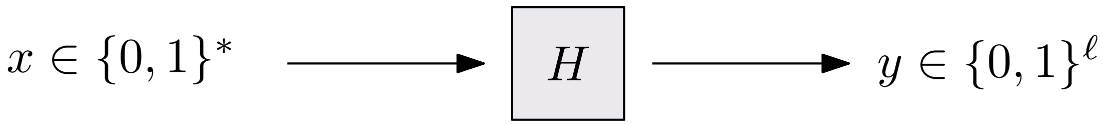

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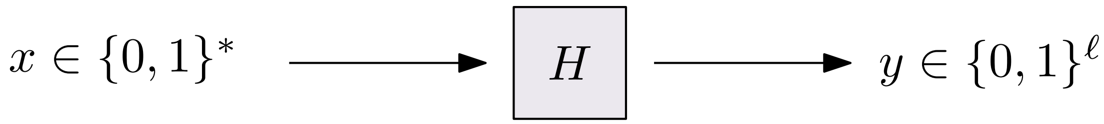

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  - **In cryptography, even few collisions are bad!**

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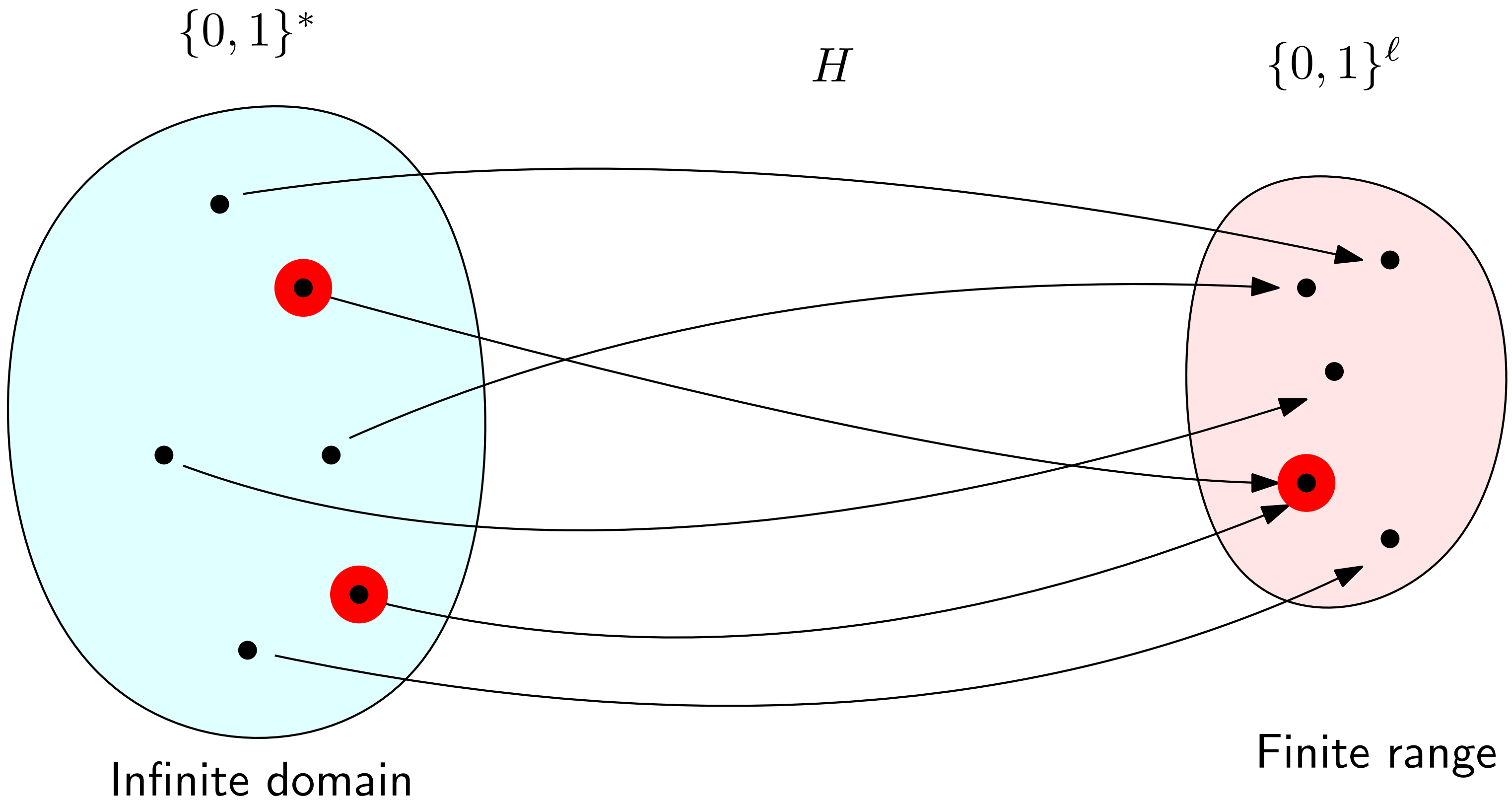

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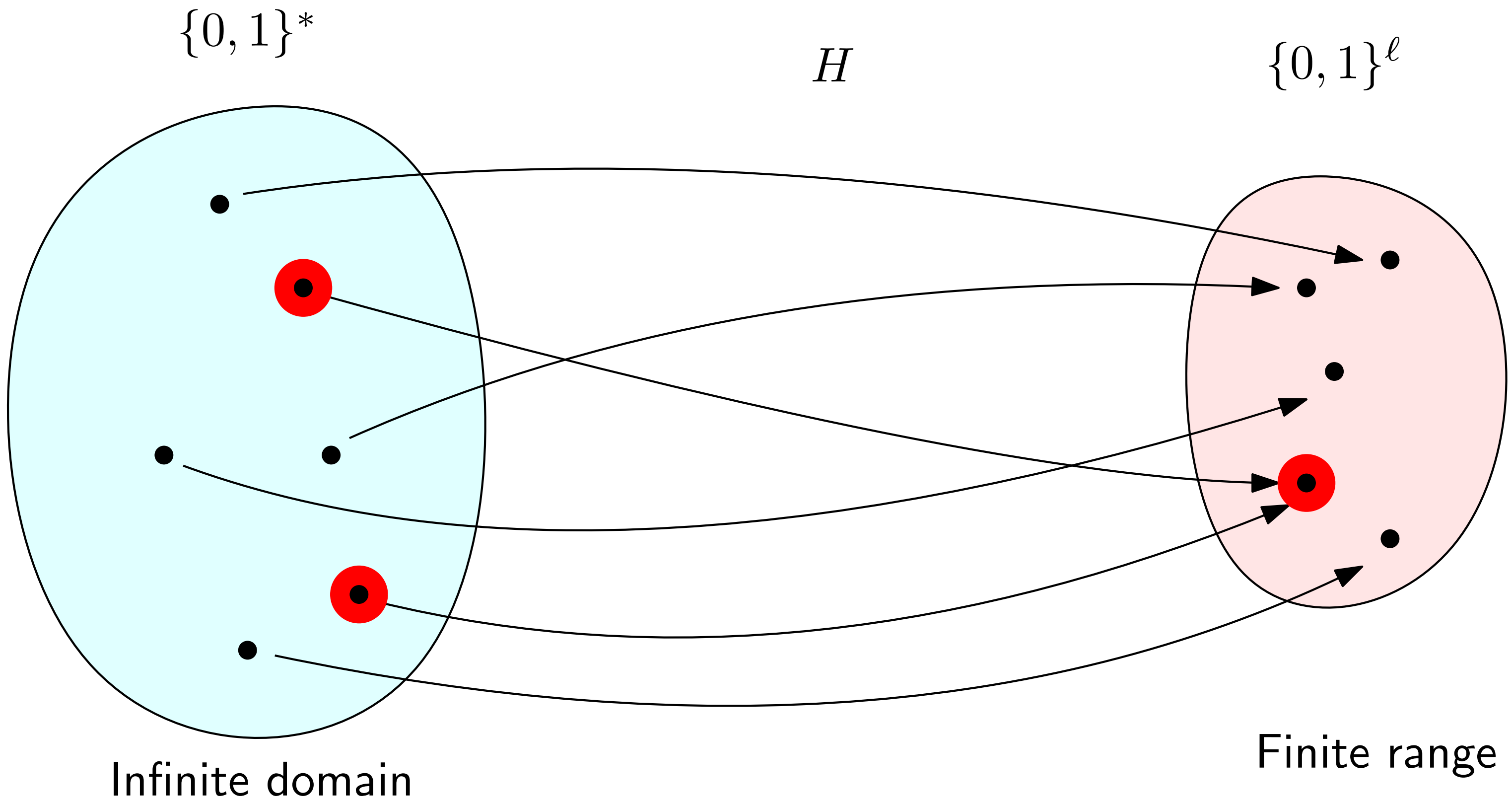

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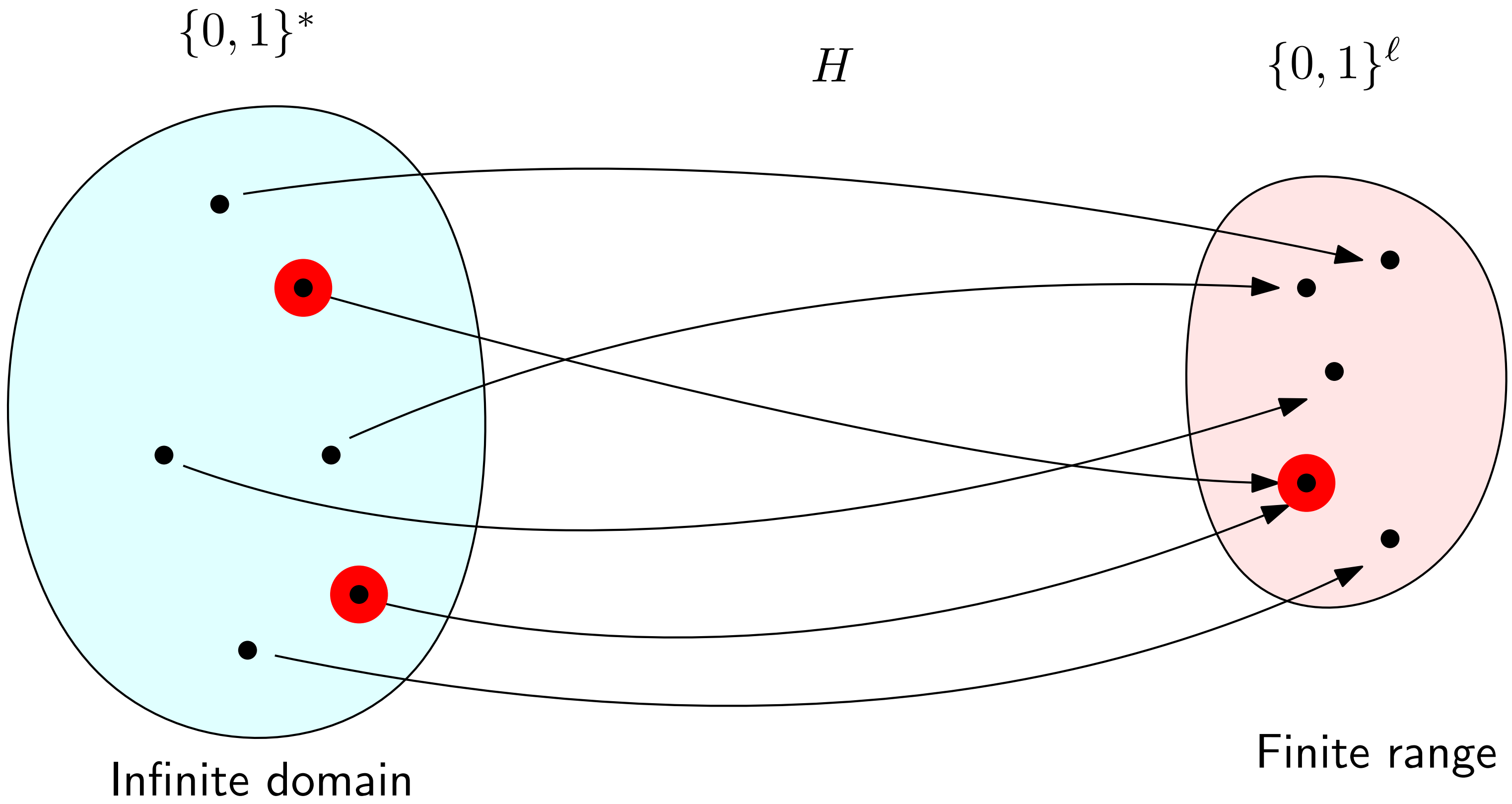

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Next best thing: Collisions are hard to find (by efficient adversaries)

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- **Gen:** is a probabilistic algorithm that takes as input 1^n and outputs a key s
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Small abuse of notation: when Gen is clear, we say that H is a hash function

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The Hash Collision experiment

Let $\mathcal{H} = (\mathsf{Gen}, H)$ be a Hash function. We name the following experiment Hash-coll$_{\mathcal{A},\mathcal{H}}(n)$:

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Definition: *A hash function $\mathcal{H} = (Gen, H)$ is* **collision resistant** *if, for every probabilistic polynomial-time adversary $\mathcal{A}$, there is a negligible function ε such that:*

$$\Pr[\textit{Hash-coll}_{\mathcal{A},\mathcal{H}}(n) = 1] \leq \varepsilon(n)$$

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Collision resistance $\implies$ **Second preimage resistance** $\implies$ **Preimage resistance**

Attacking Hash Functions: Birthday Attack

Let $H^s : \{0,1\} \to \{0,1\}^\ell$ be **some** hash function.

