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- For EAV-security we had to rely on PRGs
- For CPA-security we need a new cryptographic primitive: **pseudorandom functions** (PRFs)

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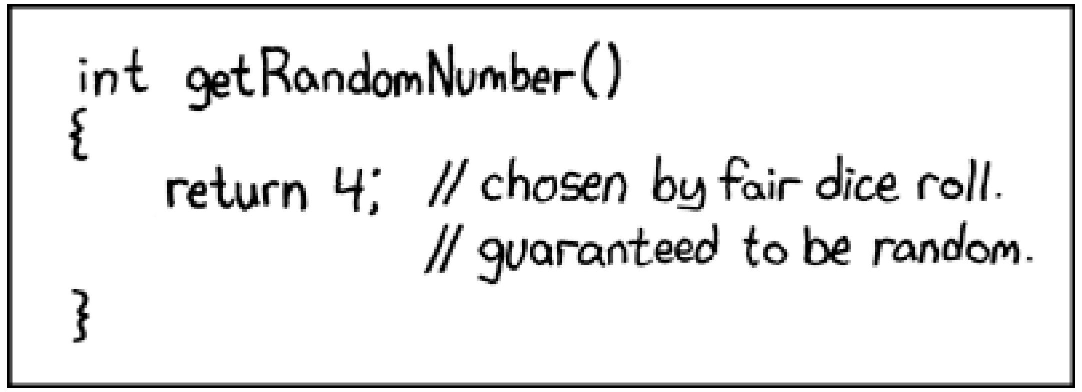

xkcd.com

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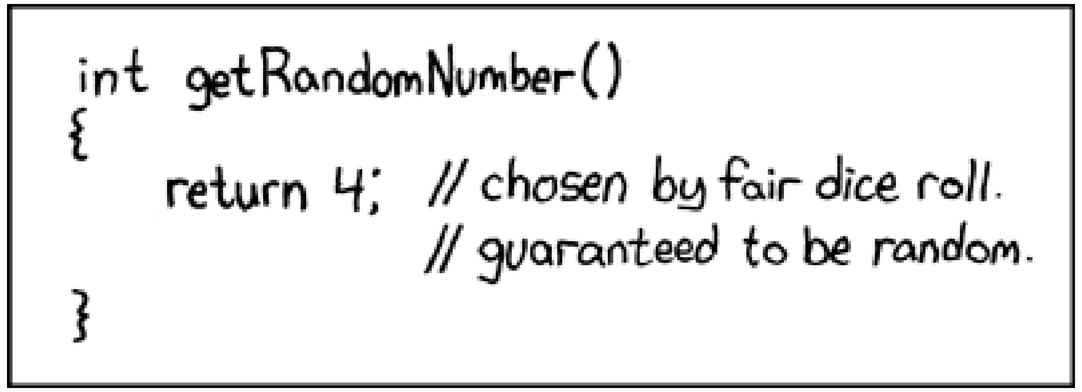

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We need to talk about probability distributions over functions instead

This is formalized using the notion of **keyed functions**

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Simplifying assumption (can be removed): F is **length-preserving**

$$\ell_{key}(n) = \ell_{in}(n) = \ell_{out}(n) = n$$

Number of functions

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2^n rows

x	$F(x)$
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- As a function whose outputs are decided **lazily**: whenever we need to evaluate $f(x)$:
  - If $f(x)$ was never evaluated before with input x:
    - Return a binary string chosen u.a.r. from $\{0,1\}^n$
  - Otherwise, return the previously chosen string for input x

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We can only sample a **tiny** fraction of the functions in Func_n!

Defining pseudorandom functions

Intuition: $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ is pseudorandom if no polynomial-time algorithm $\mathcal{D}$ can distinguish the function F_k (where k is chosen u.a.r.) from a random function $f \in \mathsf{Func}_n$, except for a negligible probability.

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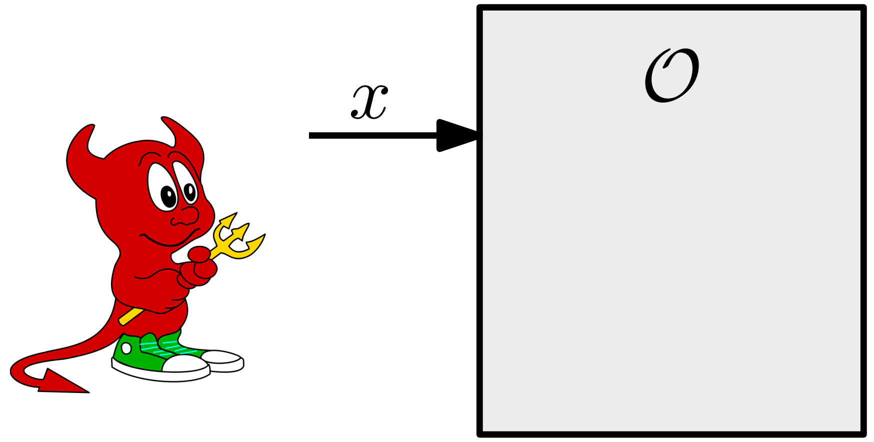

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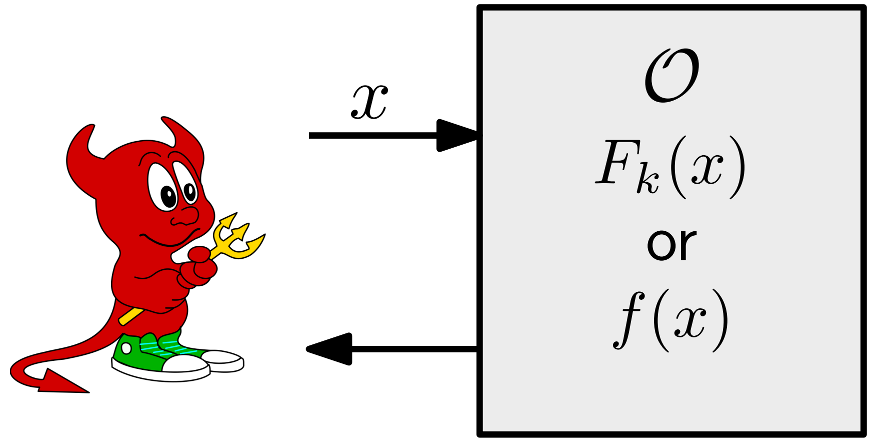

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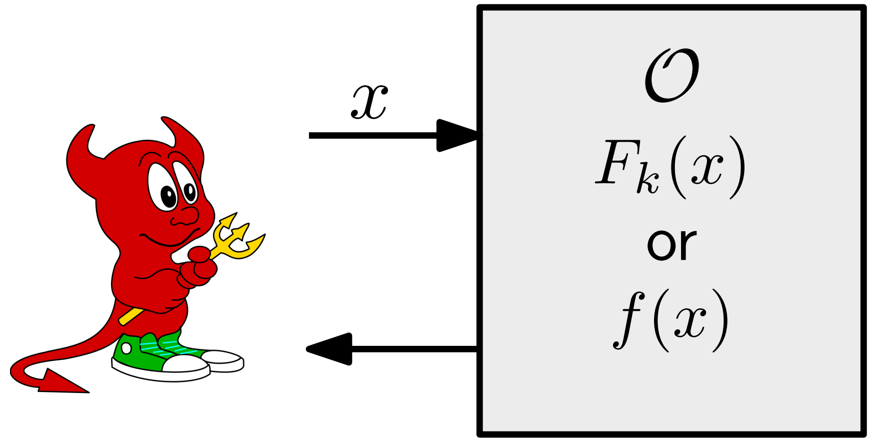

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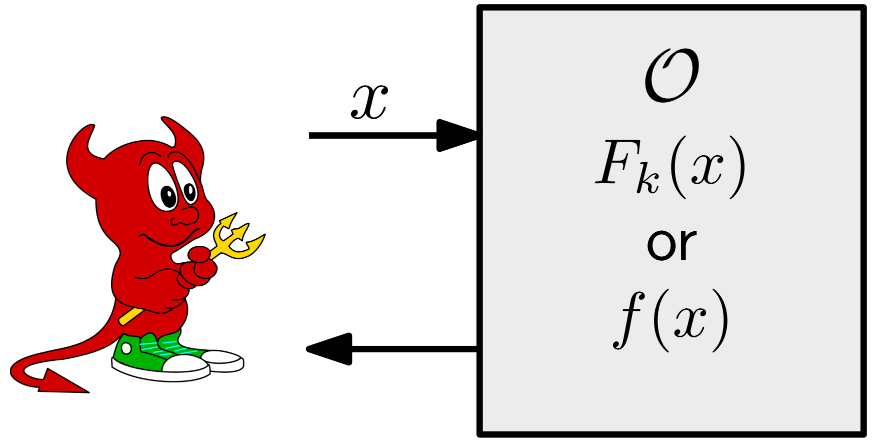
- $\mathcal{D}$ needs to guess whether $\mathcal{O}$ is evaluating F_k or f



Defining pseudorandom functions

"World 1":

k is chosen u.a.r. in $\{0,1\}^n$

$\mathcal{O}$

Evaluates F_k

Distinguisher $\mathcal{D}$

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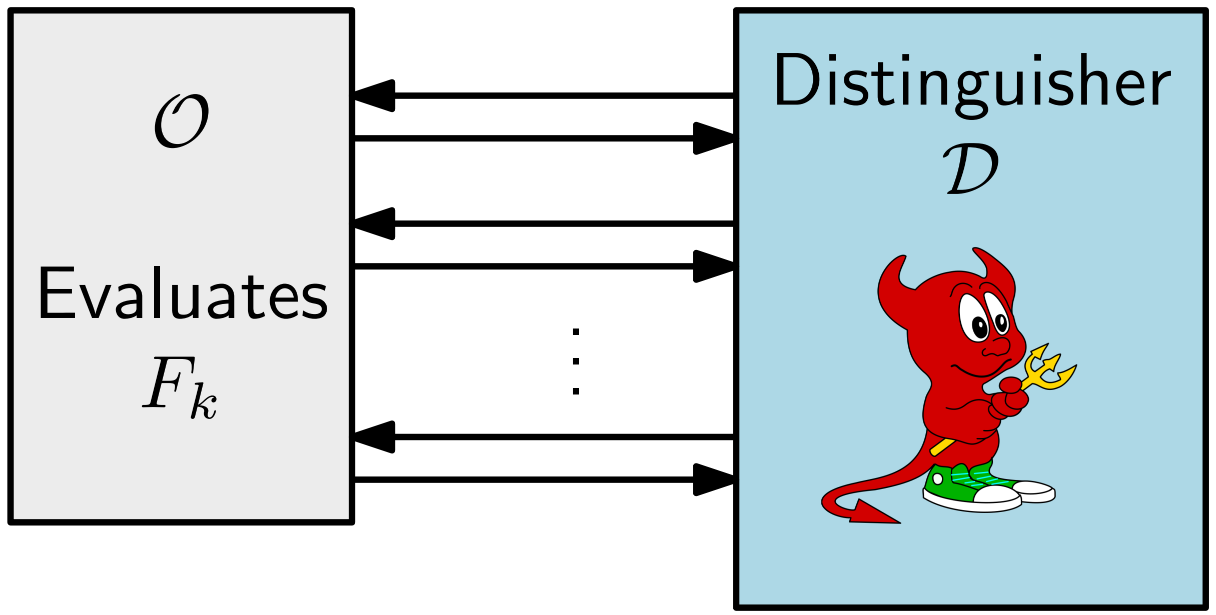

