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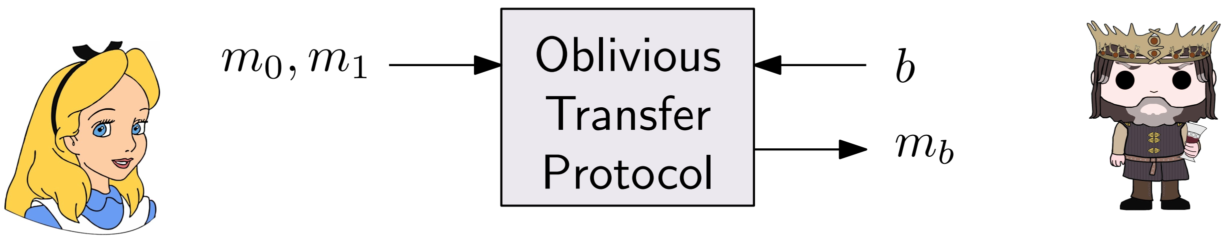
- In the oblivious transfer protocol Alice has two messages m_0, m_1 of length $\ell(n)$
- Bob wants to learn one of them, say m_b, without revealing which one he is interested in to Alice
- Alice wants to be sure that Bob learns exactly one of the two values



Reminder: DDH-Based Key Encapsulation Mechanism

Gen(1^n):

- Run $\mathcal{G}(1^n)$, where $\mathcal{G}$ is a group generation algorithm, to obtain (G, q, g) where G is a group of order q and $g \in G$ is a generator
- Pick some key derivation function $H : G \to \{0,1\}^{\ell(n)}$
- Choose a uniform x u.a.r. from $\{0, \ldots, q-1\}$
- Compute $h = g^x$
- Output (pk, sk) where $pk = (G, q, g, h, H)$ and $sk = (G, q, g, x, H)$.

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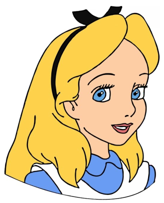
$\mathsf{Encaps}_{pk}(1^n)$:

- Here $pk = (G, q, g, h, H)$
- Choose y u.a.r. from $\{0, \ldots, q-1\}$
- Output the pair (c, k) with $c = g^y$ and $k = H(h^y) = H(g^{xy})$



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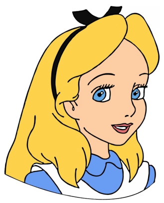

Decaps$_{sk}(c)$:

- Here $sk = (G, q, g, x, H)$
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Reminder: DDH-Based KEM & Hybrid Encryption

We can build a CPA-secure PKE scheme by combining a CPA-secure KEM with an EAV-secure DEM

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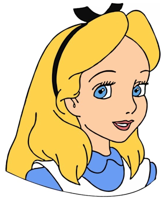
The resulting scheme is as follows:

Gen(1^n):

- Pick a group G, its order q, a generator $g \in G$, a key-derivation function $H : G \to \{0, 1\}^{\ell(n)}$
  We think of these as fixed public values agreed upon in advance between Alice and Bob.
- Pick a random $x \in G$, the **public-key** is $h = g^x$ and the **secret-key** is x

Enc$_h(m)$:

- Choose a uniform $y \in \{0, \ldots, q - 1\}$. Return the pair $c = (g^y, H(h^y) \oplus m)$



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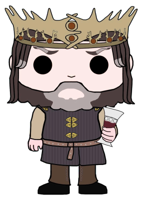

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Idea:

- We pick **two** public keys h_0, h_1 for Bob
- We ensure that Bob knows the secret key x_b corresponding to **exactly** one of the public keys (of his choice)
- Alice encrypts m_0 with h_0 and m_1 with h_1
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How can Bob "prove" to Alice that he knows exactly one private key?

