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Can we relax the security definition in a meaningful way?

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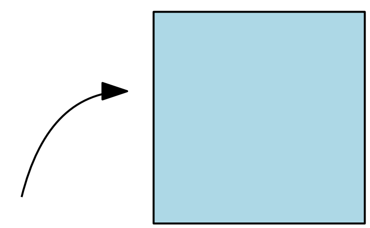

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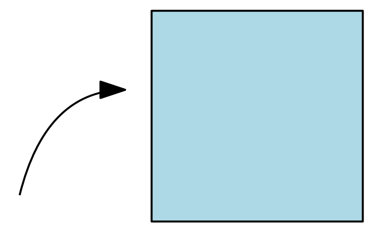

Do we need to be concerned?

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We relax perfect secrecy in two ways:

- We allow secrecy to fail with some tiny probability
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Our starting point is the following (equivalent) definition of perfect secrecy:

Definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *with message space* $\mathcal{M}$ *is* **perfectly indistinguishable** *if for every* $\mathcal{A}$ *it holds:*

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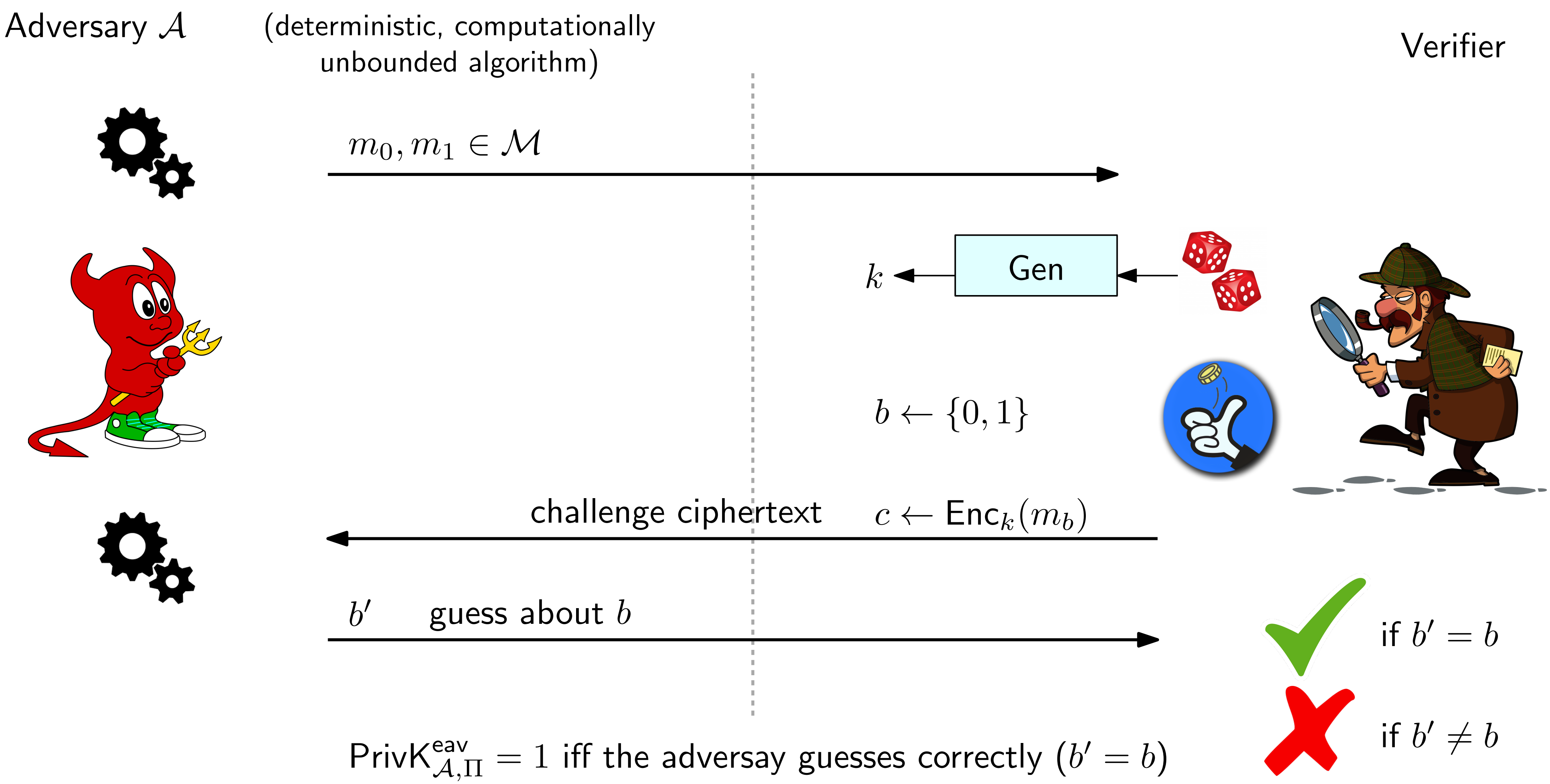
Two possible approaches:

- **Concrete**
- **Asymptotic**

Reminder: Perfect indistinguishability



Computational secrecy (concrete)

Candidate definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *is* (t, ε)*-indistinguishable if for every attacker* $\mathcal{A}$ *running in time at most* t*, it holds that:*

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Observation: $(\infty, 0)$-indistinguishability is equivalent to perfect indistinguishability

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Does not lead to a clean theory

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  Measure probabilities and running times as a function of n

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"$\eta(n)$ approaches 0 faster than the inverses of all polynomials in n"

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As a special case, the product of two negligible functions is negligible

Negligible and polynomially bounded functions

Which of the following functions are polynomially bounded? Which are negligible?