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- Worst-case approach
- Model H as a random function

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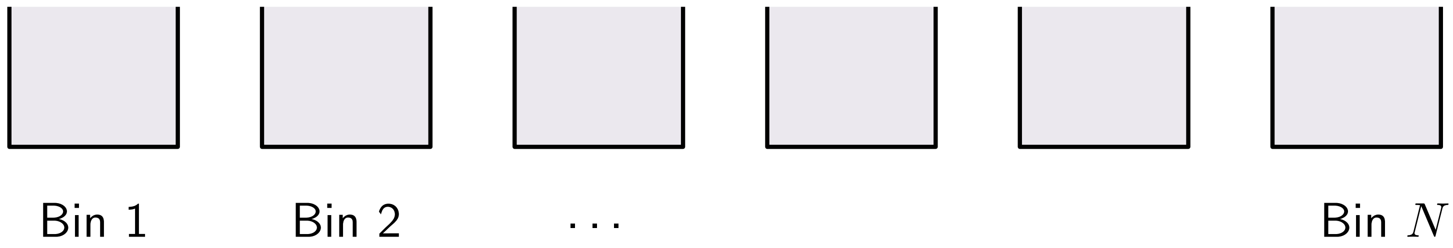
Can be thought of as a **balls into bins** experiment

Repeatedly throw a ball into a one out of N possible bins, chosen u.a.r.



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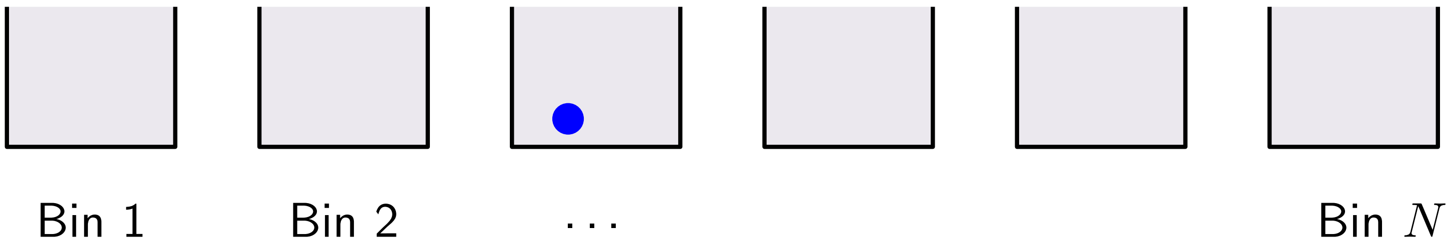

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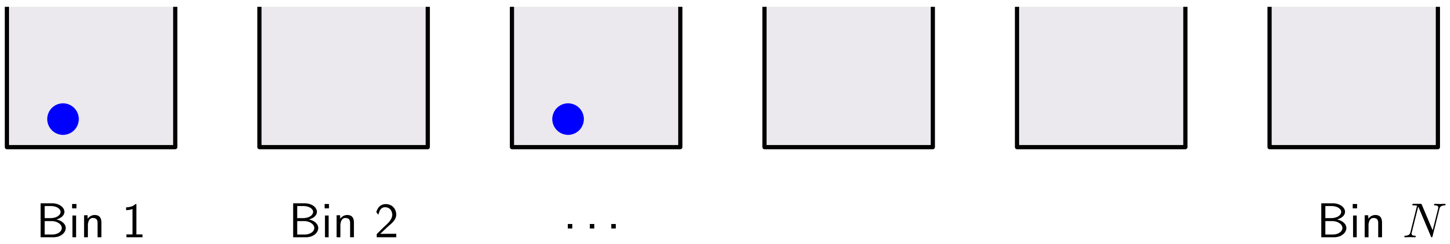

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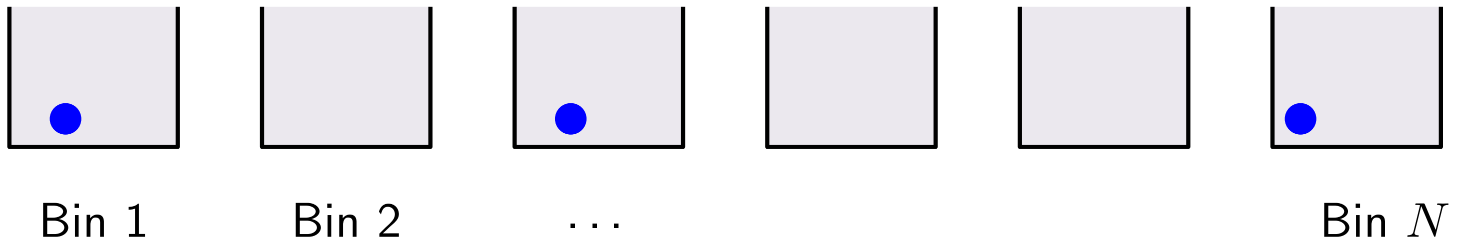

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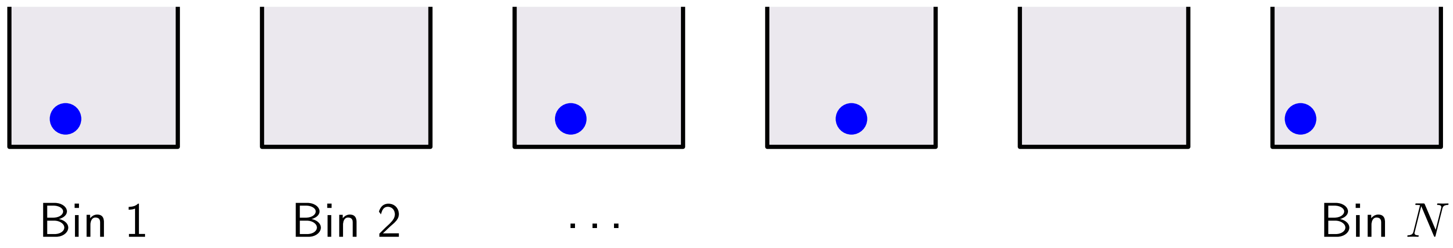

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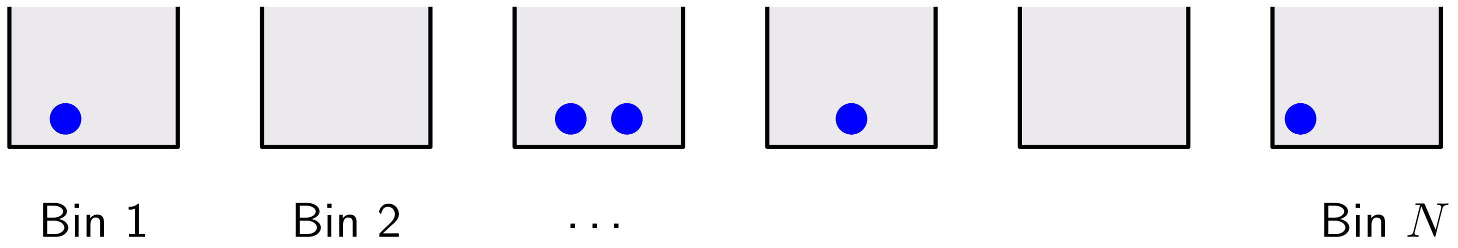

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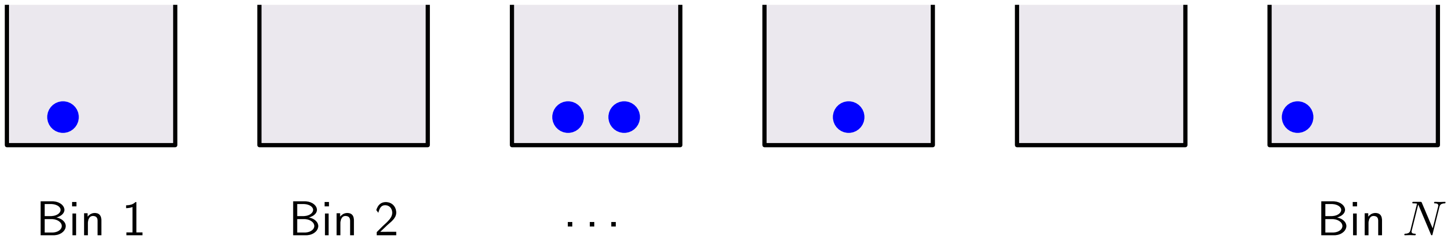

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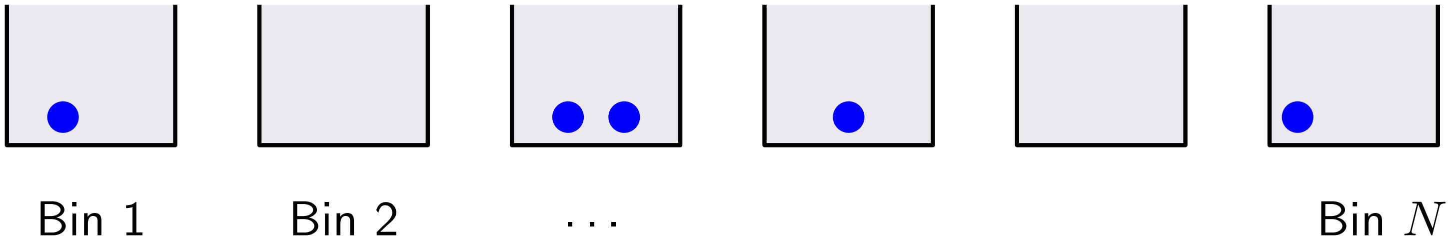

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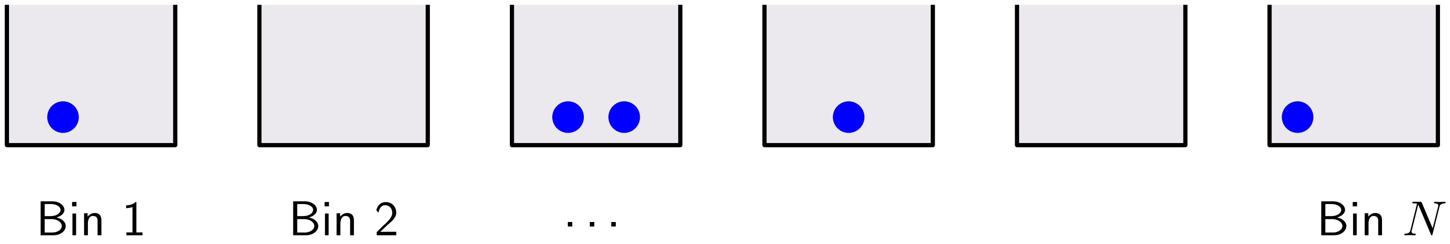

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We want to know: If we throw q balls, what's the chance that some bin contains at least 2 balls?

Probability of a collision

Theorem: Let Coll denote the event "at least one bin contains at least 2 balls". If $q \leq \sqrt{2N}$, then $\frac{q(q-1)}{4N} \leq \Pr[\mathsf{Coll}] \leq \frac{q(q-1)}{2N}$.