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Secure under the Random Oracle model and the CDH assumption

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Multiparty Computation

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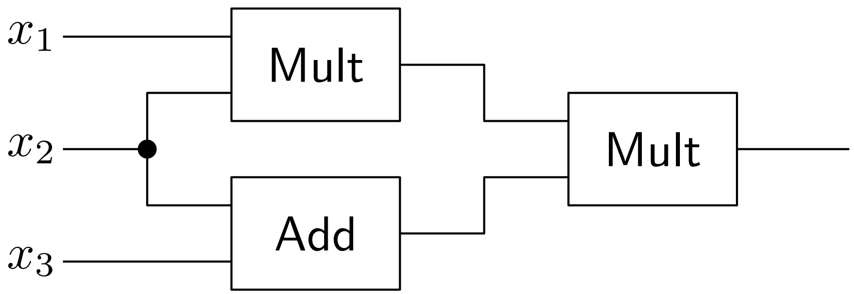

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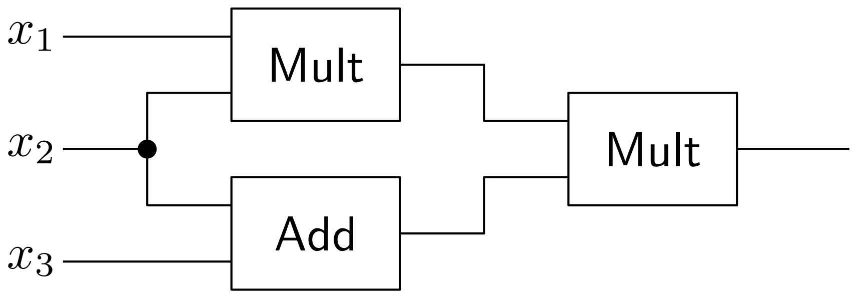

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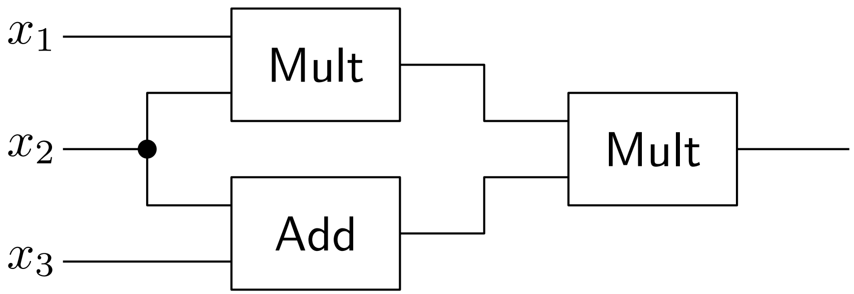

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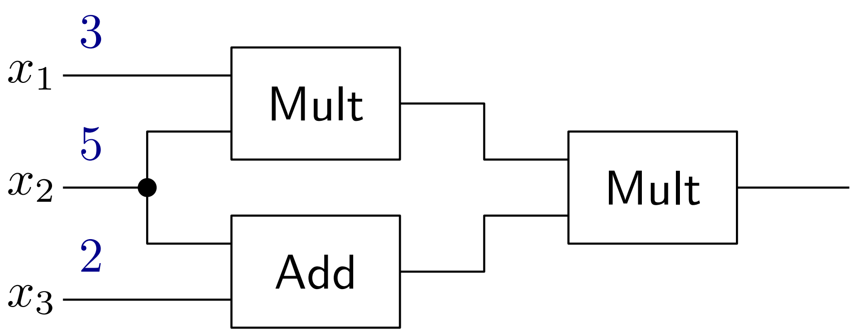