$n^2 + 4n - 2$	n^{100}	$n^3 + \cos(n)$	$n!$
$\frac{1}{n^{10}} + 2^{-n/2}$	2^n	3^{-n}	$\sqrt[3]{n} + \frac{1}{n}$
$n^{-n} \cdot (n^5 + n^2)$	$2^{\sqrt{n}}$	$\sqrt{n}$	$42 - \frac{1}{1+\log n}$
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$\frac{1}{n^{10}} + 2^{-n/2}$	2^n	3^{-n}	$\sqrt[3]{n} + \frac{1}{n}$
$n^{-n} \cdot (n^5 + n^2)$	$2^{\sqrt{n}}$	$\sqrt{n}$	$42 - \frac{1}{1+\log n}$
$4^{\sqrt{\log n}}$	n^{-5}	$2^{-\log n \cdot \log\log n}$	$(1 + \frac{1}{n})^n$

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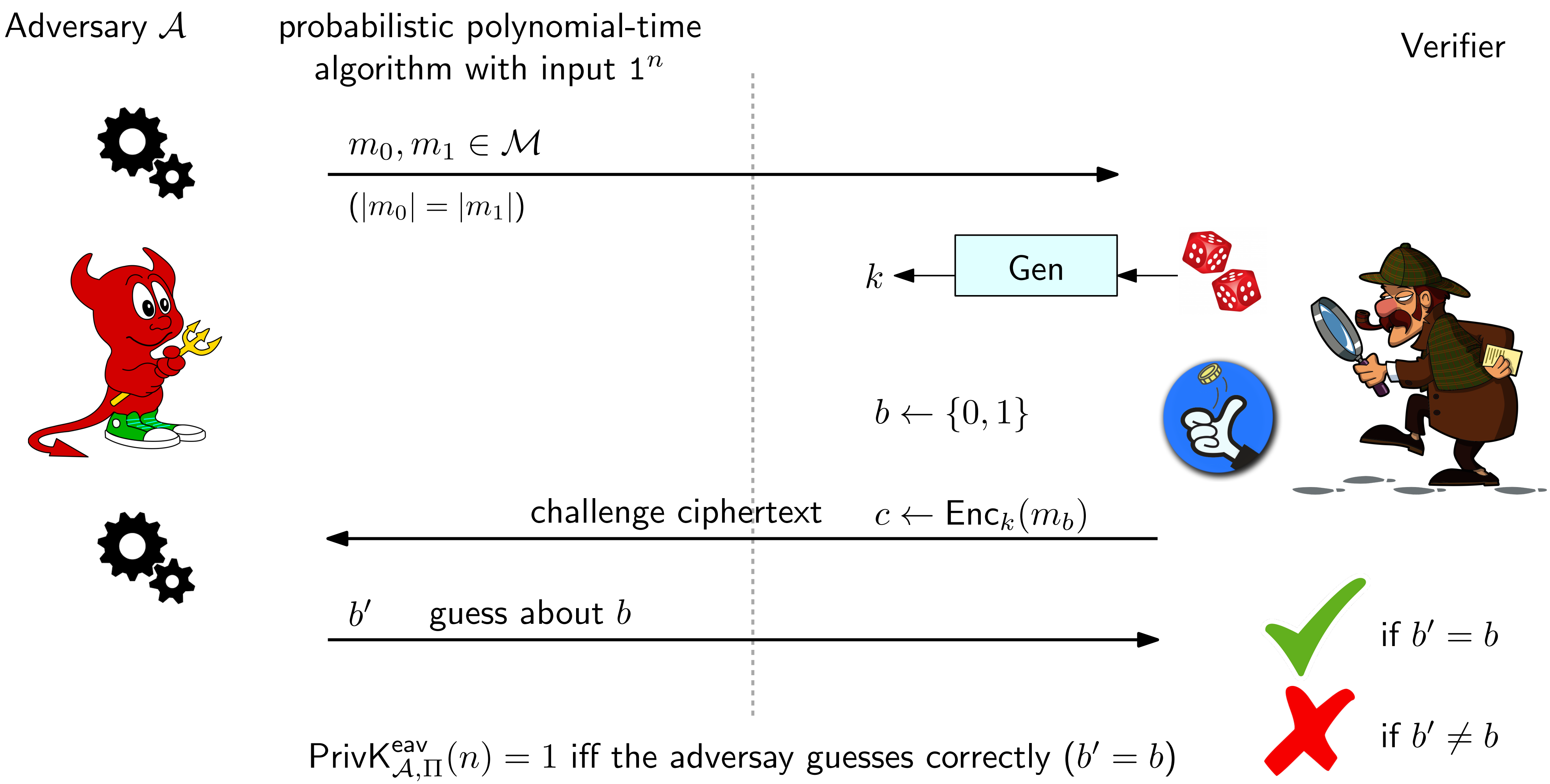
If $M = \{0,1\}^{\ell(n)}$ then $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is a **fixed-length** private-key encryption scheme

(for messages of length $\ell(n)$)