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- For hash functions, if we want to withstand attacks running in time $\approx 2^n$ we need $\ell \geq 2n$

Birthday Attack: Finding Meaningful Collisions

The collisions found by the birthday attack do not seem very useful

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$A =$ {Today, This morning} I {took, went for} a {walk, stroll} in the city {center, park}. While there, I {had, drank} {a coffee, an espresso} and ate a {cream, sweet} {doughnut, donut}. $|A| = 2^8$

$B =$ This is to {inform, notify} you that I am {resigning, quitting} from my {job, position} {effective immediately, at once}. Please {give, send} me my {final, last} paycheck as {soon, quickly} as possible. {Goodbye, Regards}. $|B| = 2^8$

(Collision Resistant) Hash Functions... do they even exist?

Let H^s be **any** function that is computable in polynomial-time

Consider the following decision problem $C^s(\alpha, \beta)$:

> Are there two distinct strings x, y s.t. $H^s(x) = H^s(y)$, $|x| = |y| = \ell + 1$, x starts with α, and y starts with β?

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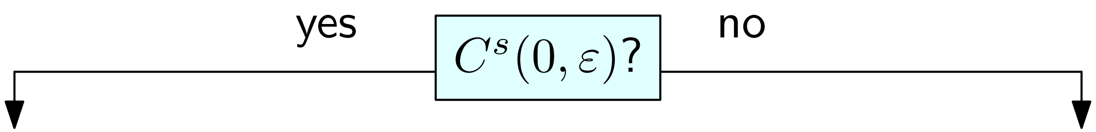

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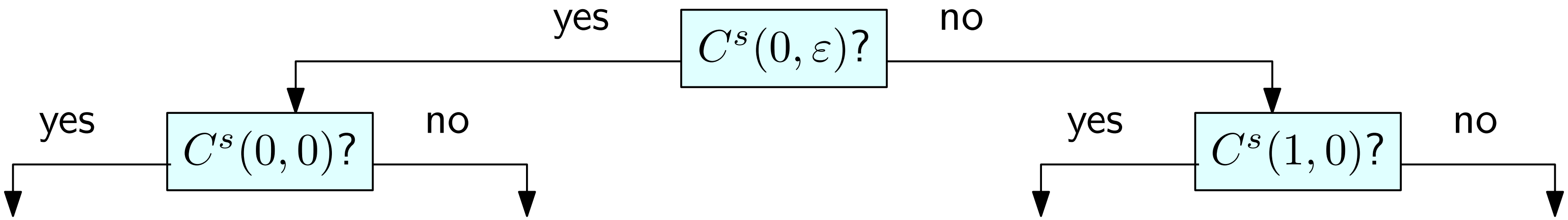

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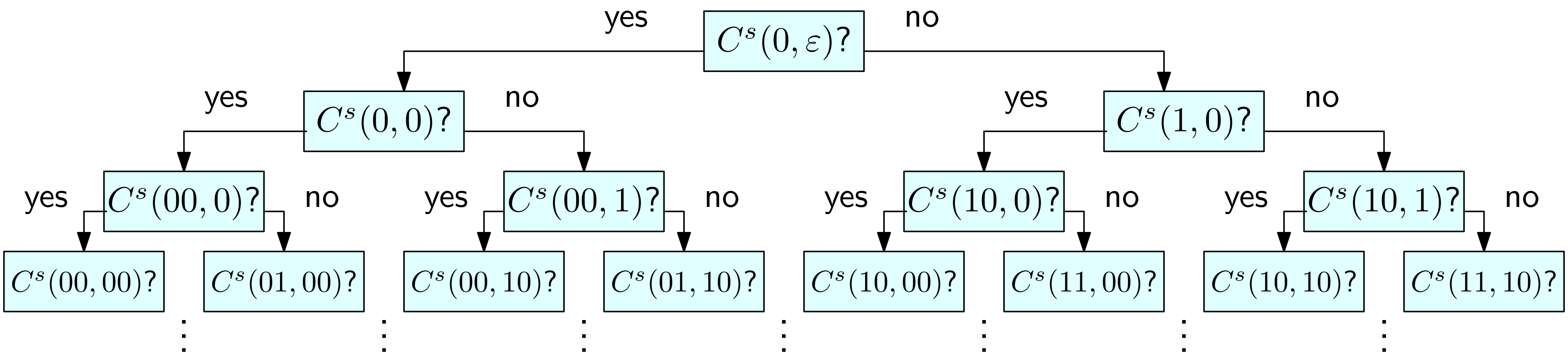

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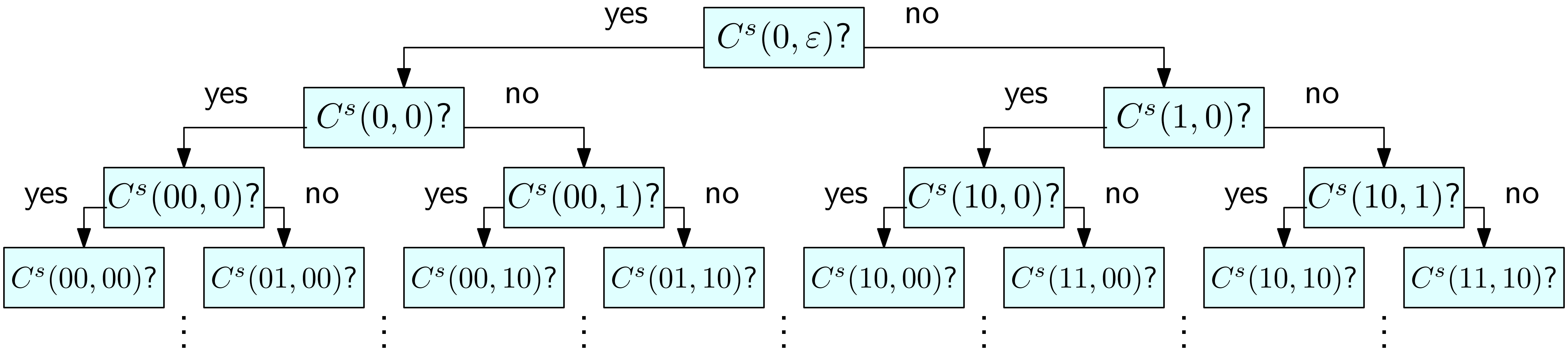

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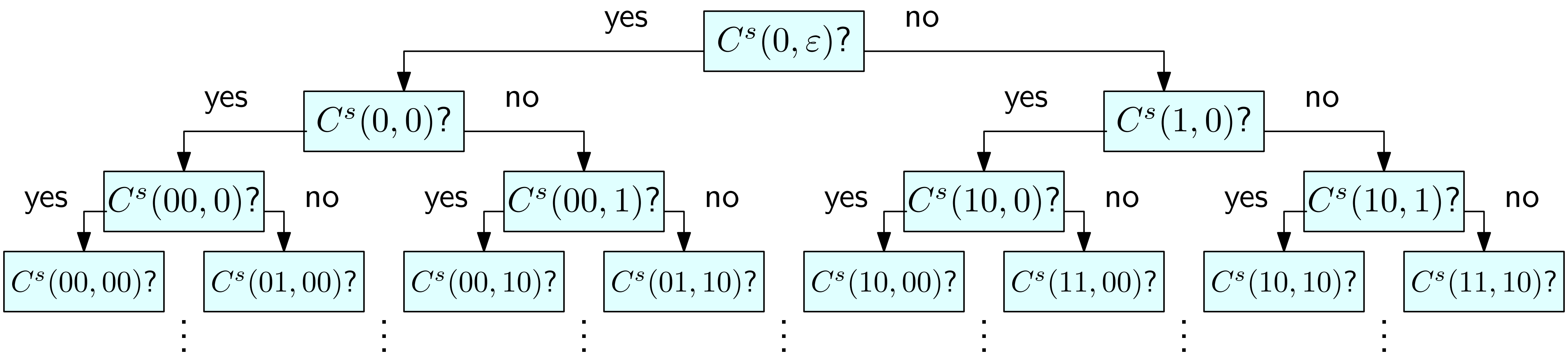

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Pragmatic approach: Pretend that Hash functions exist & use practical constructions

Constructing a Hash Function

Step 1: Start with a collision-resistant compression function for short, fixed-length, inputs

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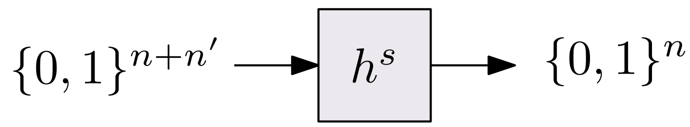

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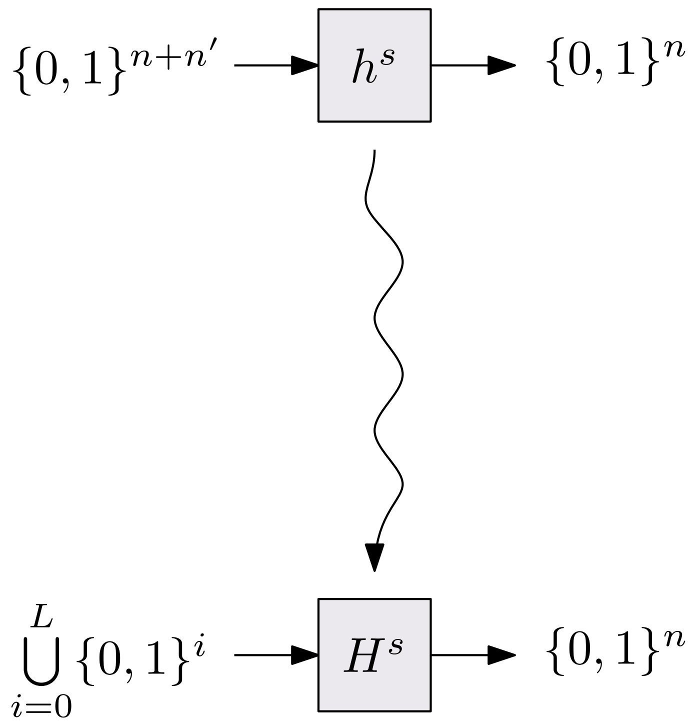

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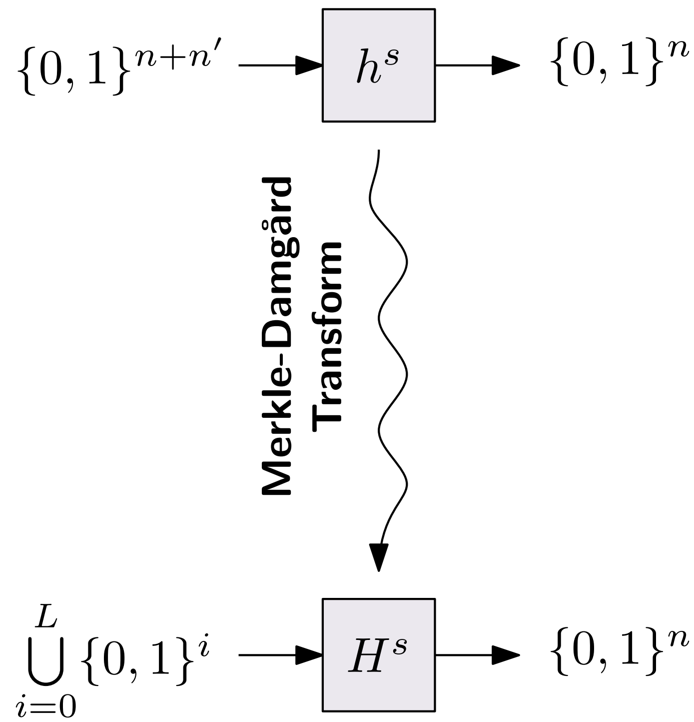
Merkle-Damgård Transform



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Ralph Merkle   Ivan Damgård

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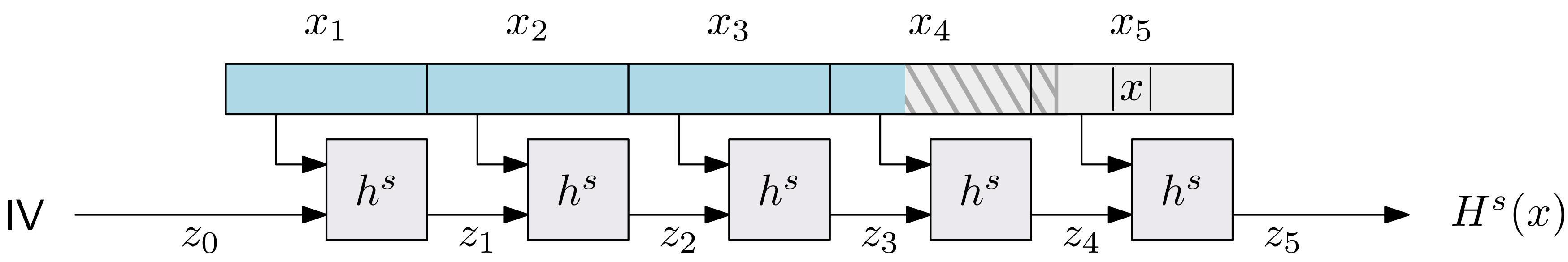
The Merkle-Damgård Transform