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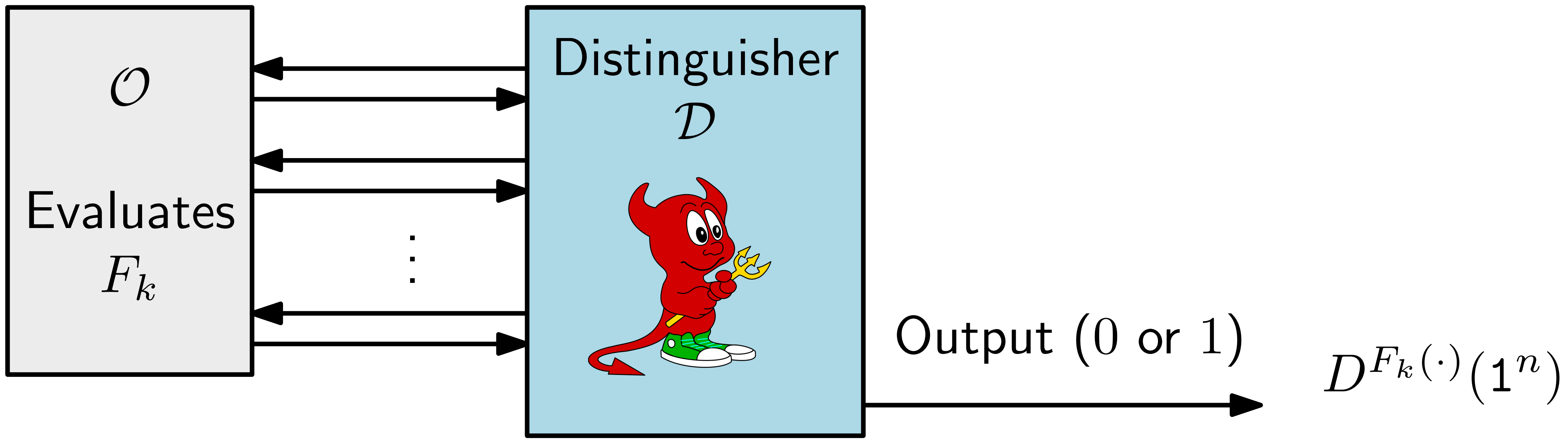

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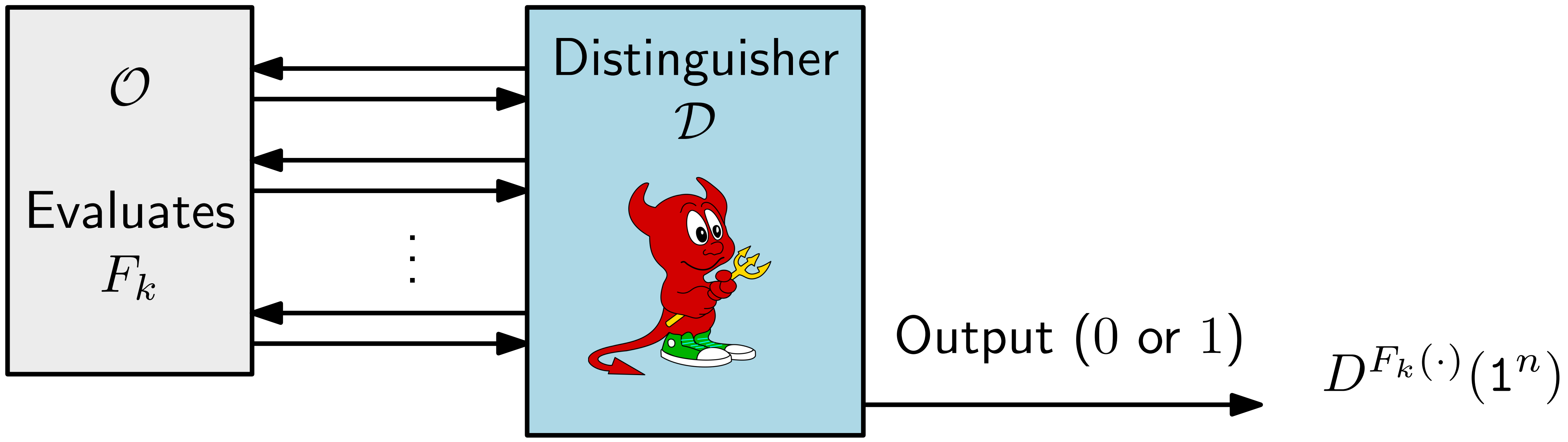
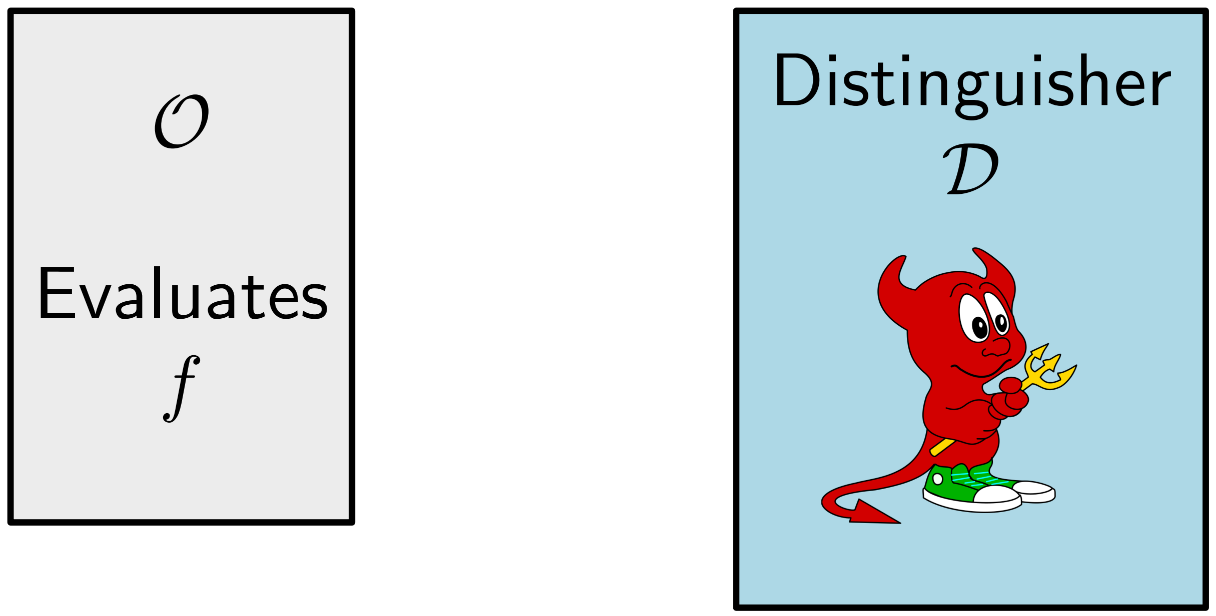


"World 0":

f is chosen u.a.r. in Func_n



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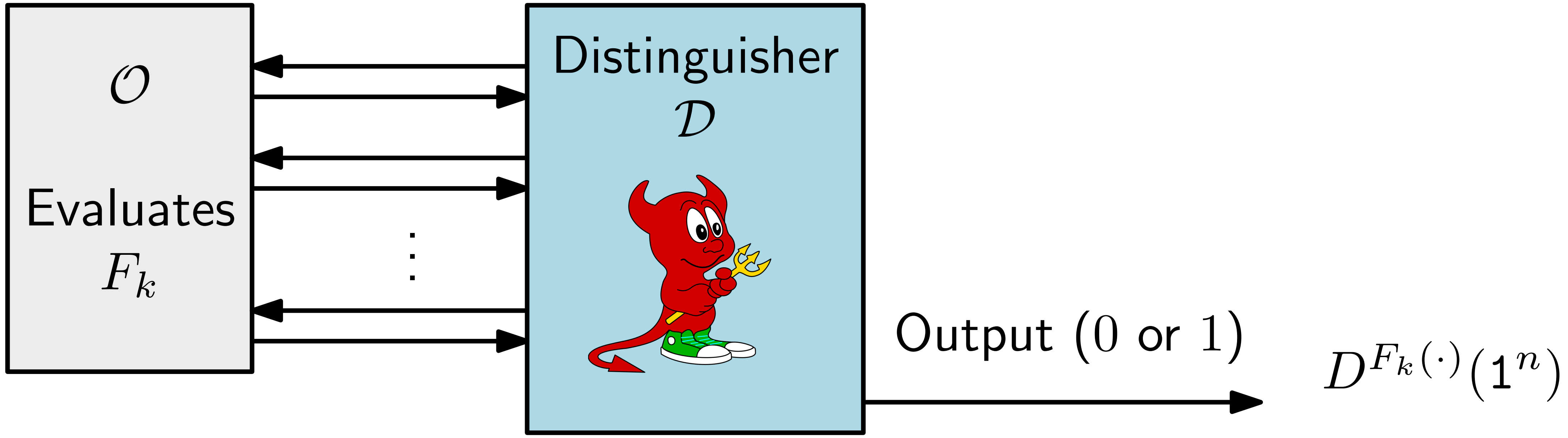