The adversarial indistinguishability experiment, revisited



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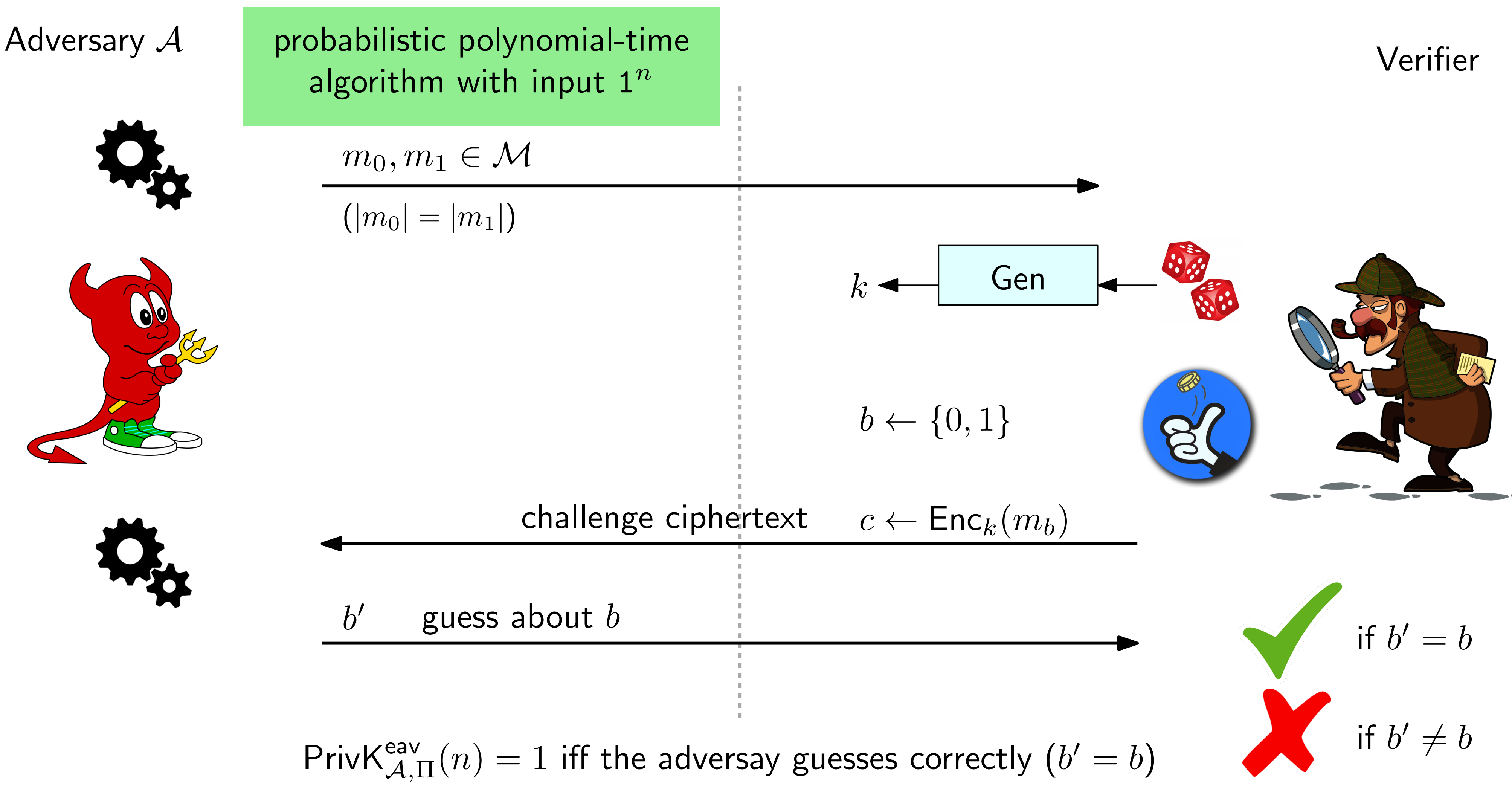

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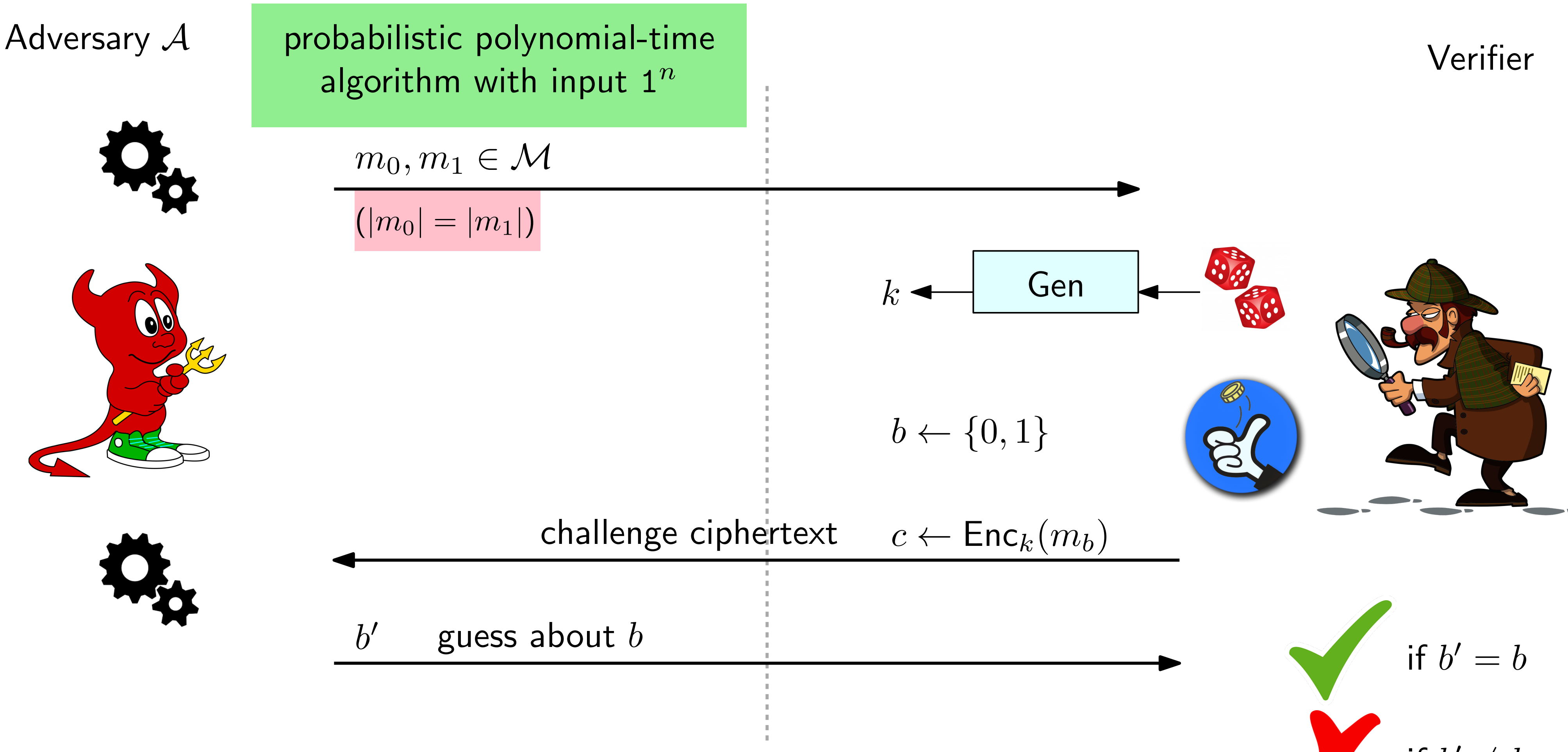