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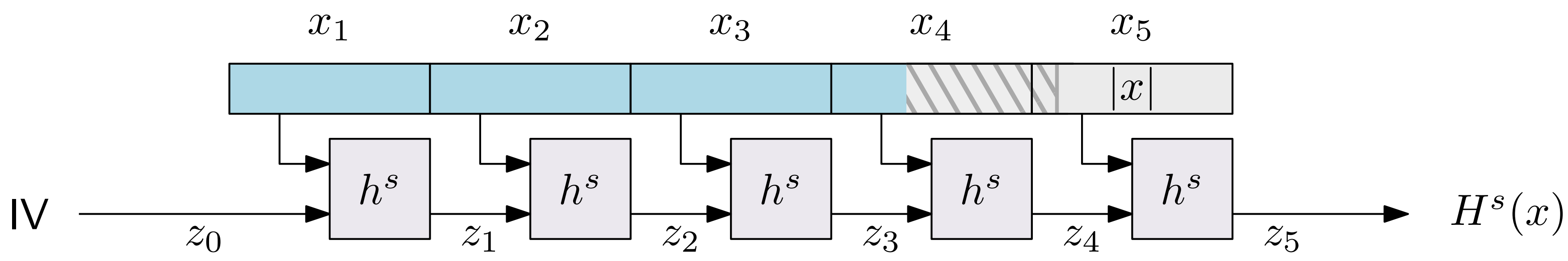

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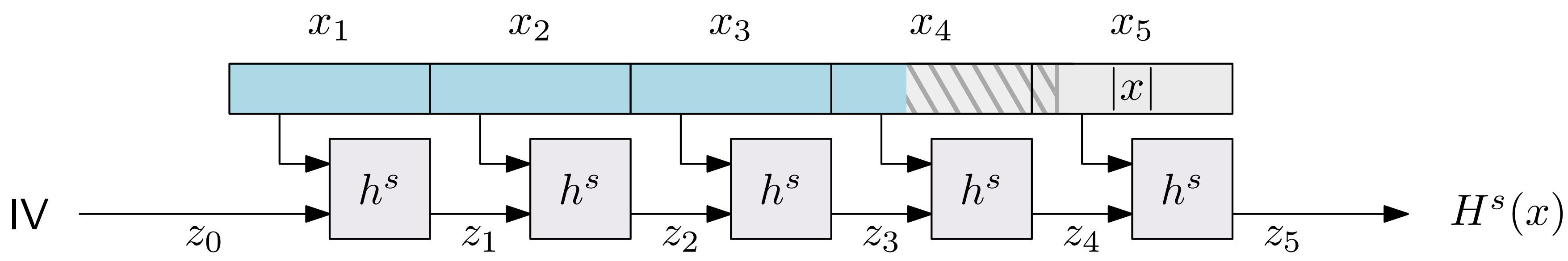

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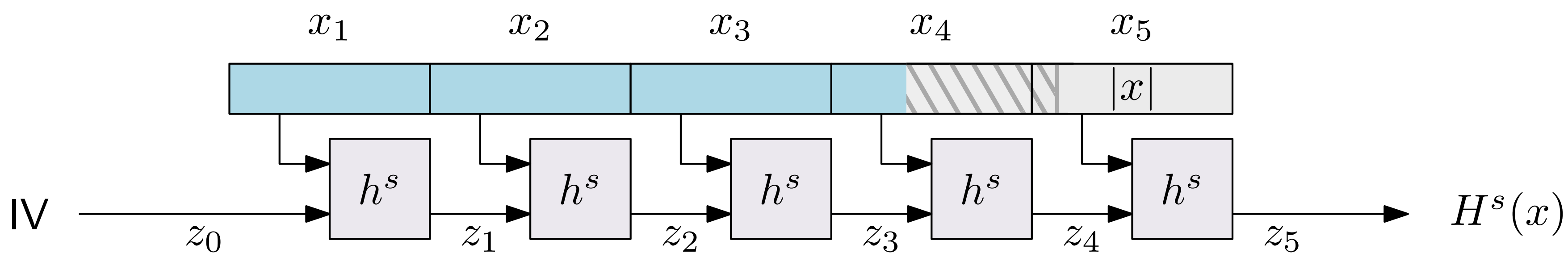

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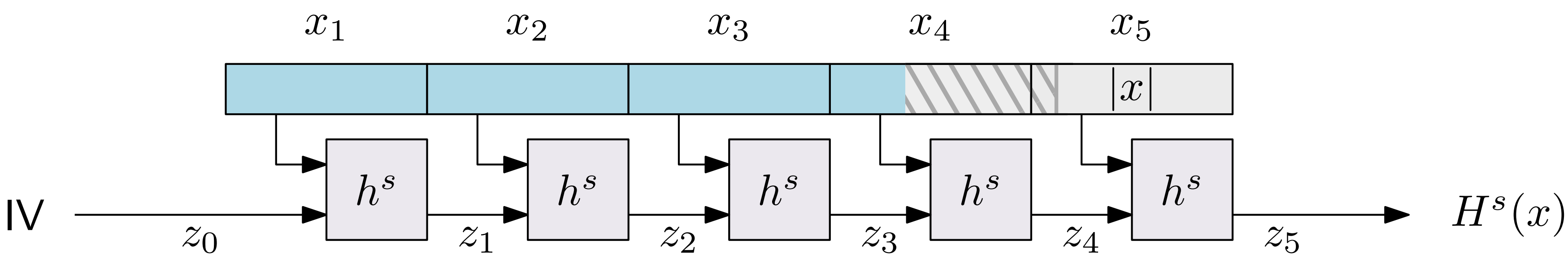

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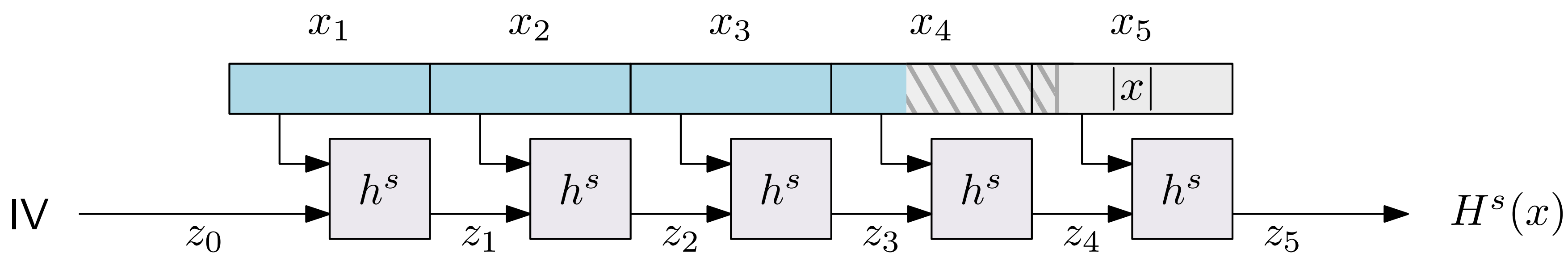

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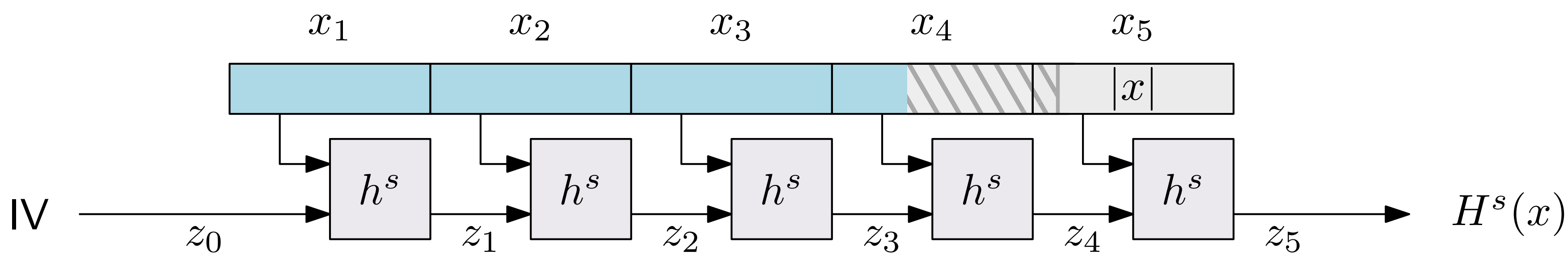

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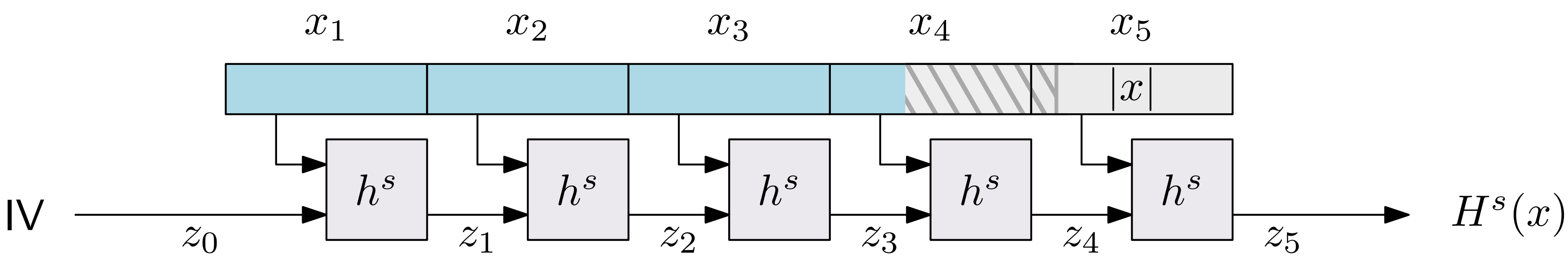
The Merkle-Damgård Transform



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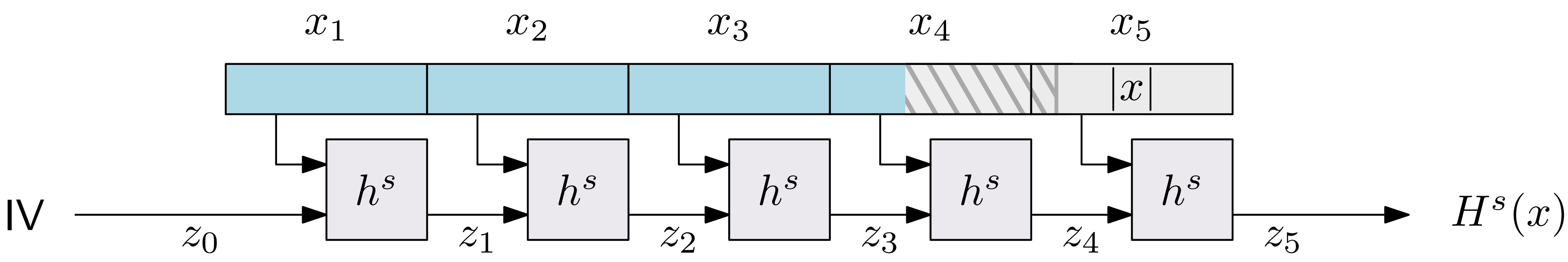

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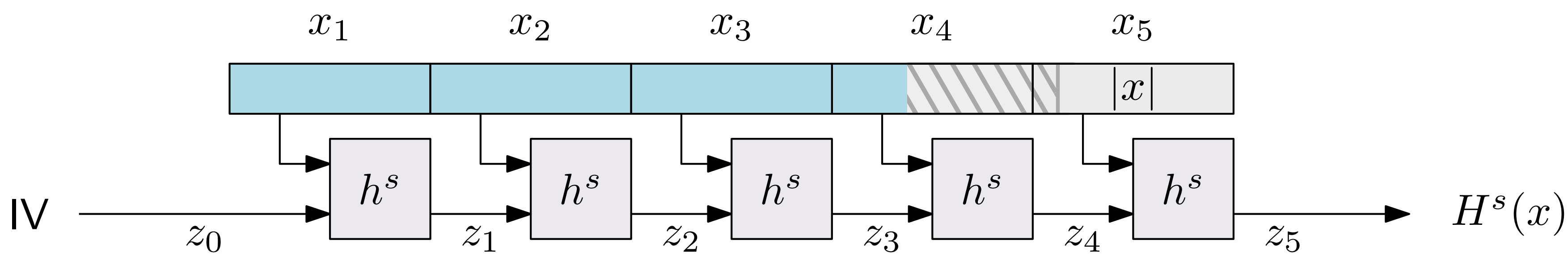