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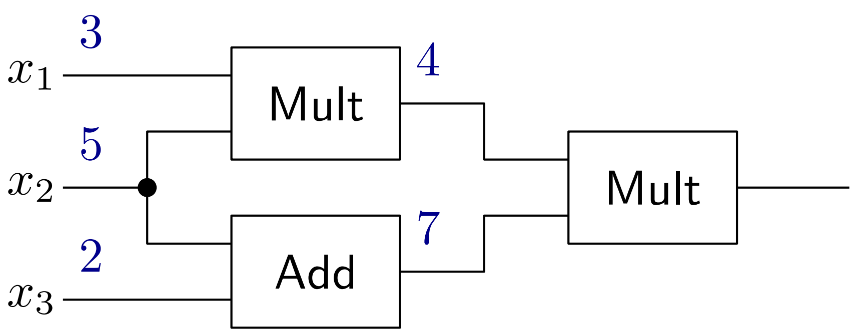

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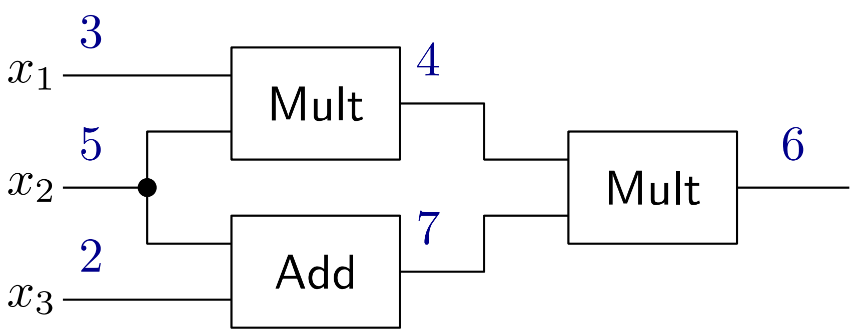

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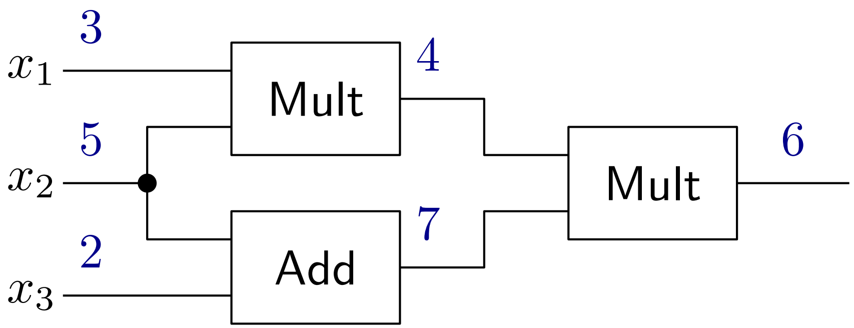

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$\Leftarrow$ **We can simulate an arithmetic circuit with a Boolean circuit:**

- Replace each wire of the arithmetic circuit with $\lceil \log p \rceil$ Boolean wires
- Replace each Addition/Multiplication gate with a Boolean circuit that computes the Sum/Product of the inputs modulo p

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- Each party shares its input with all other parties using Shamir's k-out-of-n threshold secret sharing scheme
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Multiparty Computation

How do we “garble” and arithmetic circuit for multiple parties?

How do we evaluate it?

> We can perform computation on shares, without having to recover their secrets first!

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- The parties combine their output shares and recover the value of $f(x_1, \ldots, x_n)$

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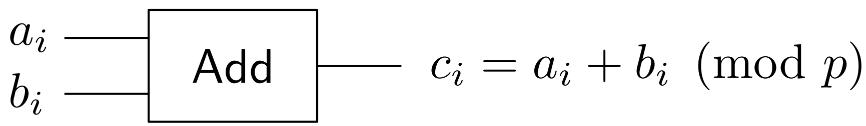

Addition gates do not require any special care!