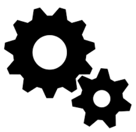
$\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1$ iff the adversay guesses correctly $(b' = b)$

The adversarial indistinguishability experiment, revisited

Adversary $\mathcal{A}$

probabilistic polynomial-time algorithm with input 1^n

Verifier

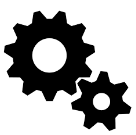


$m_0, m_1 \in \mathcal{M}$

$(|m_0| = |m_1|)$

k ← Gen ←

$b \leftarrow \{0, 1\}$



challenge ciphertext $c \leftarrow \mathsf{Enc}_k(m_b)$

b' guess about b

if $b' = b$

if $b' \neq b$

Notation includes the security parameter

$\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1$ iff the adversay guesses correctly $(b' = b)$

Computational indistinguishability (asymptotic)

Definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *has indistinguishable encryptions in the presence of an eavesdropper (is* **EAV-secure***) if, for every probabilistic polynomial-time adversary* $\mathcal{A}$*, there is a negligible function* ε *such that:*

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Observation: perfect indistinguishability implies EAV-security

Example 1

Consider a scheme where:

- $\mathsf{Gen}(1^n)$ returns a key chosen uniformly at random in $\{0,1\}^n$
- The best possible adversary $\mathcal{A}$ performs a brute-force search over the key space
- If the running time of the adversary is $t(n)$ then:

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Is this scheme EAV-secure? Yes!

For all polynomial running times $t(n)$, all functions in $O\left(\frac{t(n)}{2^n}\right)$ are negligible

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Consider a scheme where:

- $\mathsf{Enc}_k(m)$ runs in $n^2 \cdot |m|$ steps
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A increase in computing power resulted in a more difficult attack!

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How large do we need to choose n?

n	48	64	128	256	512	1024
running time	2.5 months	6 months	4 years	32 years	255 years	2041 years
probability of success	1 in 256	$\approx$ 1 in 17 mil	$\approx$ 3 in 10^{26}	$\approx$ 3 in 10^{65}	$\approx$ 1 in 10^{142}	$\approx$ 2 in 10^{296}

Computational secrecy (asymptotic)

We relax perfect secrecy in two ways:

- We allow secrecy to fail with some tiny probability ⟵ probabilities that are *negligible* in n
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Not negligible!

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  In Google maps, the map tiles are compressed and (essentially) static. The size of the ciphertext can be used to determine the viewed location

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- . . . but it requires long keys
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It depends...

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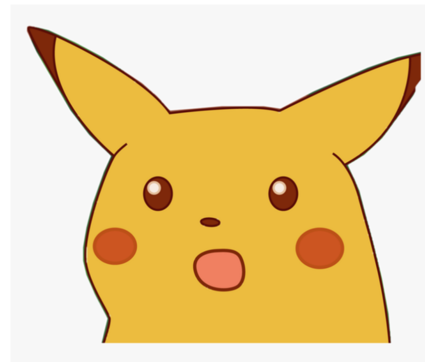

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Equivalent assumption: one-way functions (OWF) exist

Inf. Functions that are easy to compute but hard to invert even "on average"

Impagliazzo's Five Worlds

A Personal View of Average-Case Complexity
Russel Impagliazzo
Proceedings of the 10th annual IEEE conference on Structure in Complexity Theory, 1995

Donald Knuth
Algorithms guy

Whitfield Diffie
Crypto guy

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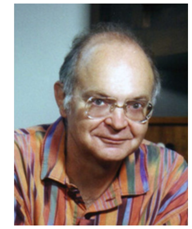

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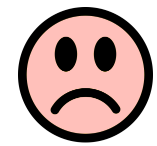

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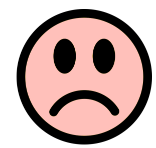

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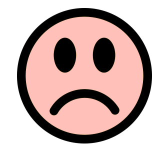

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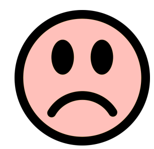

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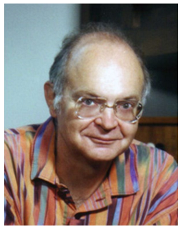
A Personal View of Average-Case Complexity
Russel Impagliazzo
Proceedings of the 10th annual IEEE conference on Structure in Complexity Theory, 1995