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Then $h^s(z_{i-1} \| x_i) = z_i = z'_i = h^s(z'_{i-1} \| x'_i)$

$\square$

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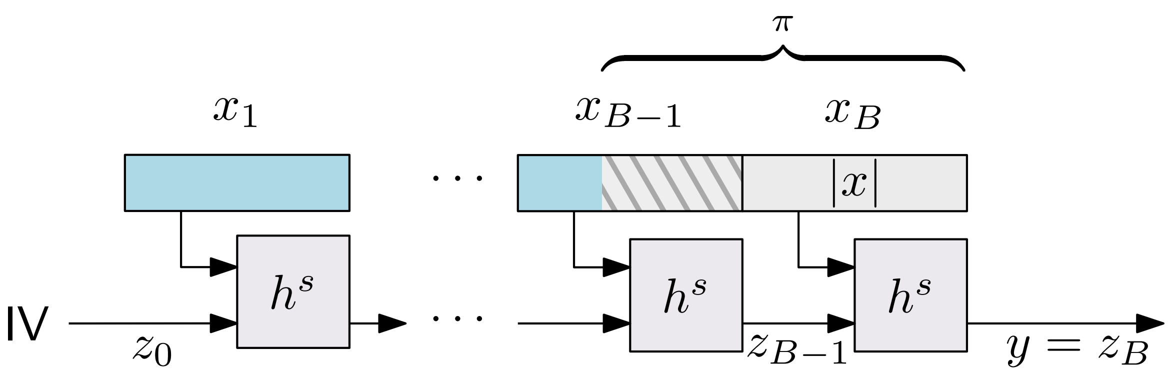

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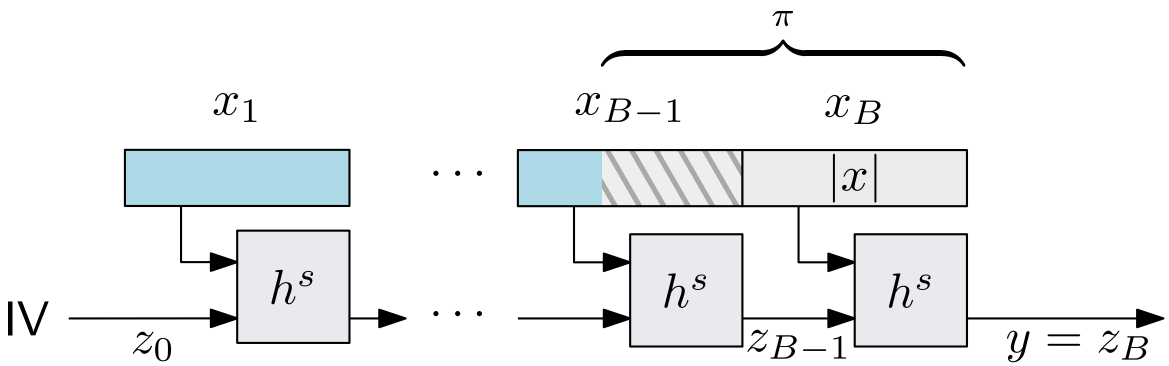

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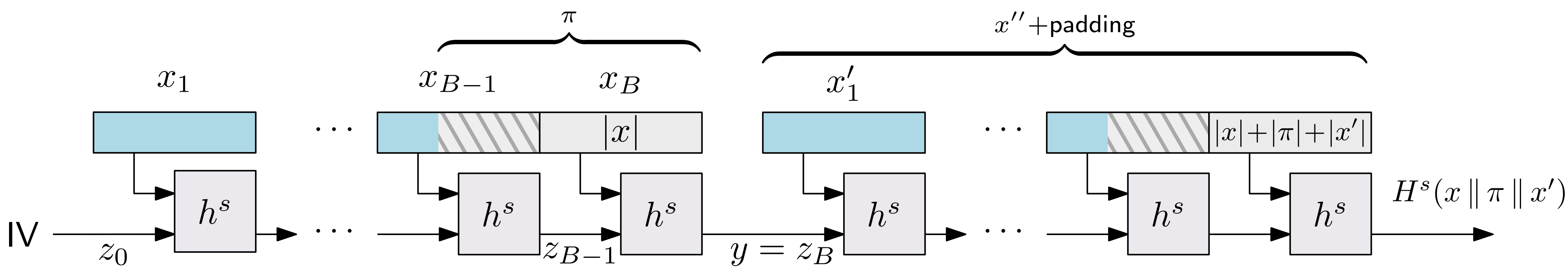

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Practical construction of hash functions are unkeyed...

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- 128 bit digest
- Birthday attack (μs), Preimage attack (theoretical)

MD5

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SHA1

- 160 bit digest
- Birthday attack (SHAttered: 110 years of computing time on GPU), improved chosen-prefix attacks

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- Actually a family of algorithms: 224, 256, 384, and 512 bit digests
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Hash-and-Mac

Suppose that we have:

- A fixed-length MAC $\Pi' = (\mathsf{Gen}', \mathsf{Mac}', \mathsf{Vrfy}')$ for messages of length ℓ
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Theorem: if Π' is a secure MAC for messages of length ℓ and $\mathcal{H}$ is collision resistant, then the hash-and-mac construction Π is a secure MAC

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We will show that an adversary $\mathcal{A}$ that breaks the security of Π can be used to either break the security of Π' or to find a collision in $\mathcal{H}$ (possibly both).

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At least one of the summands is non-negligible

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This contradicts the collision resistance of $\mathcal{H}$!

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Adversary $\mathcal{A}''(1^n)$:

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In practice:

- Prove security in the Random Oracle model
- Replace the Random Oracle with a concrete hash function
- Cross your fingers...

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- There are no known "natural" schemes that have been attacked while proven secure in the Random Oracle model
- A security proof is in the random oracle model is better than no security proof at all... maybe?

Applications of Hash Functions: Fingerprinting & Deduplication

If H is a collision-resistant hash function, and x is a (part of) a file, then we can think of $H(x)$ as a unique identifier of that (part of the) file

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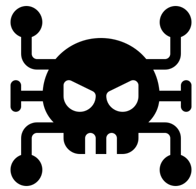

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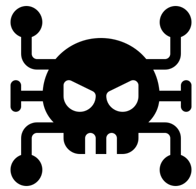

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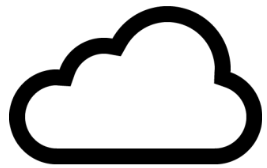

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Peer-to-peer file sharing: Hashes are used to uniquely identify files (and chunk of files) in peer-to-peer file-sharing networks.

`magnet:?xt=urn:btih:C9A337562CB0360FD6F5AB40FD2B1B81D5325DBD`

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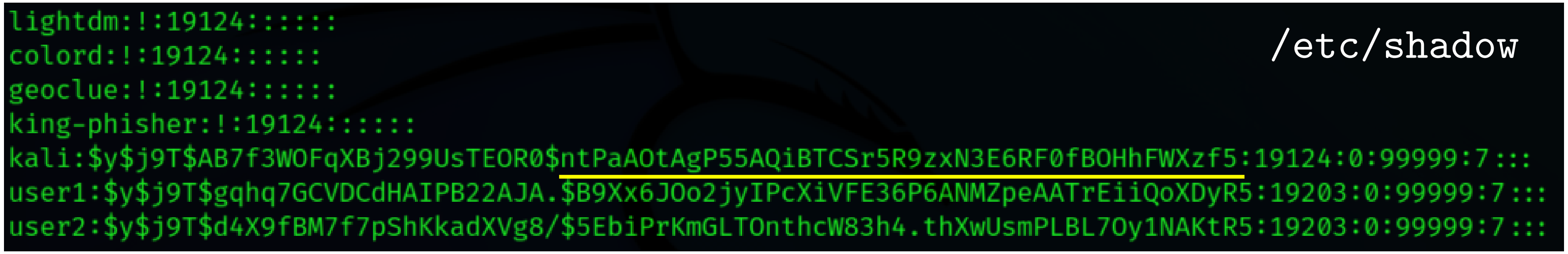

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kali:$y$j9T$AB7f3WOFqXBj299UsTEOR0$ntPaAOtAgP55AQiBTCSr5R9zxN3E6RF0fBOHhFWXzf5:19124:0:99999:7:::
user1:$y$j9T$gqhq7GCVDCdHAIPB22AJA.$B9Xx6JOo2jyIPcXiVFE36P6ANMZpeAATrEiiQoXDyR5:19203:0:99999:7:::
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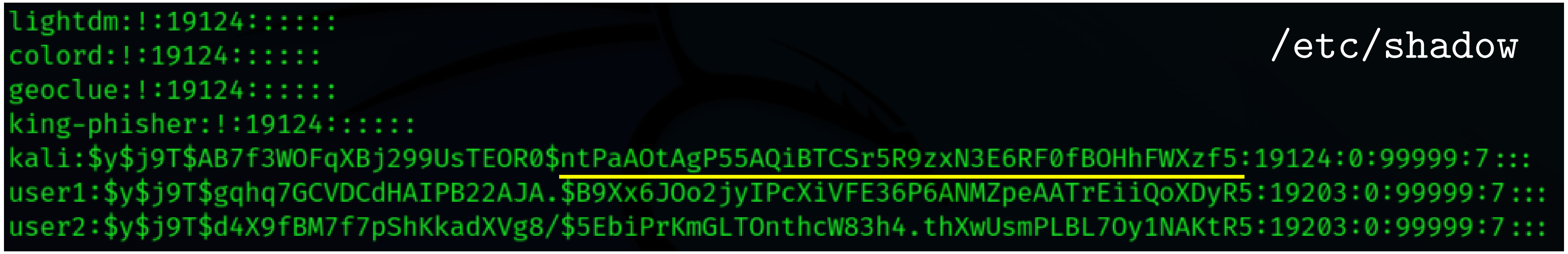

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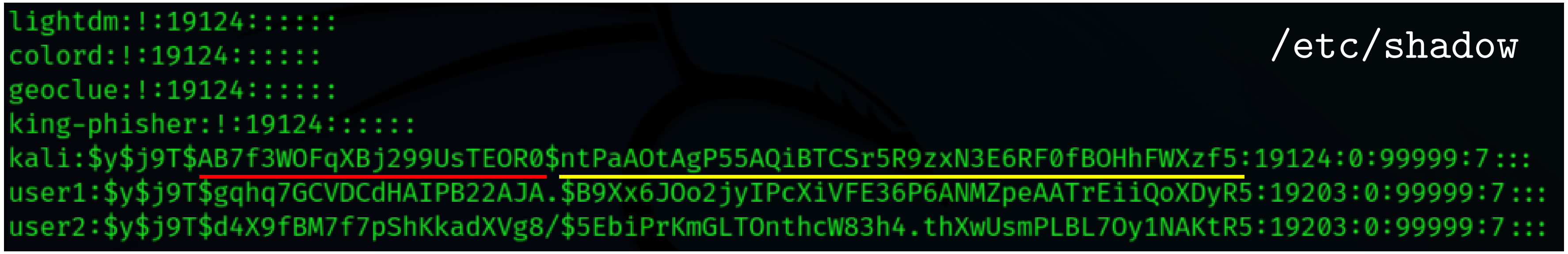

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- **Solution:** pick a random string z called **salt**.
  Compute $y = H(z \| x)$ and store the pair (z, y).

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Choose $k = H(x)$

- If H is a random oracle, then $H(x)$ is uniform as long as the attacker does not query H with x.
- If an attacker makes q queries to $H(\cdot)$, it will query H with x with probability at most $q \cdot 2^{-m}$.