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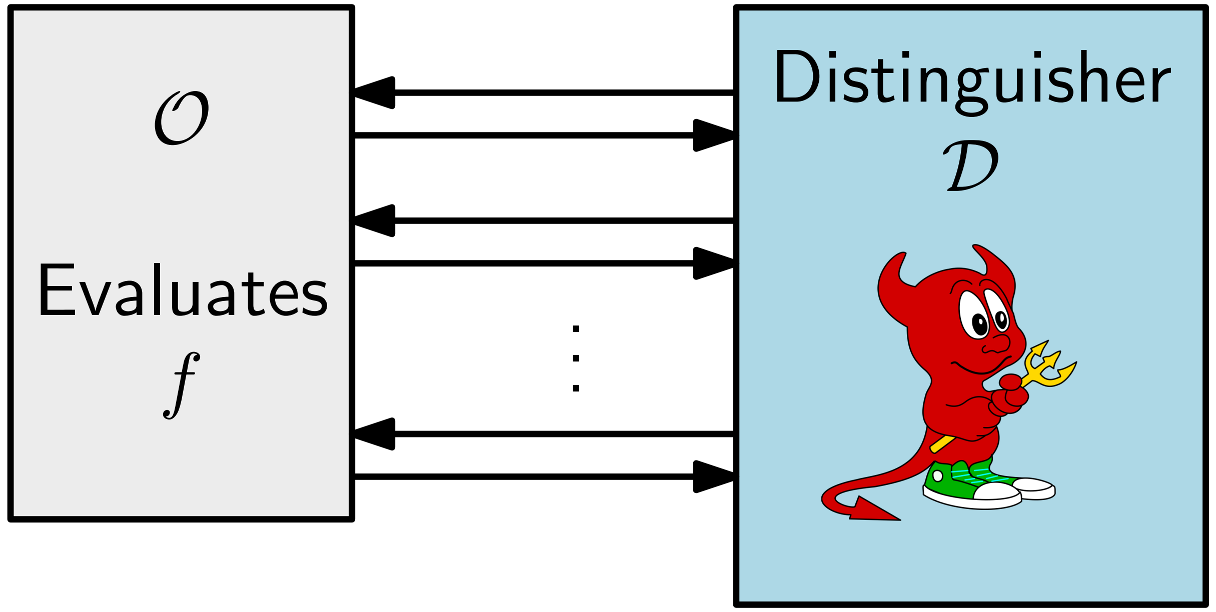

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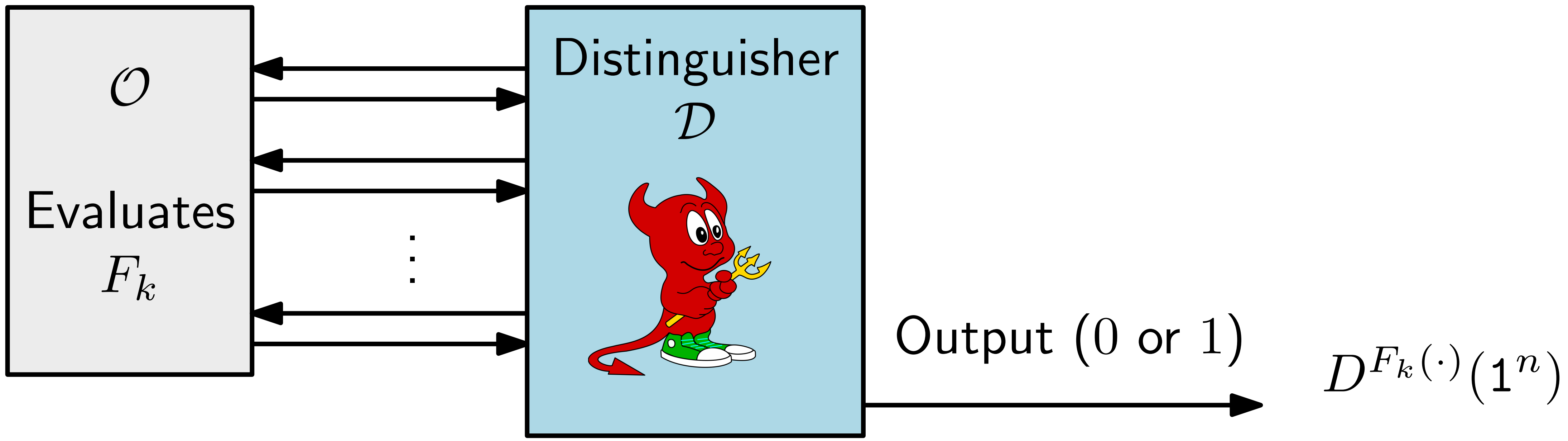

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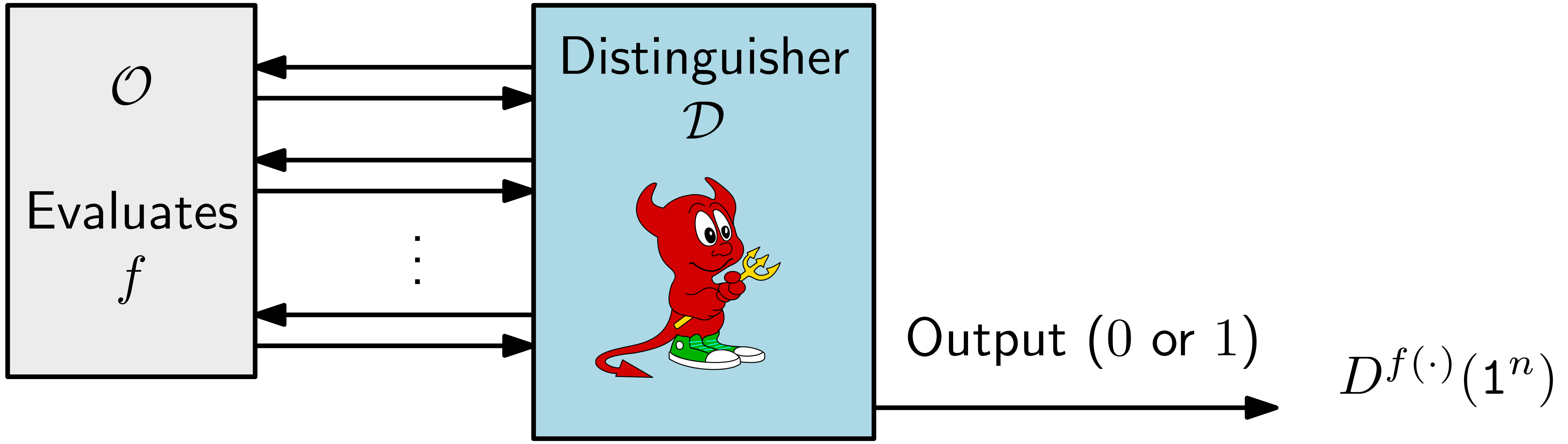

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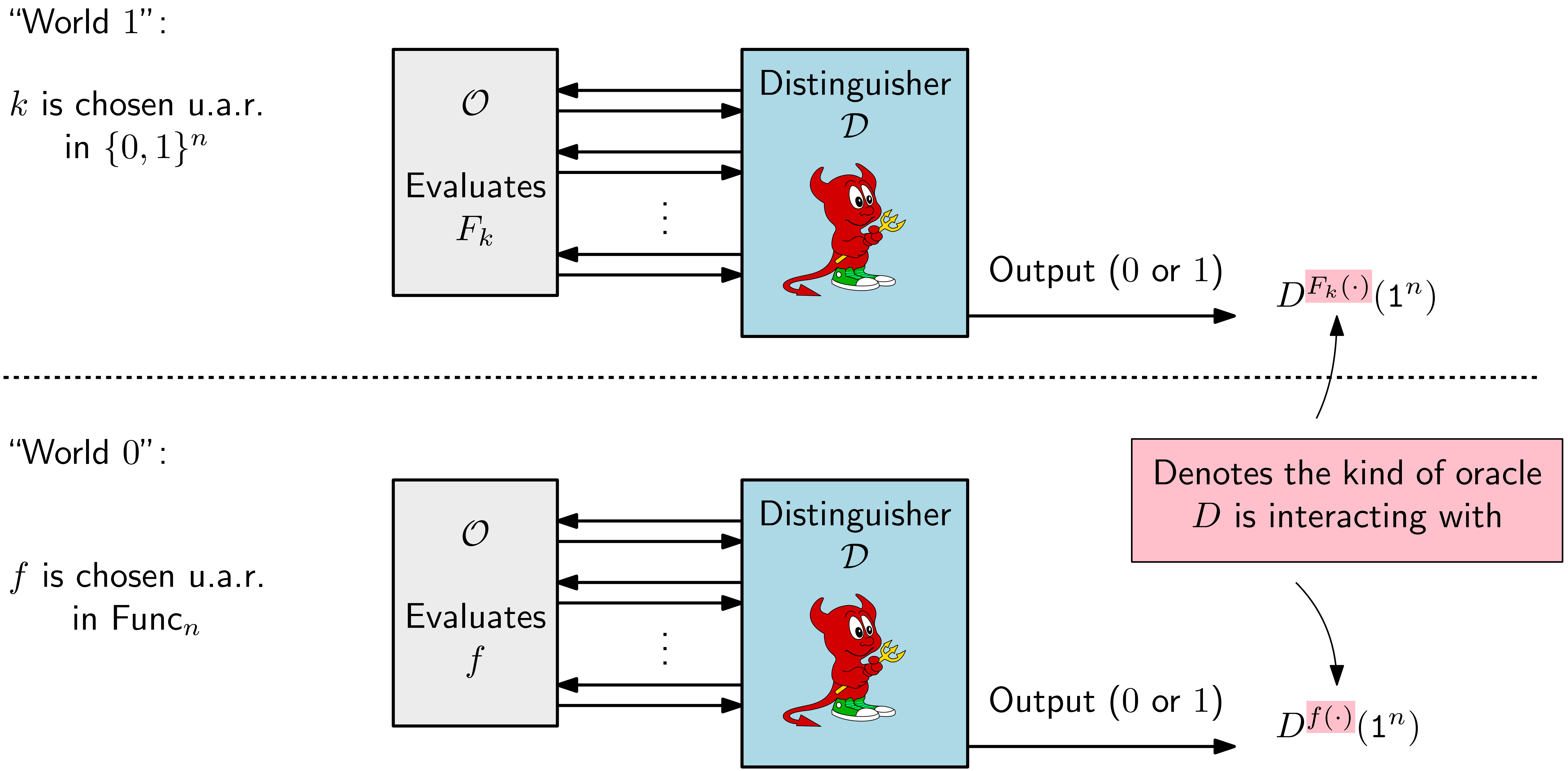