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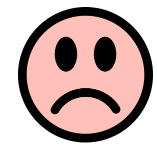

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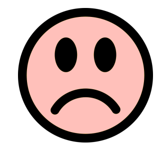

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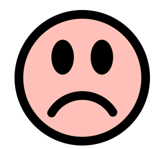

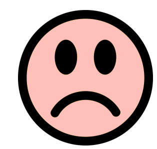

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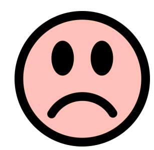

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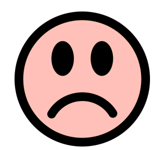

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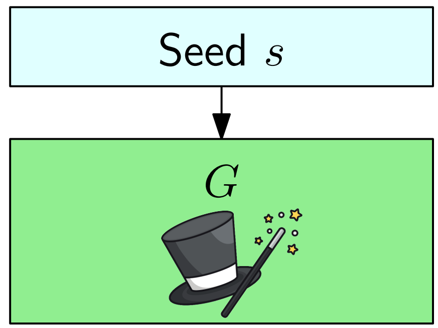

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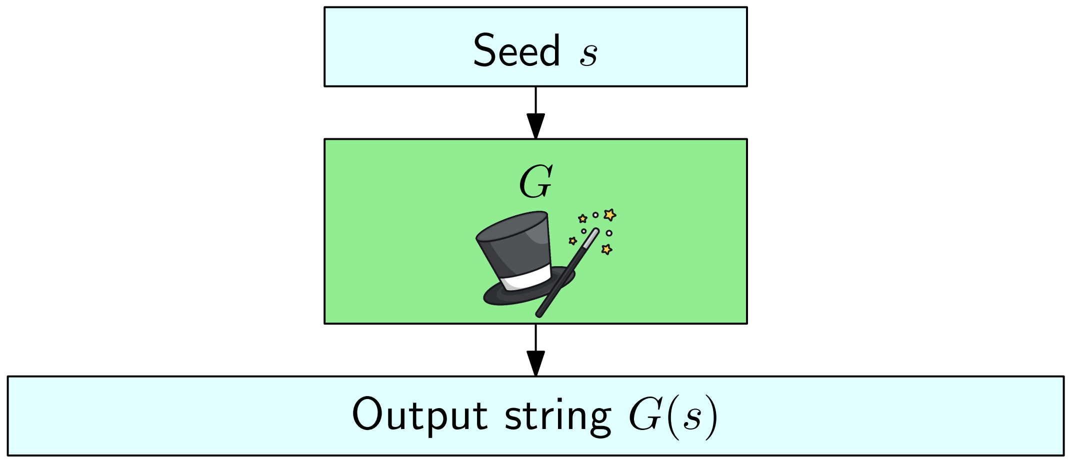

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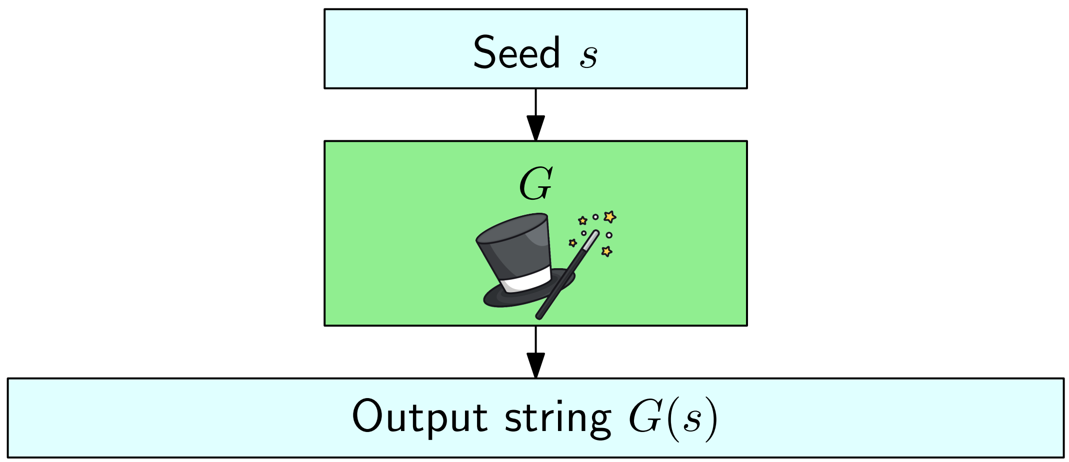

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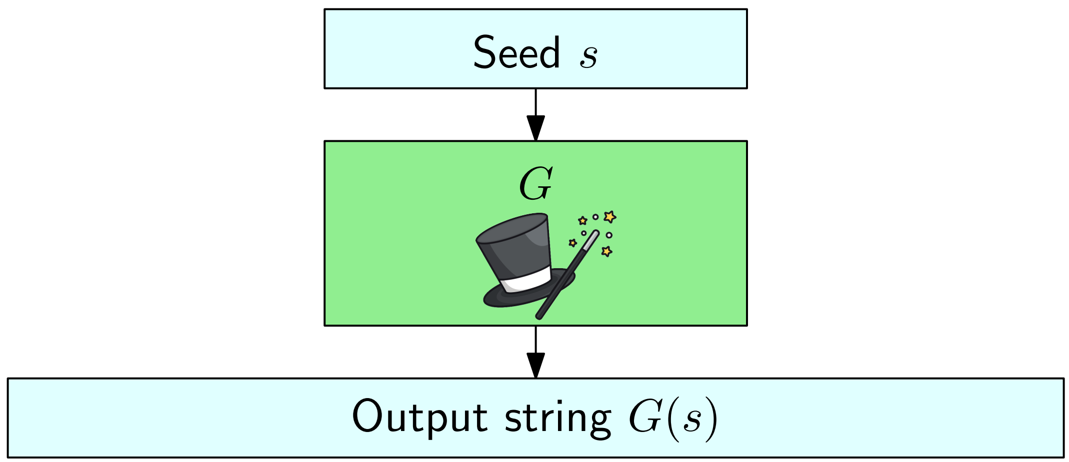