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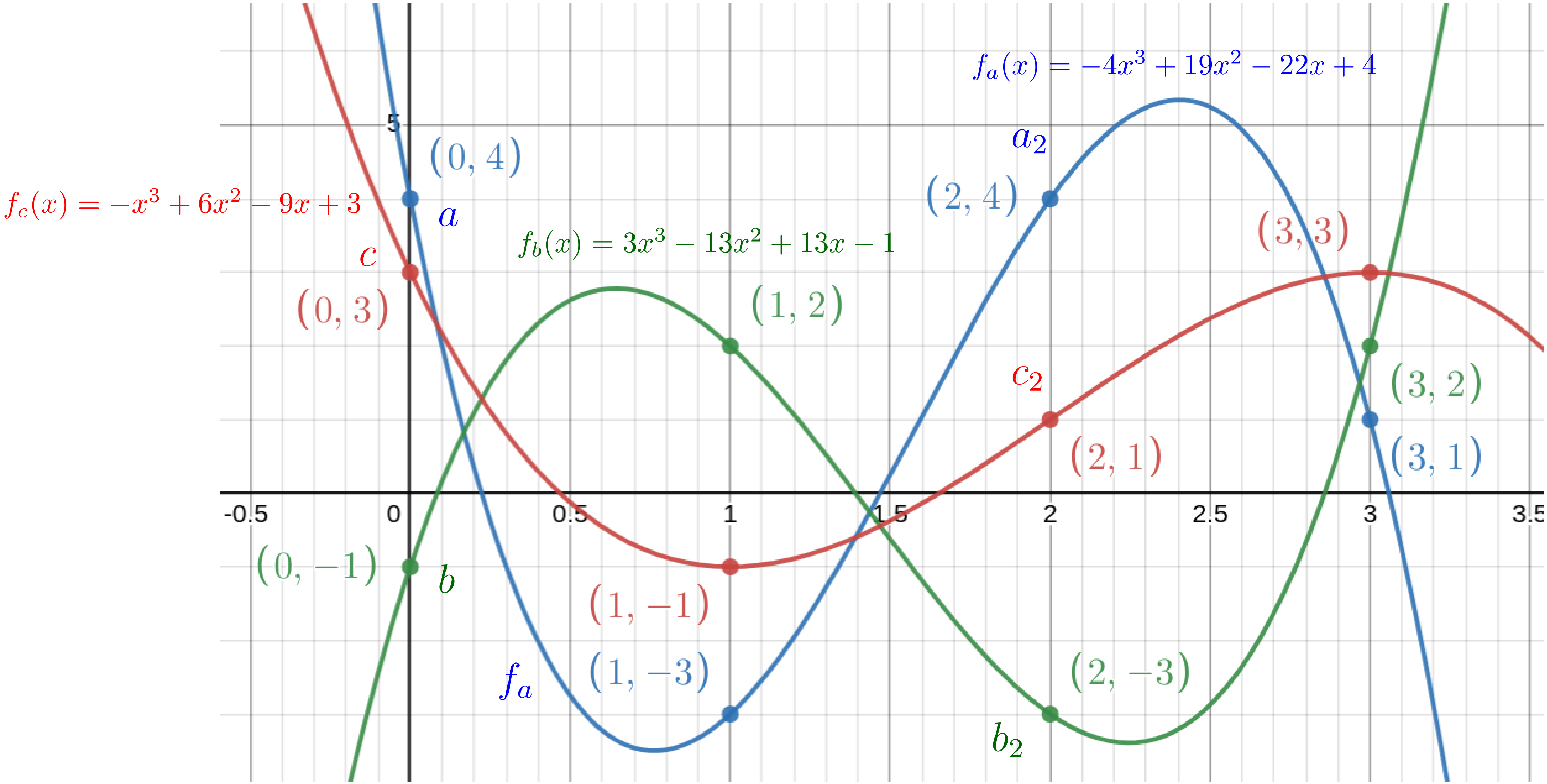

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We need to use another property of interpolating polynomials...

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Lemma: Given distinct $x_1, \dots, x_n$, define $r_j = \prod_{\substack{i=1,\dots,n \\ i \neq j}} x_i \cdot (x_i - x_j)^{-1}$.

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- h is a polynomial of degree at most $k-1$ s.t. $h(0) = c$ and c_i is exactly the i-th share of h