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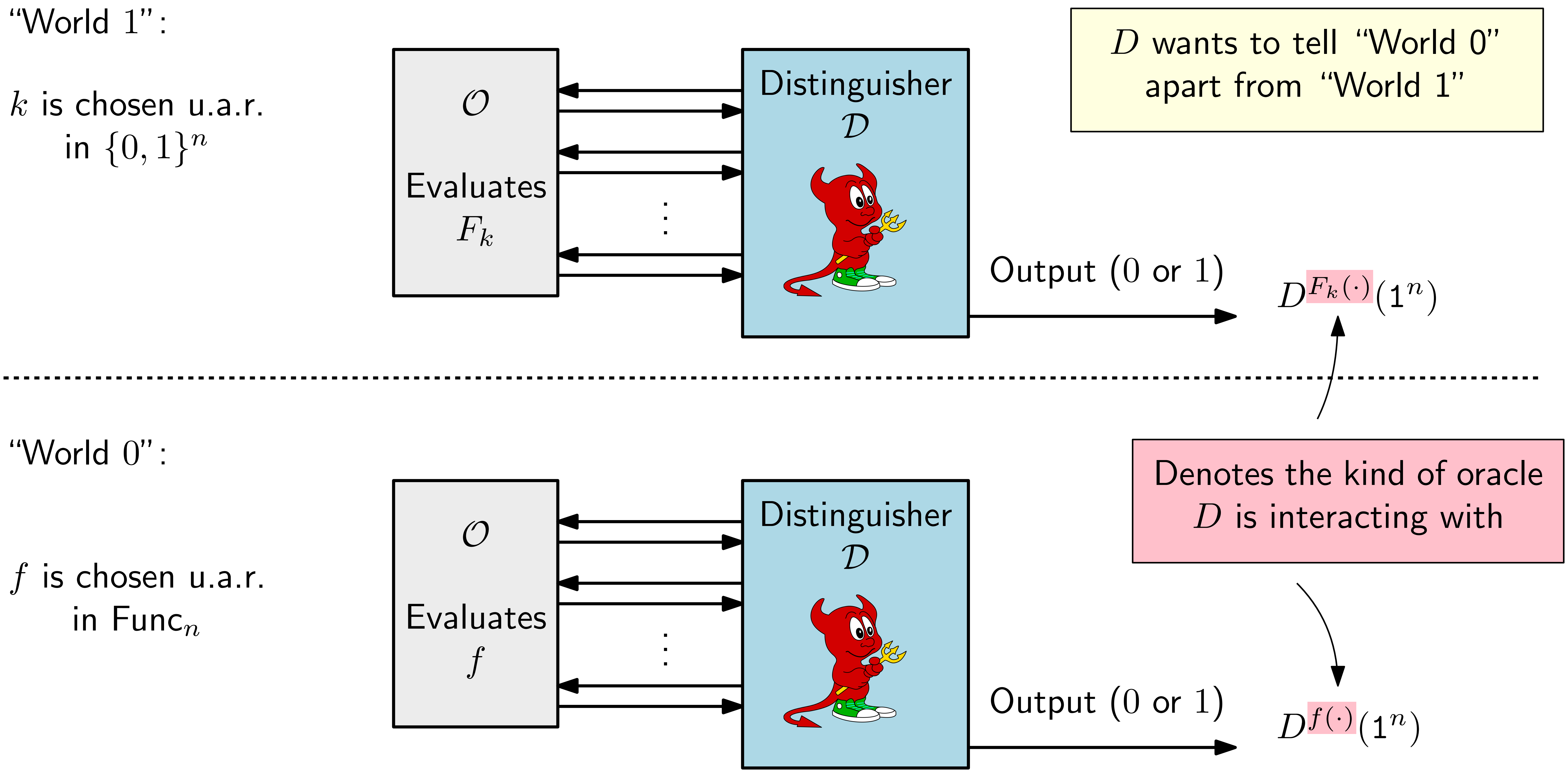

Defining pseudorandom functions (formal)

Definition: An efficient, length preserving, keyed function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ is a **pseudorandom function** if for all probabilistic polynomial-time distinguishers D, there is a negligible function ε such that:

$$\Big|\ \Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]\ \Big| \leq \varepsilon(n)$$

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Probability over the randomness of the distinguisher and the uniform choice of $f \in \mathsf{Func}_n$

Examples

What are some possible distinguishers from the following (failed attempts at) pseudorandom functions?

- $F(k, x) = 1^n$
- $F(k, x) = k$
- $F(k, x) = k \vee x$
- $F(k, x) = k \wedge x$
- $F(k, x) = k \oplus x$

PRFs and PRGs

If we have a PRF $F(k, x)$ we can use it to build a PRG G.

$G(s)$:

- Return $F_s(\texttt{0...000}) \,\|\, F_s(\texttt{0...001})$

expansion factor $\ell(n) = 2n$

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- Return $F_k(\langle 0 \rangle) \,\|\, F_k(\langle 1 \rangle) \,\|\, \ldots \,\|\, F_k(\langle L \rangle)$

$\langle x \rangle$ = binary encoding of x with n bits

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expansion factor $\ell(n) = n \cdot L$ (for $L = O(\text{poly}(n))$)

Proof that G is a PRG? Security reduction ("breaking G implies breaking F")

- Suppose that G is not a PRG, then there is some distinguisher D for G (with non negligible gap)
- Use D to build a distinguisher $\mathcal{A}$ for F (with non negligible gap)
- This contradicts the fact that F is a PRF (i.e., no such D can exist)

PRFs and PRGs

$G(k)$:

- Return $F_k(\langle 0 \rangle) \parallel F_k(\langle 1 \rangle) \parallel \ldots \parallel F_k(\langle L \rangle)$

- Suppose that G is not a PRG, then there is some D such that:

$$|\Pr[D(G(k)) = 1] - \Pr[D(r) = 1]| = \varepsilon(n) \text{ where } \varepsilon(n) \text{ is not negligible}$$

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- We design a distinguisher $\mathcal{A}$ for F. $\mathcal{A}^{\Phi}$ has access to an oracle Φ and returns:

$$D(\ \Phi(\langle 0 \rangle) \,\|\, \Phi(\langle 1 \rangle) \,\|\, \dots \,\|\, \Phi(\langle L \rangle)\ )$$

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- Therefore F is not a PRF. ↯ $\square$

PRFs and PRGs

If we have a PRF $F(k, x)$ we can use it to build a PRG G.

Are PRFs a stronger cryptographic primitive than PRGs?

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Caveat: To construct the table in polynomial time we need $t(n) = O(\log n)$ $\implies$ F has short inputs

PRFs and PRGs

Proof of security: $G(k) = \texttt{11011010010010110000101001011110}$

- Suppose that F is not a PRF, then there is $\mathcal{A}$ such that

$$\left| \Pr[\mathcal{A}^{F_k(\cdot)}(\texttt{1}^n) = 1] - \Pr[\mathcal{A}^{f(\cdot)}(\texttt{1}^n) = 1] \right| = \varepsilon(n) \text{ for non-negligible } \varepsilon(n)$$

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$\Longrightarrow$ G is not a PRG

↯ $\square$

The Goldreich-Goldwasser-Micali construction

Let G be a *length-doubling* PRG, i.e., $\ell(n) = 2n$.

$$G(s) = G_0(s) \,\|\, G_1(s)$$

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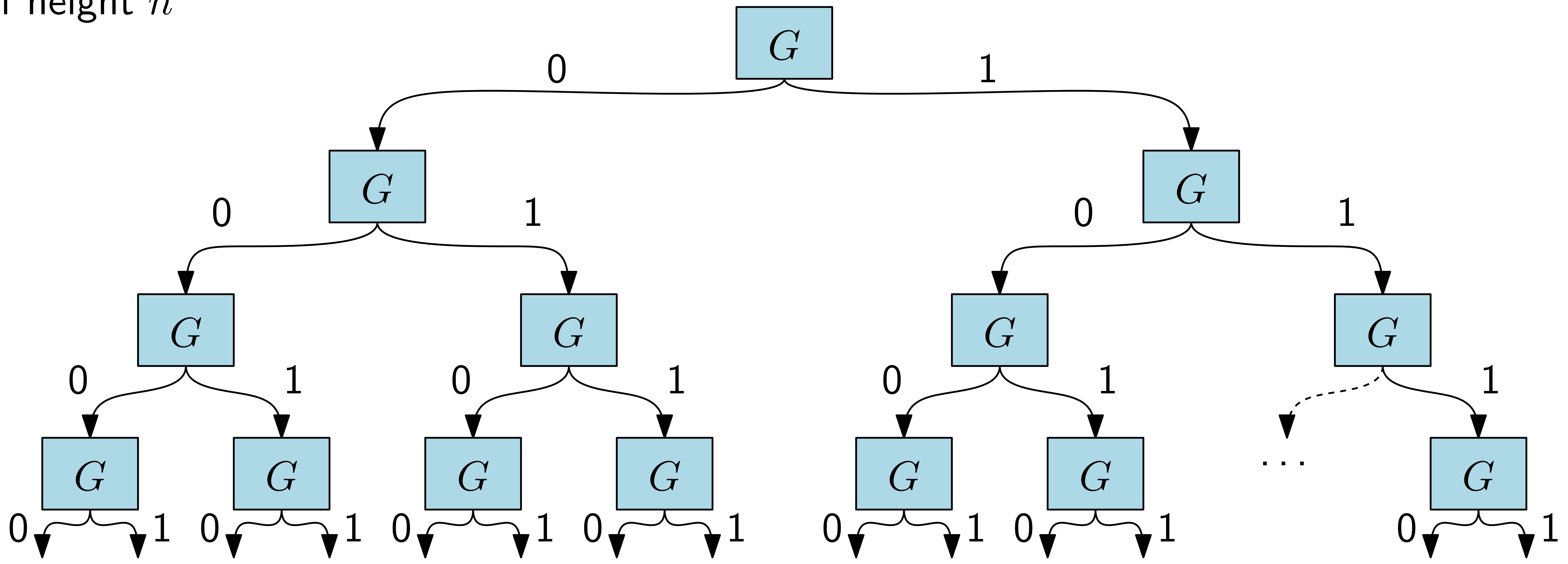
Imagine the following complete binary tree of height n