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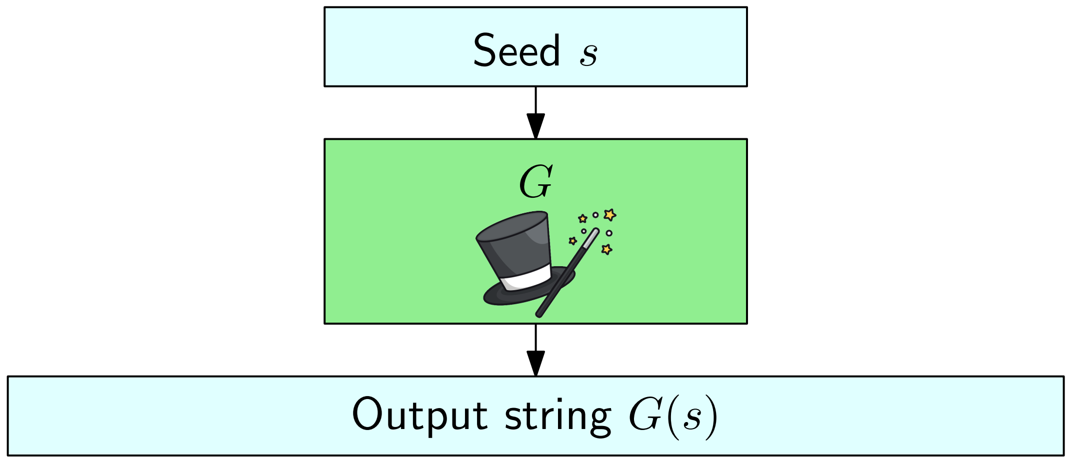

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...and that x is "pseudorandom" if it is the output of a PRG

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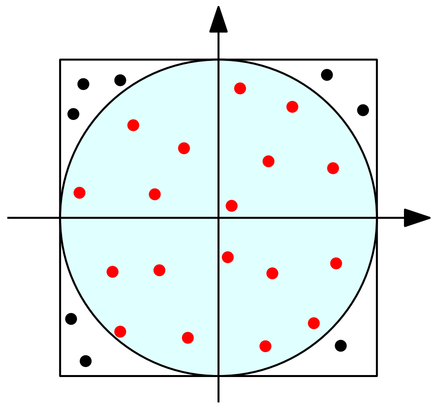

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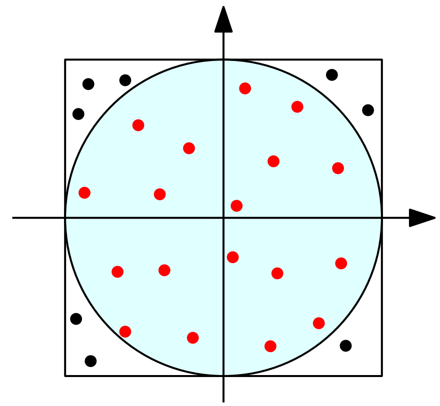

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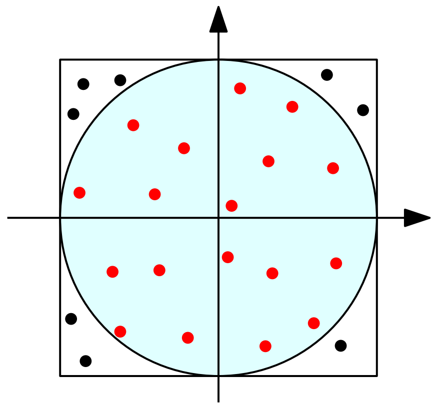

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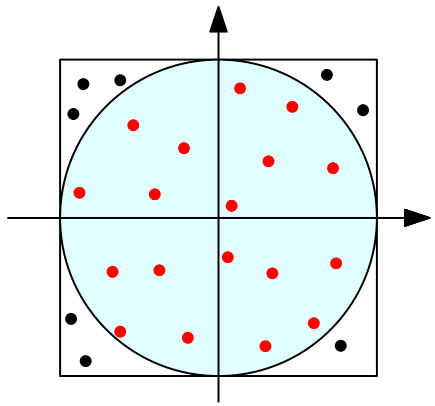

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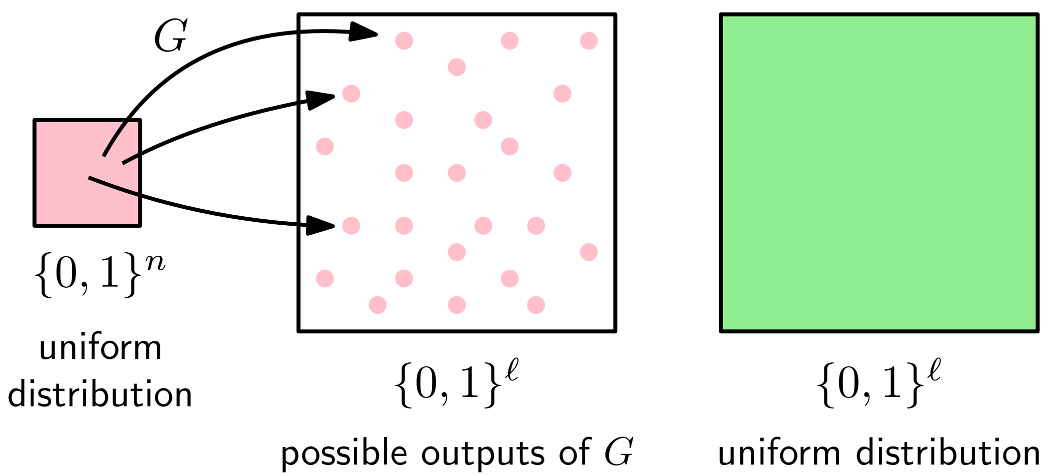
- At least half of the ℓ-bit strings (actually a $\frac{2^{\ell-n}-1}{2^{\ell-n}}$-fraction) can never be output by G !