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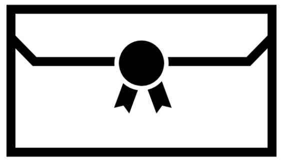
In some sense: a digital equivalent of placing a message in a sealed envelope (hiding), which is opened at a later time



Applications of Hash Functions: Commitment Schemes

A commitment scheme allows a party to

- Commit to a value m
- At a later time, "open" the commitment to reveal m

The commitment scheme must be:

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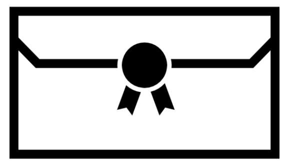

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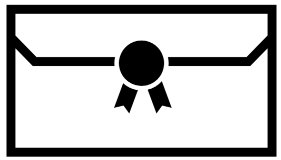

To commit to m:

- Pick a random string r and compute com $= H(m \,\|\, r)$

To open the commitment:

- Send m and r. Given some m' and r' one can easily check whether com $= H(m' \,\|\, r')$

Applications of Hash Functions: Merkle Trees

- Alice wants to compute some fingerprint h of a list of strings $\langle x_1, \ldots, x_t \rangle$ to send to Bob
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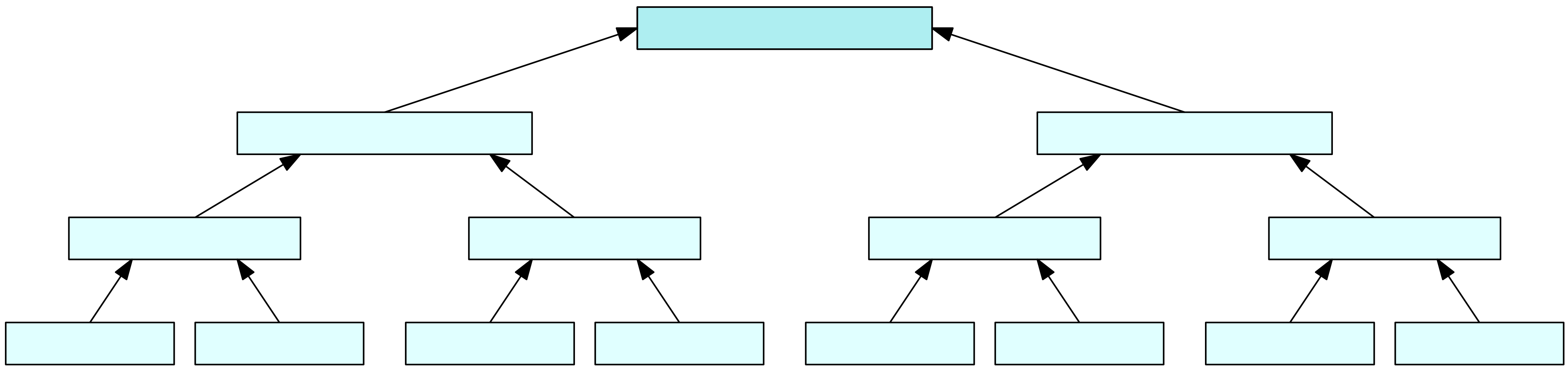
Solution 3: Merkle trees

Applications of Hash Functions: Merkle Trees

- Build a complete binary tree with t leaves
- Each node u stores a hash



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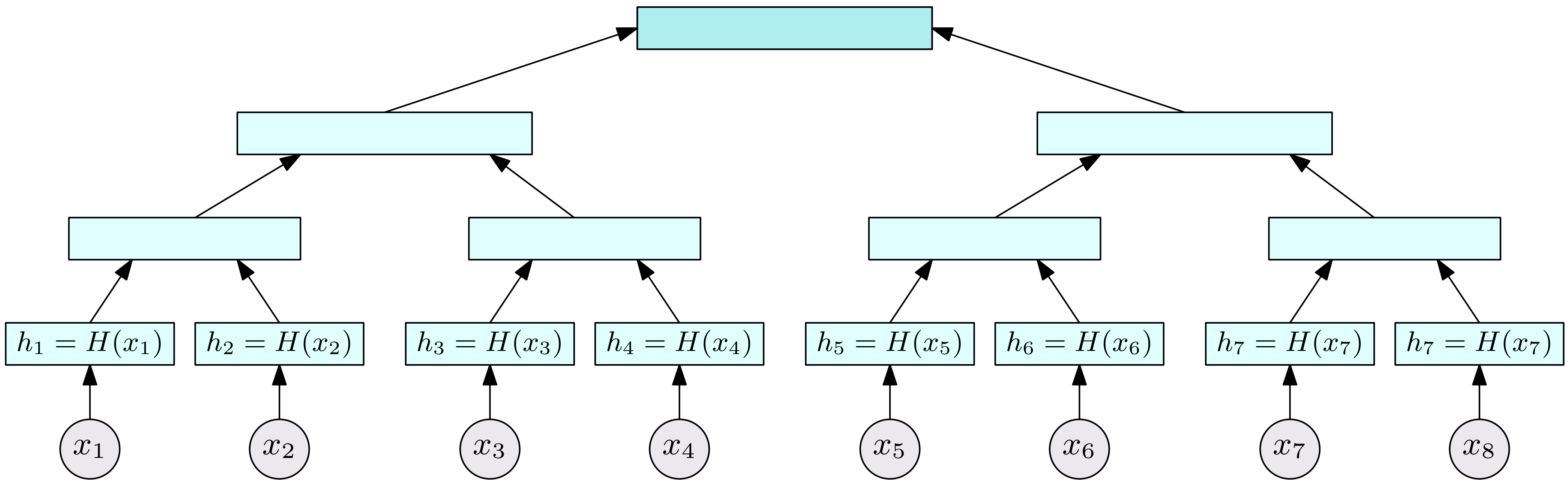

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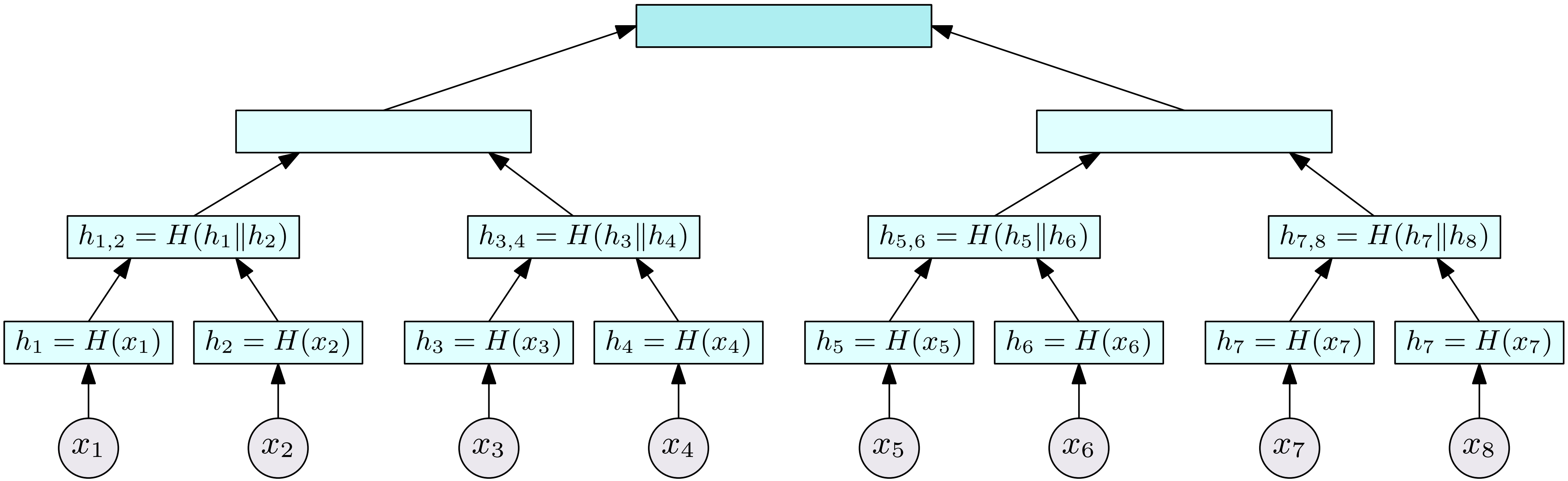

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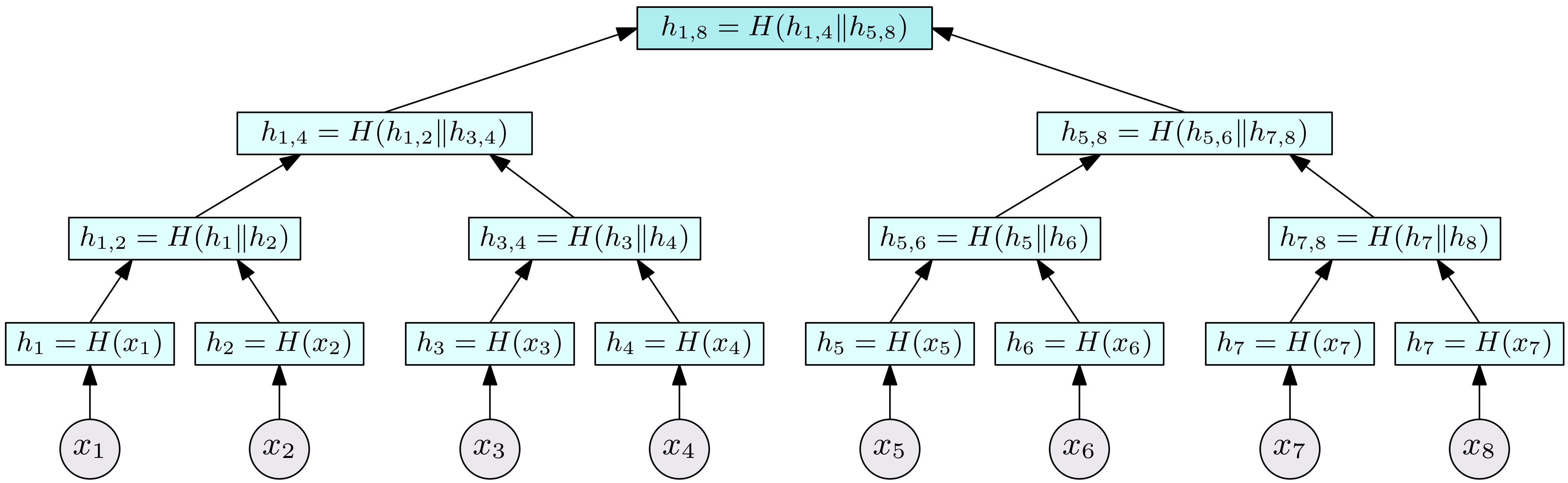

Applications of Hash Functions: Merkle Trees

- Build a complete binary tree with t leaves
- Each node u stores a hash
- The final hash of the whole list $\langle x_1, \ldots, x_t \rangle$ is the hash stored in the root.
- The hash stored in the i-th leaf is $H(x_i)$
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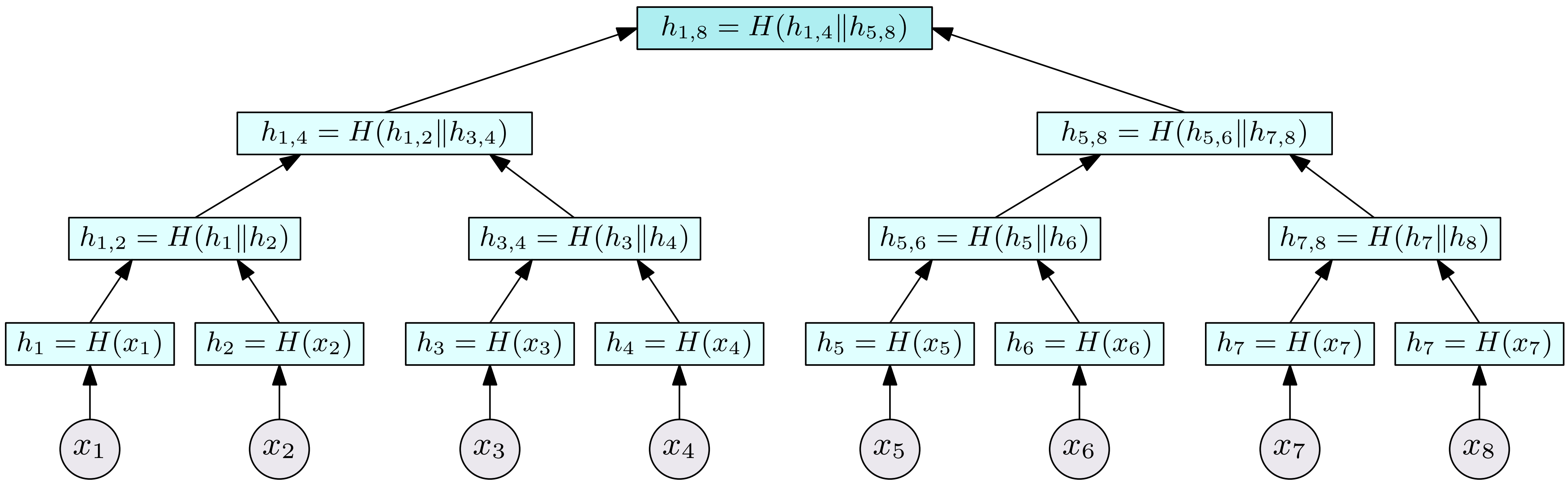