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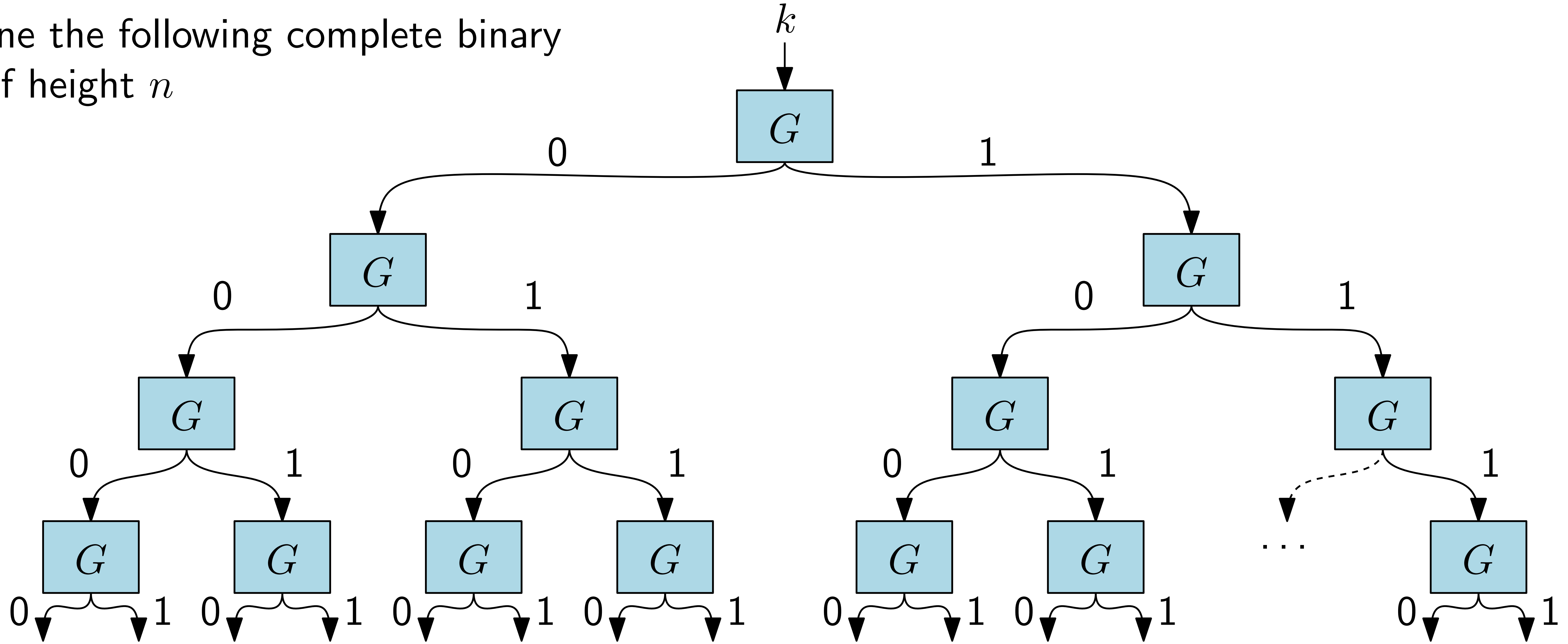

Interpret the key k of $F(k, x)$ as the seed of the root of the tree

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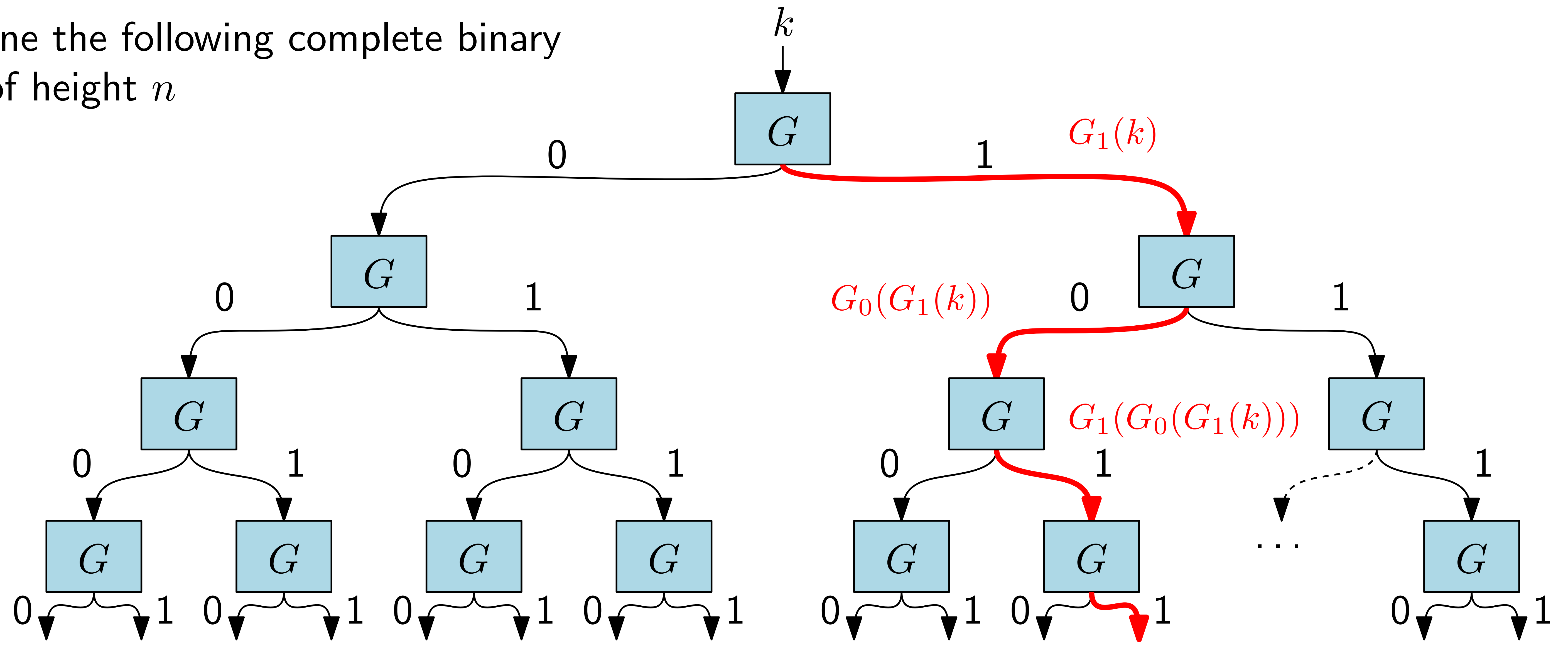

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$F(k, 1011)$

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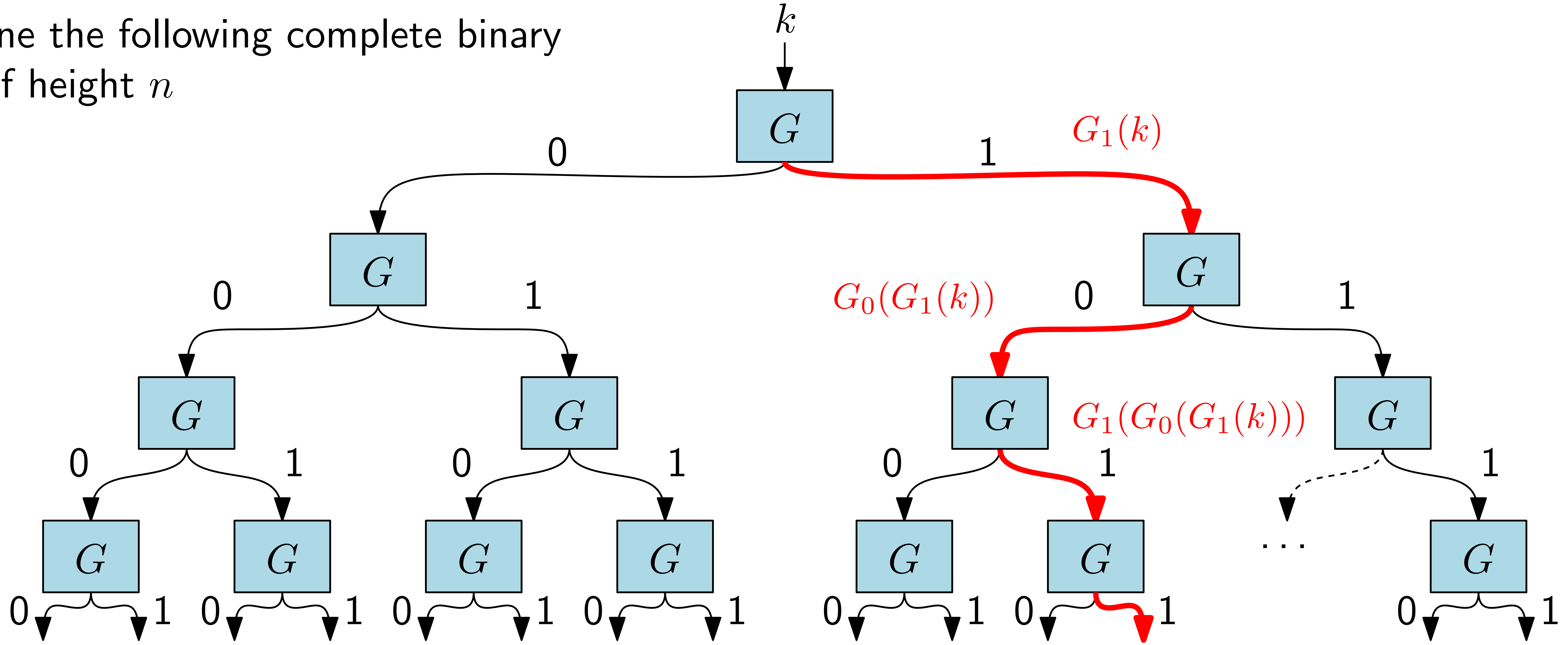

Interpret the key k of $F(k, x)$ as the seed of the root of the tree

Interpret the binary digits of x as a path in the tree

Interpret the output of the leaf as the output of $F(k, x)$

$F(k, 1011) = G_1(G_1(G_0(G_1(k))))$

The Goldreich-Goldwasser-Micali construction

If G is a secure length-doubling PRG, then the Goldreich-Goldwasser-Micali construction is a PRF

We won't see a proof of this fact (see Section 8.5 of the textbook if interested).

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What if don't have a length-doubling PRG?