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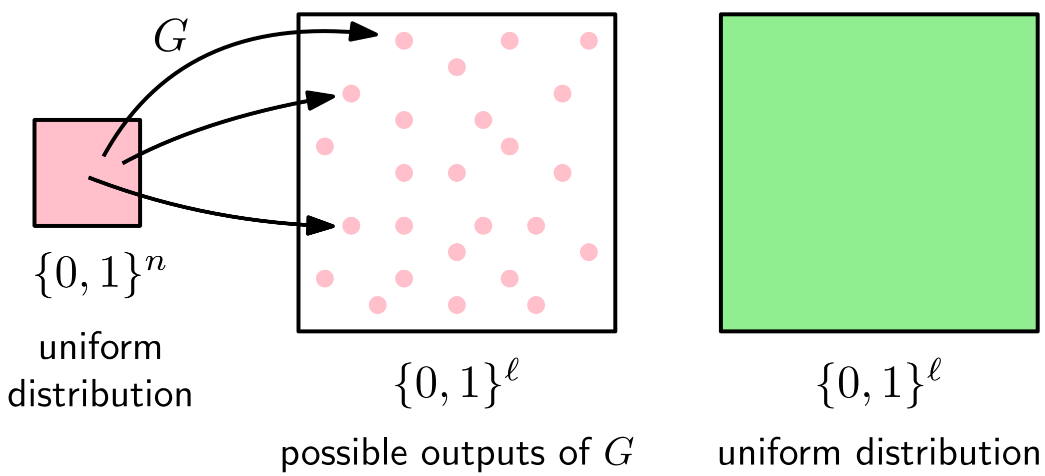

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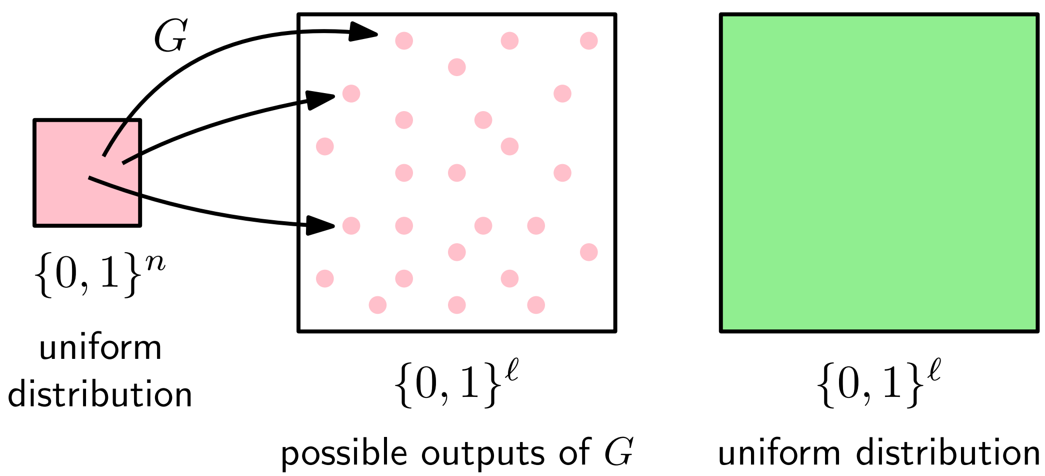
Pseudorandom strings are correctly identified with probability $\frac{2}{3}$

Random strings are correctly identified with probability $\geq \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot 1 = \frac{2}{3}$



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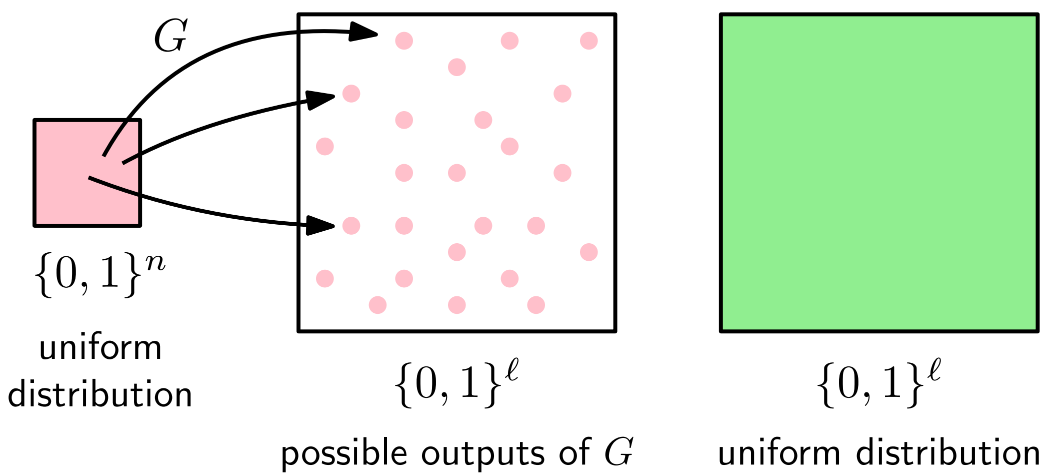