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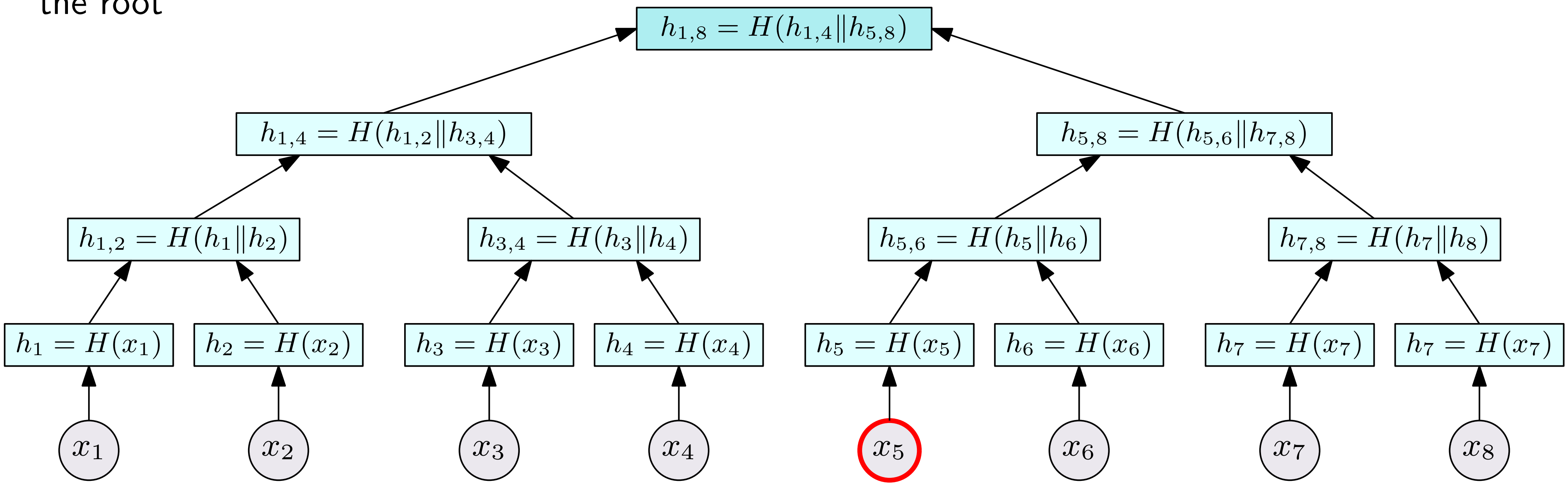

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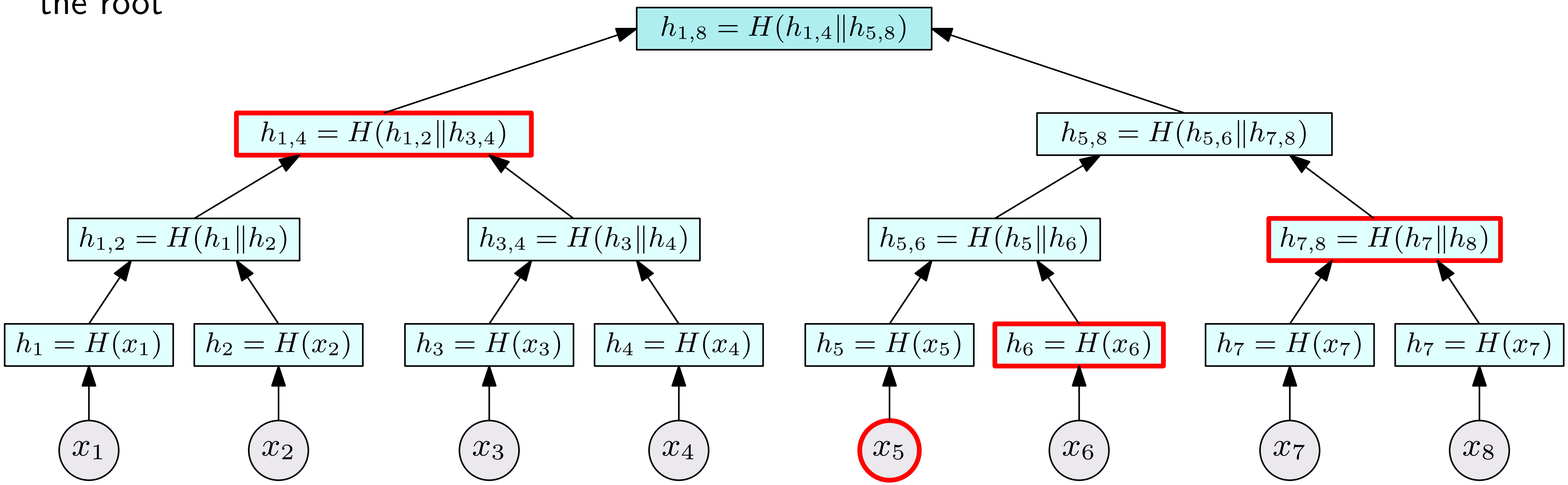

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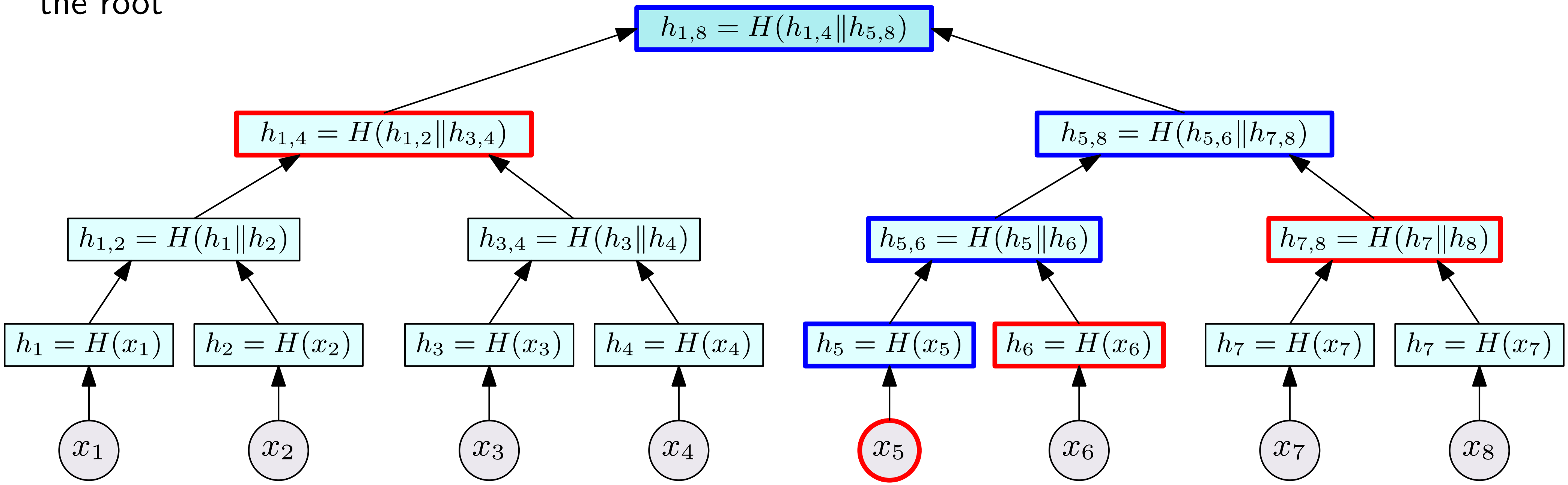

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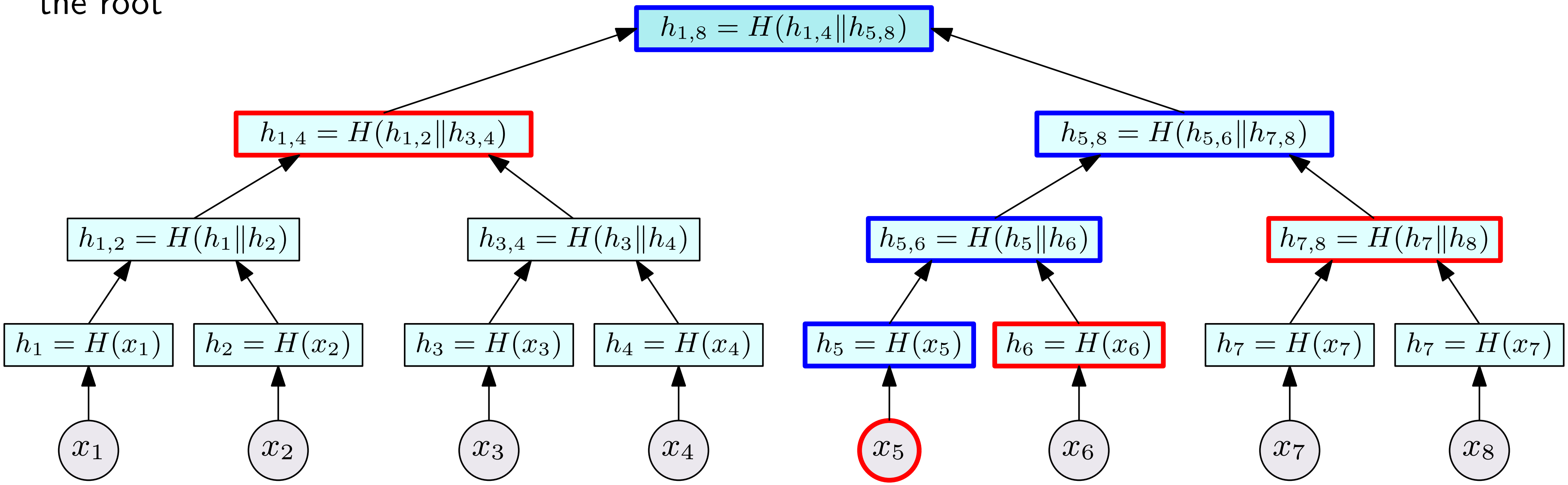

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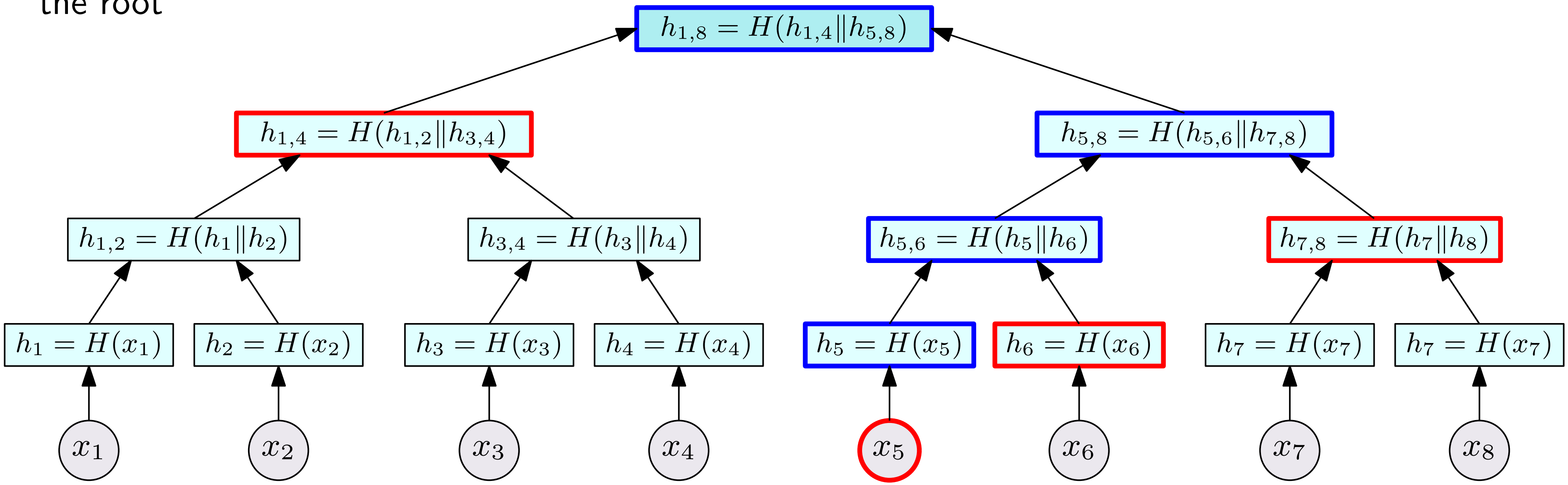