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What if don't have a length-doubling PRG?

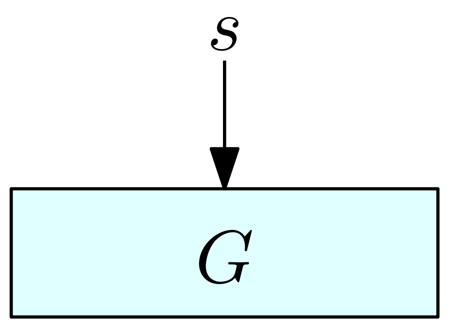
We can build one from any PRG, even if the expansion factor is just $\ell(n) = n + 1$

In fact, we can build a PRG with expansion factor $n + p(n)$ **for any polynomial** $p(n)$

Increasing the expansion factor

An easy case: increasing the expansion factor by 1

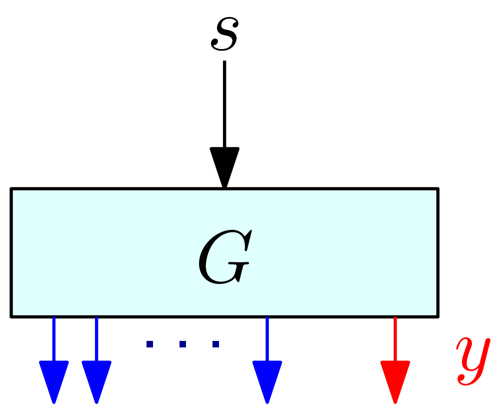
- Start from a PRG G with expansion factor $\ell(n) = n + 1$



Increasing the expansion factor

An easy case: increasing the expansion factor by 1

- Start from a PRG G with expansion factor $\ell(n) = n + 1$
- Call $G(s)$ and interpret the first n bits $x_1 x_2 \ldots x_n$ of the output as a new seed
- Let the last bit of $G(s)$ be y



$$G(s) = x_1 x_2 x_3 \ldots x_n y$$

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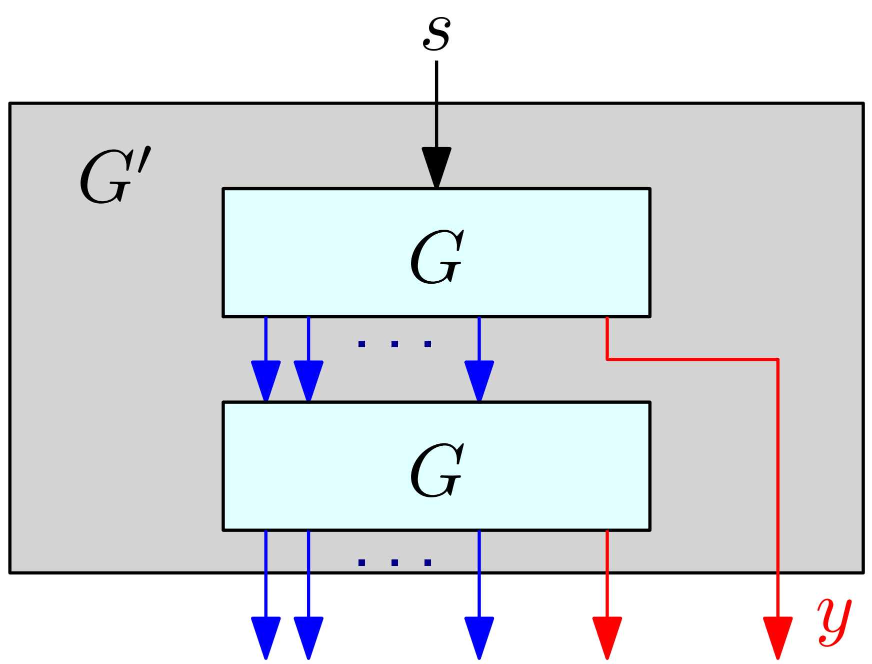
- Return $G(x_1 x_2 \ldots x_n) \,\|\, y$



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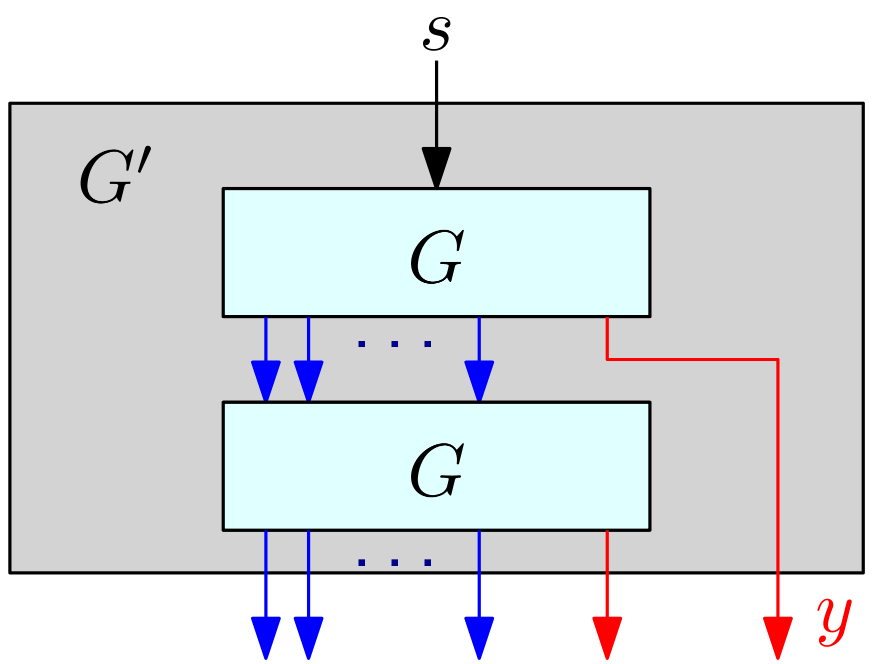


$G(s) = x_1 x_2 x_3 \dots x_n y$

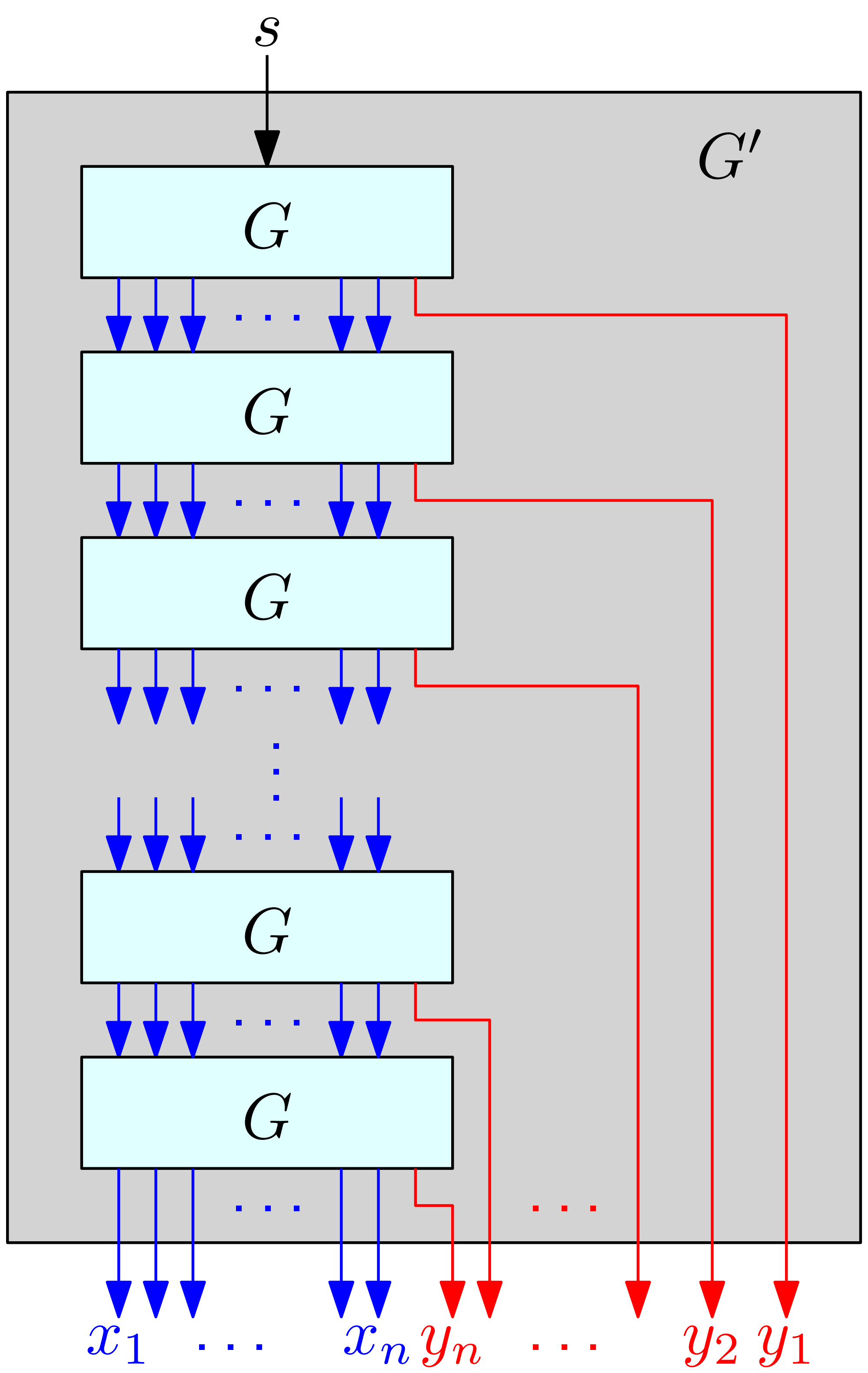
Overall expansion factor $\ell(n) = n + 2$

Increasing the expansion factor (length-doubling)

Increasing the expansion factor from $n+1$ to $2n$

- Start from a PRG G with expansion factor $\ell(n) = n+1$
- Repeat the previous idea for n levels
- The i-th intermediate level outputs $n+1$ bits
  - n bits are used as a seed for the next level
  - The $(n+1)$-th bit y_i will be part of the output of the whole construction
- The last level outputs $n+1$ bits $x_1 x_2 \dots x_n y_n$
- The final output is $x_1 x_2 \dots x_n y_n y_{n-1} \dots y_1$