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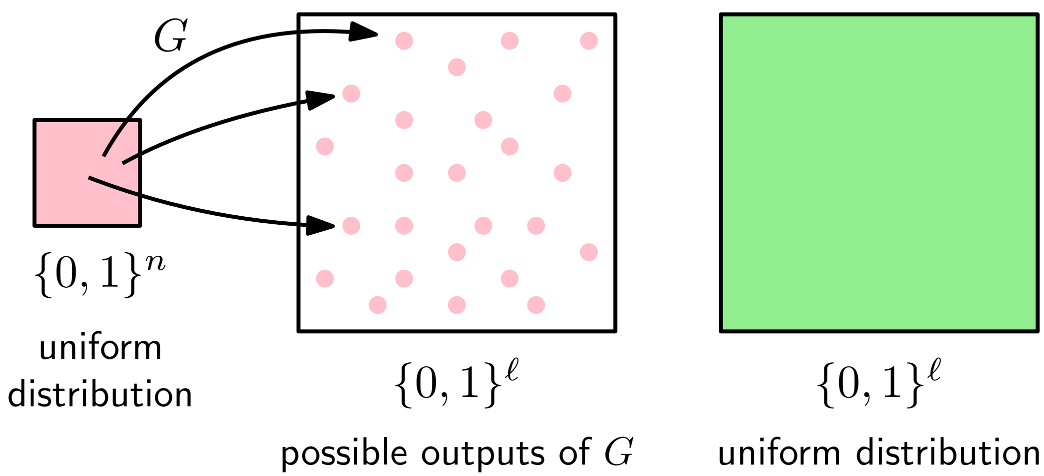
Idea: If adversaries are polynomially bounded, we only need to pass statistical tests that run in polynomial time



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$\left|1 - \frac{1}{2^{\ell(n)}}\right|$ is not negligible

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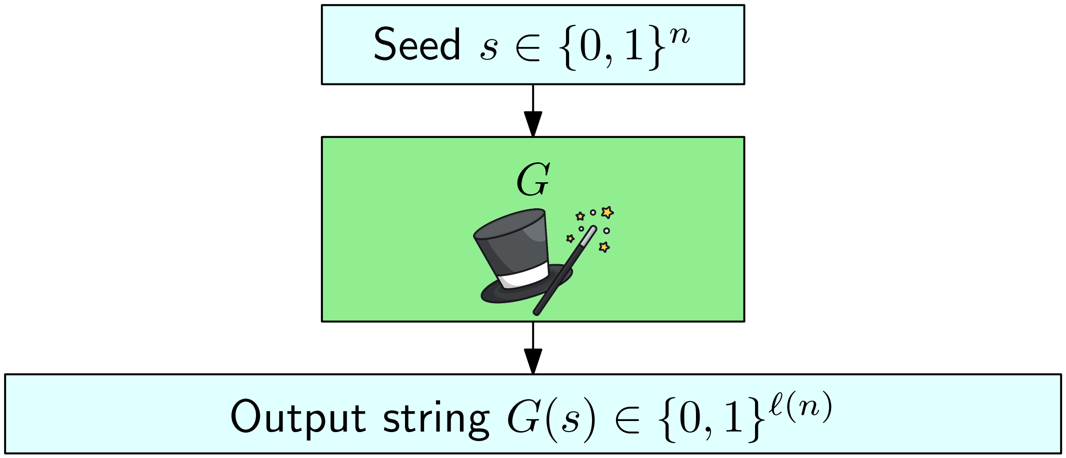

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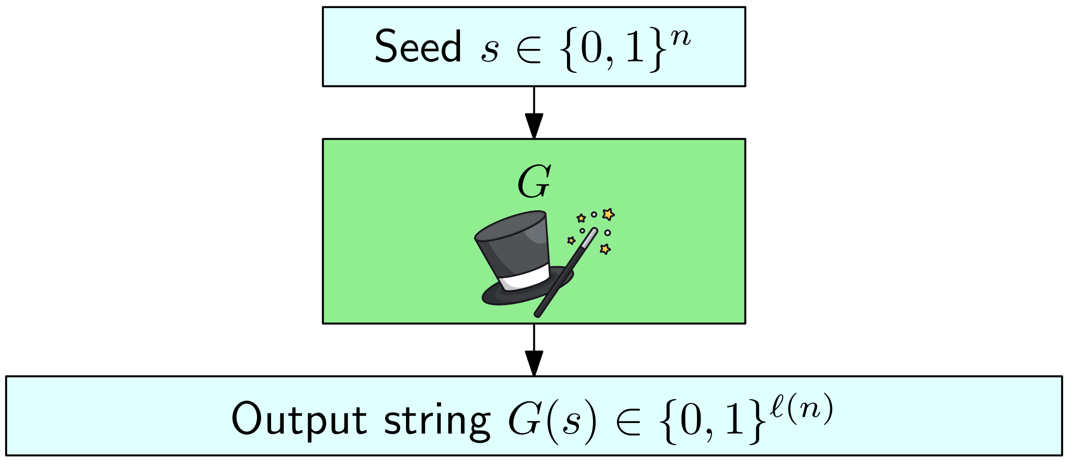

If we have a randomized polynomial-time algorithm that uses $\ell(n)$ random bits, and we replace those random bits with the output of $G(s)$, the resulting (randomized) algorithm "behaves the same" except for a negligible probability