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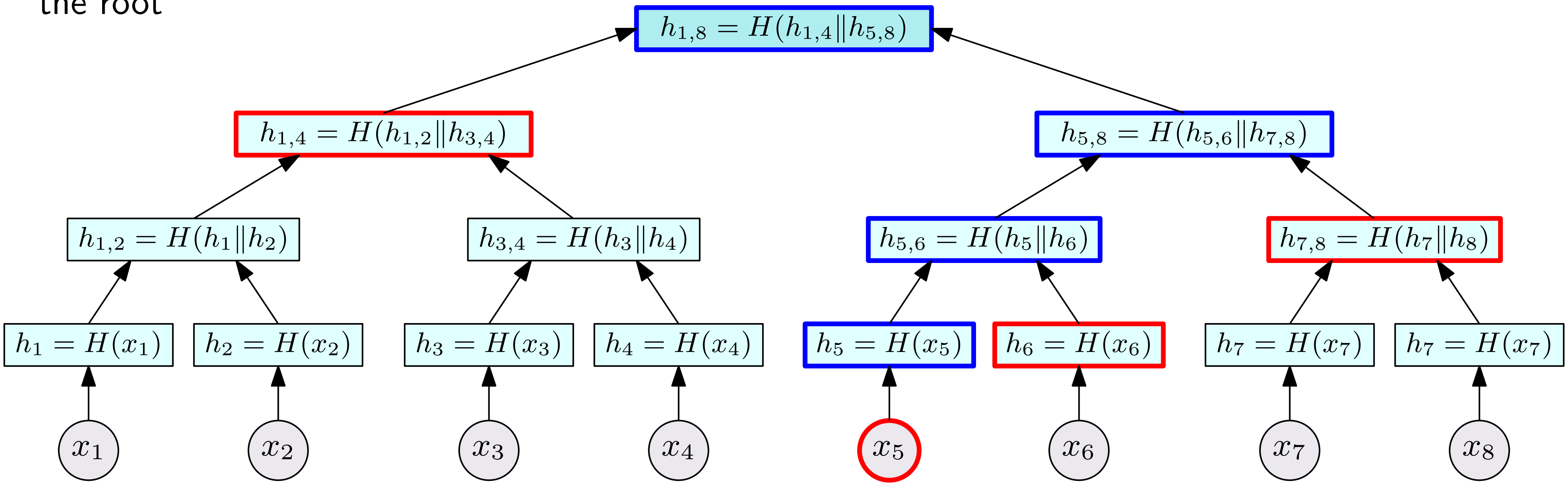

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Commitment scheme with partial reveal!

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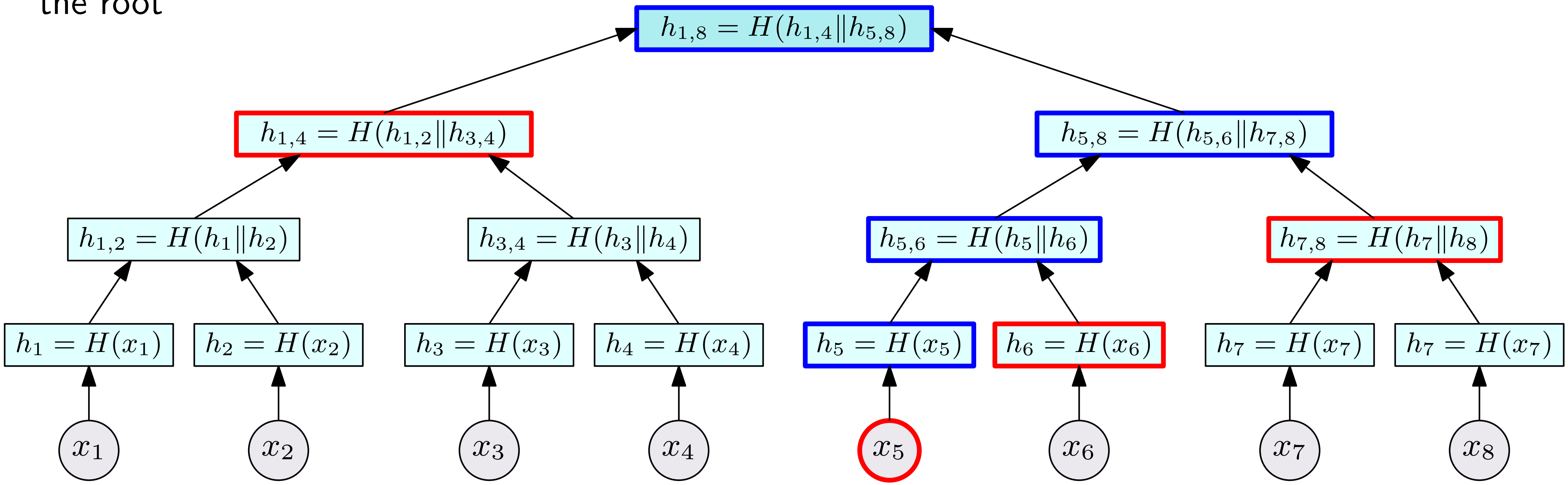

If H is collision resistant, then the hash function computed by the above Merkle tree construction is collision resistant for any fixed t.

The construction can be generalized to handle nonconstant t.

Merkle Trees: Bitcon & SPV

In Bitcoin:

- Each block of the blockchain contains list of transactions $x_1, \ldots, x_t$
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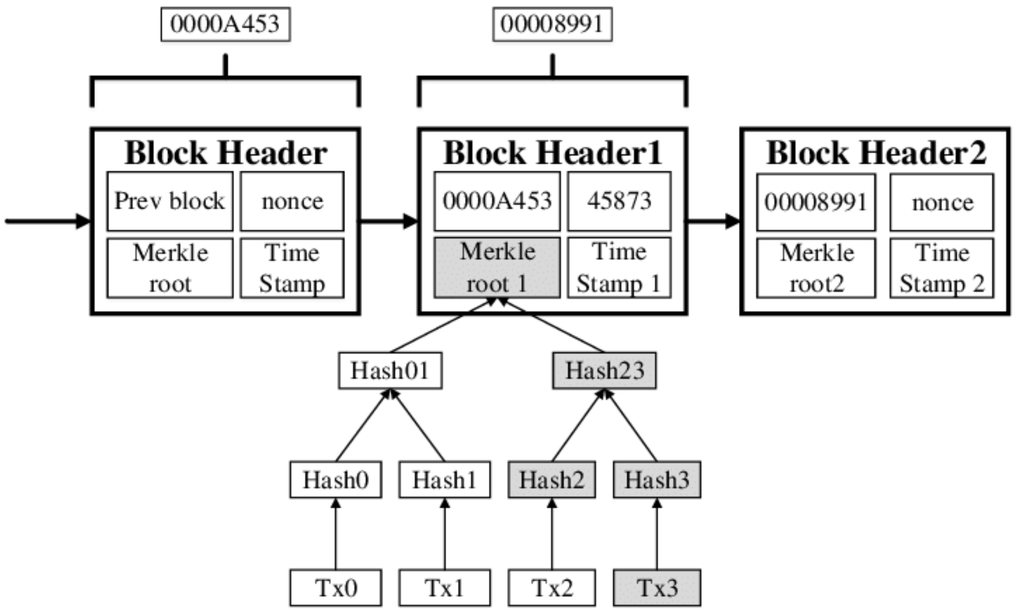

Credit: W. Dai, J. Deng, Q. Wang, C. Cui, D. Zou, H. Jin

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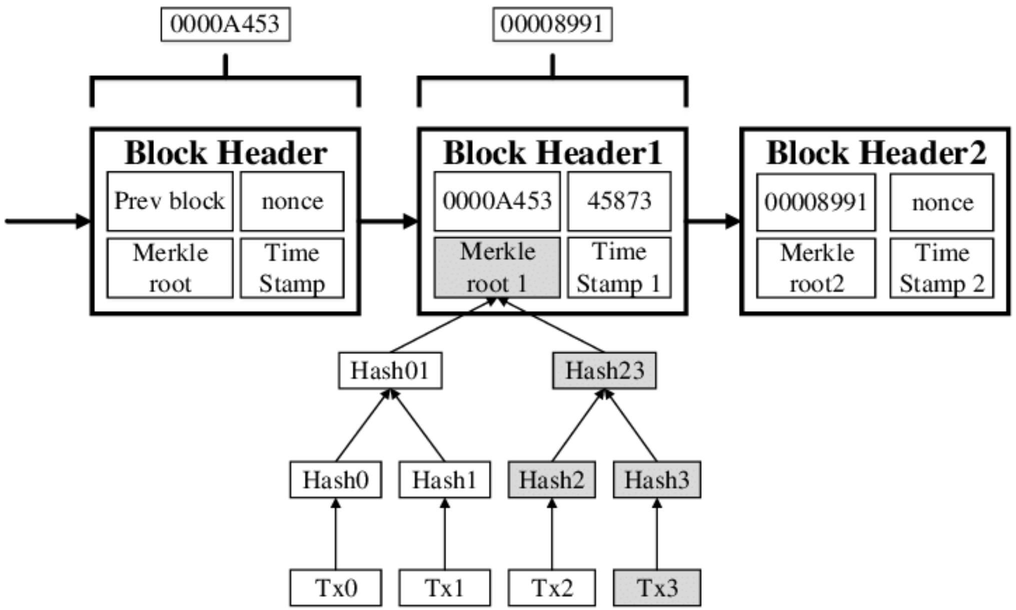

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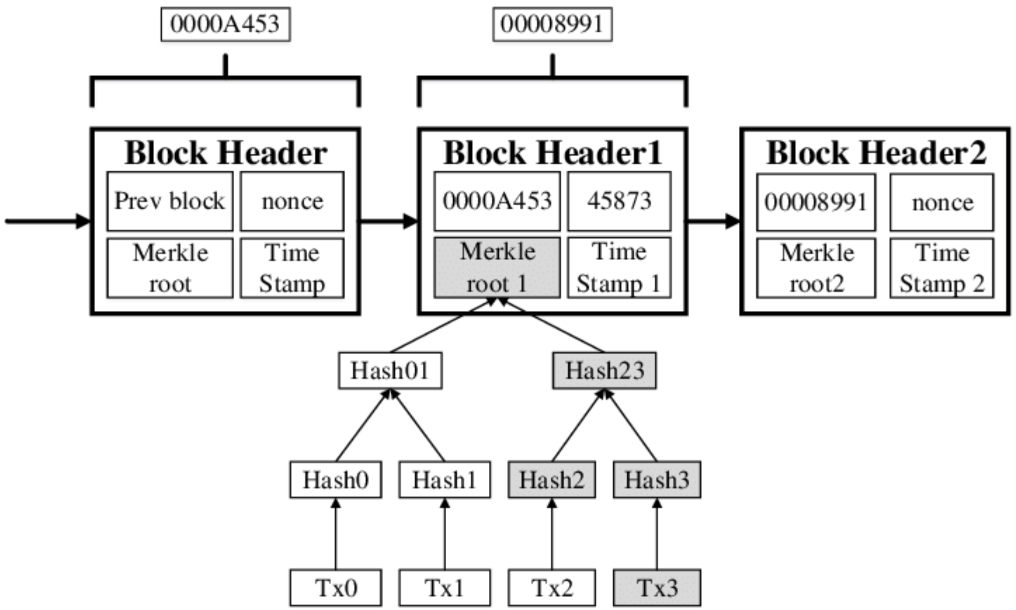
- The hash of the block is computed using a Merkle tree
- Some nodes (called SPV nodes, from simple payment verification) only store the hashes of the blocks in the blockchain (and not their contents)
- Easy to convince a SPV node that a given transaction belongs to a block in the blockchain



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