Overall expansion factor: $\ell(n) = n + n = 2n$



Increasing the expansion factor to $n + p(n)$

Repeat the previous idea $p(n)$ times

Algorithm $\widehat{G}(s)$: (here $s \in \{0,1\}^n$)

- $t_0 \leftarrow s$
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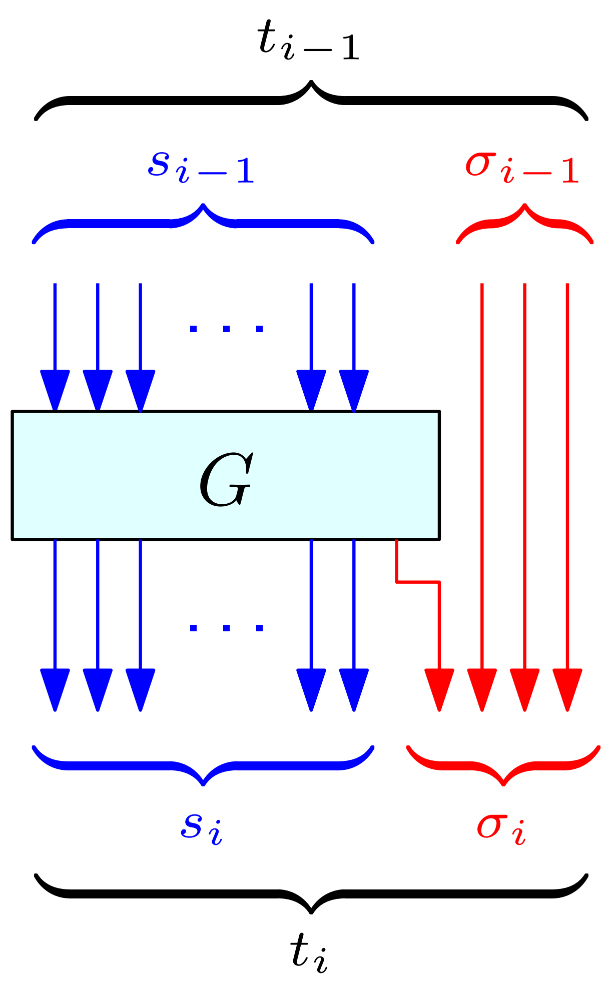

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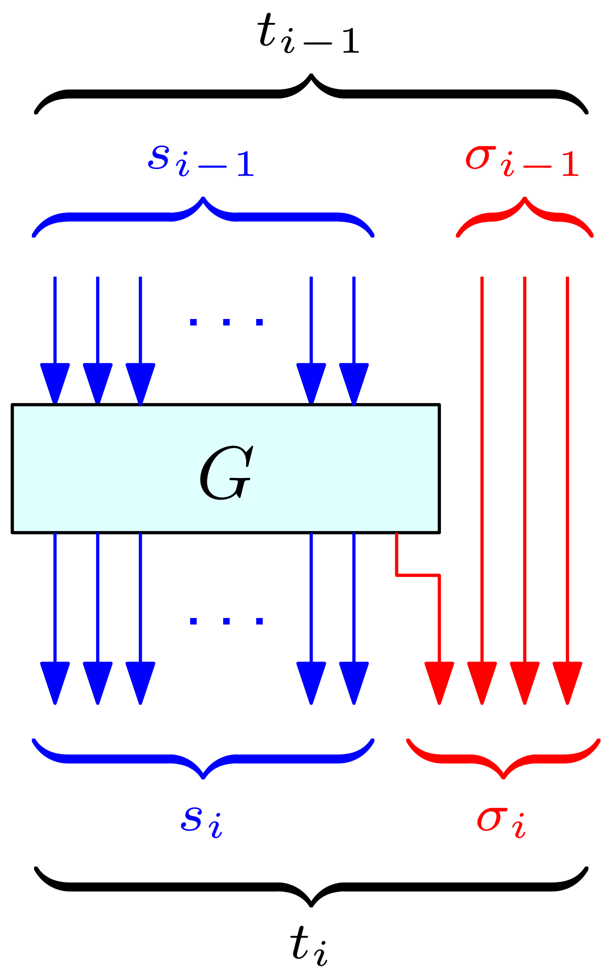

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Let D be a distinguisher such that:

$$\left| \ \Pr_s[\widehat{D}(\widehat{G}(s))] - \Pr_r[\widehat{D}(r)] \ \right| = \varepsilon(n) \text{ for some non-negligible } \varepsilon(n)$$

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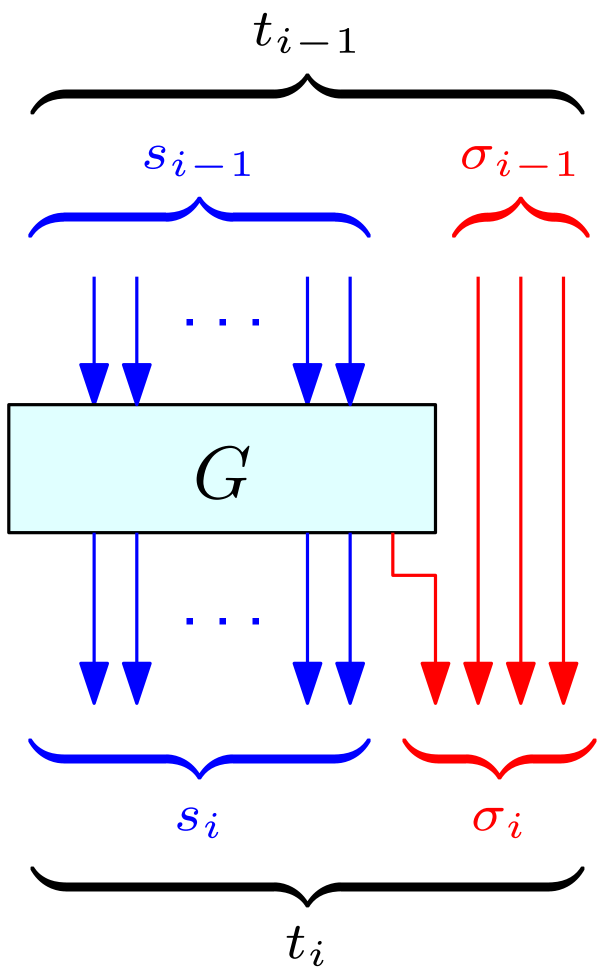

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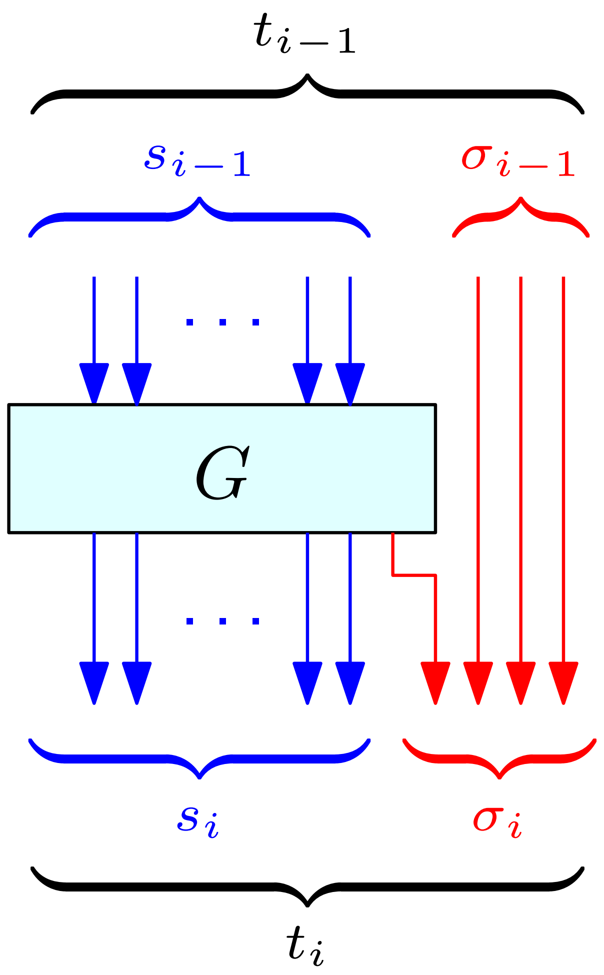

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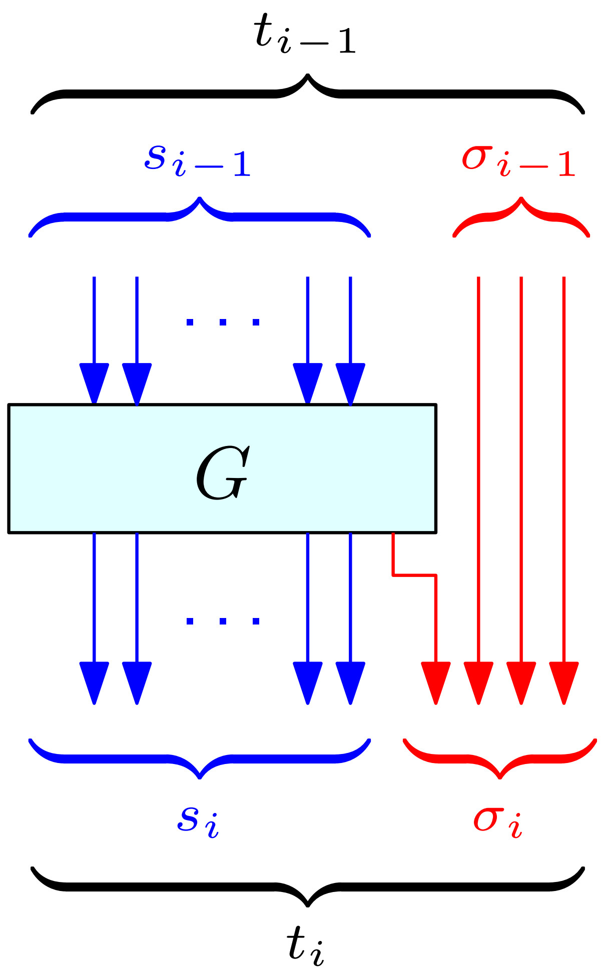

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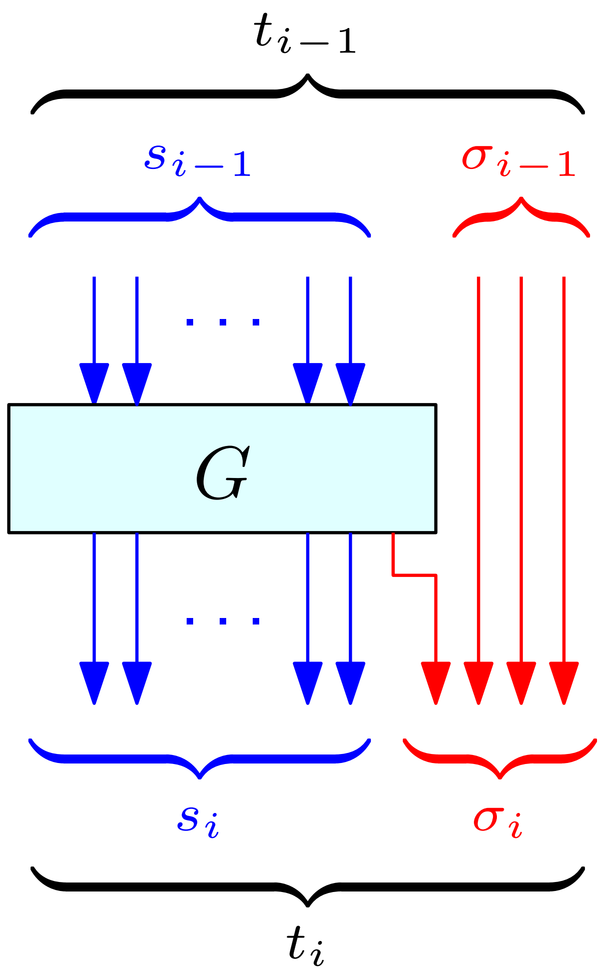

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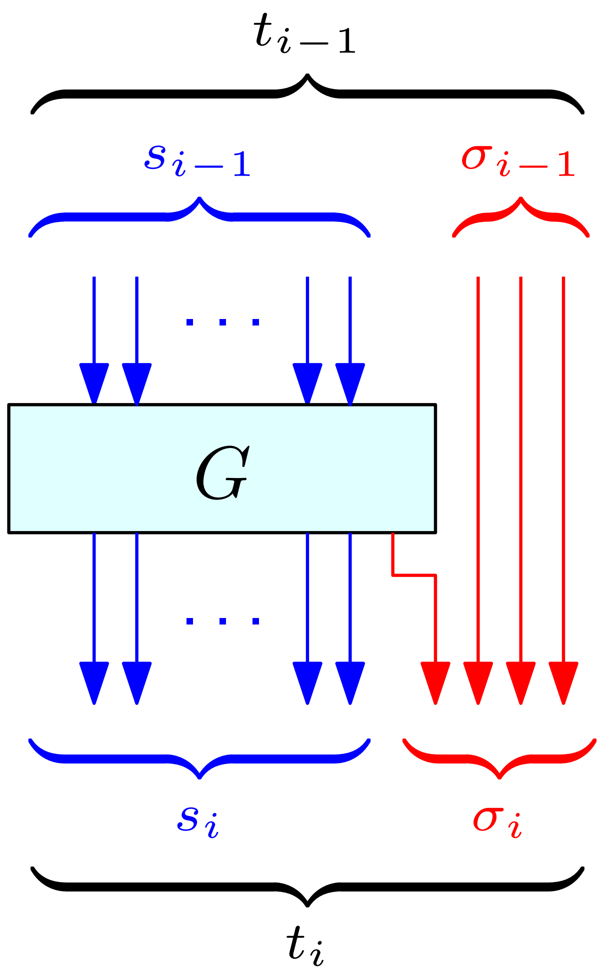

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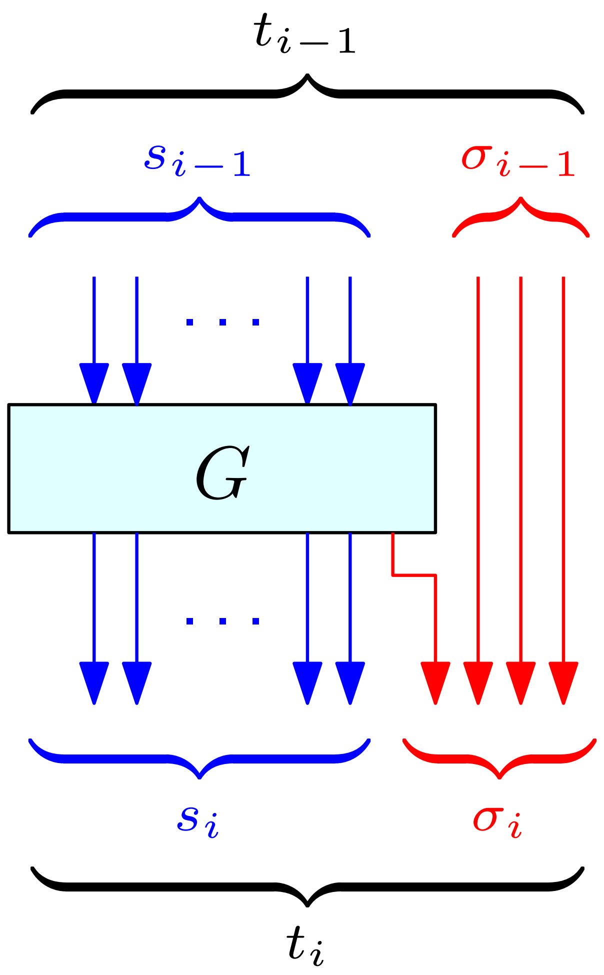

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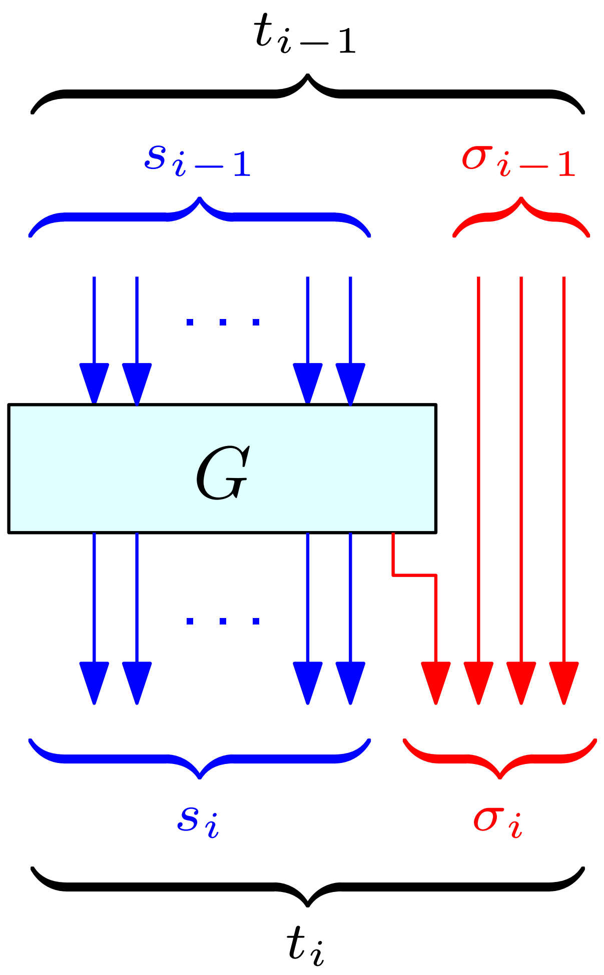

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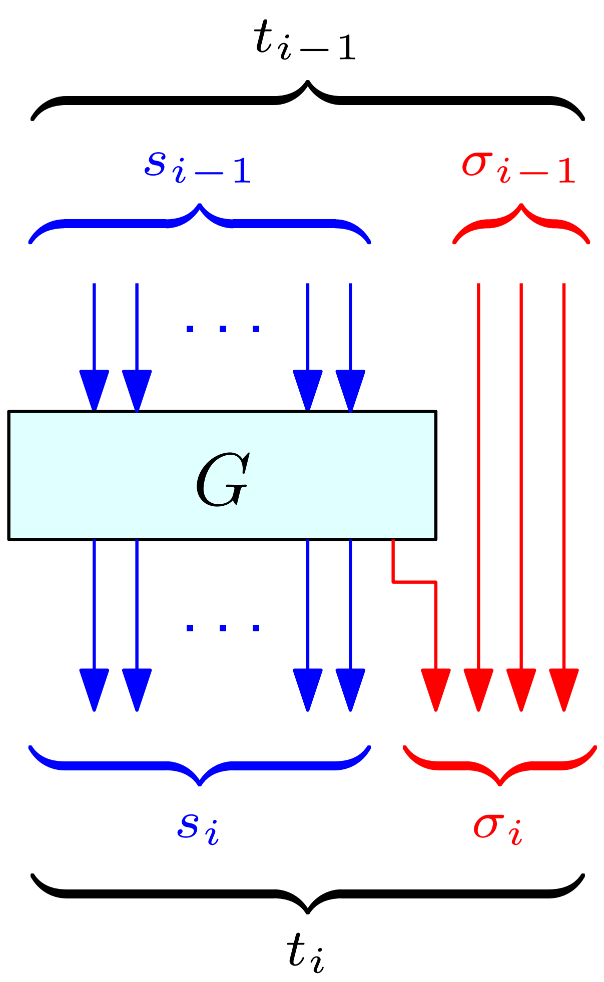

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Increasing the expansion factor to $n + p(n)$

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Not negligible!

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↯ □

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2^n rows

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$$|\text{Perm}_n| = 2^n \cdot (2^n - 1) \cdot \dots \cdot 1$$
$$= (2^n)!$$

Number of Permutations vs Number of Functions

Since a function $F \in \mathsf{Perm}_n$ is bijective, it must be **invertible**

$$F^{-1} \text{ exists and } F(x) = y \iff F^{-1}(y) = x$$

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Asymptotically, almost no function in Func_n is a permutation!

Keyed permutations

A **keyed permutation** is a keyed function $F : \{0,1\}^{\ell_{key}(n)} \times \{0,1\}^{\ell_{in}(n)} \to \{0,1\}^{\ell_{out}(n)}$ such that:

- $\ell_{in}(n) = \ell_{out}(n)$ (this quantity is called the **block length**); and
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A keyed permutation is **efficient** if:

- There is a polynomial-time algorithm that computes $F(x)$ given x; and
- There is a polynomial-time algorithm that computes $F^{-1}(y)$ given y

Pseudorandom permutations, formal definition

Definition: An efficient, length preserving, keyed permutation $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ is a **pseudorandom permutation** if for all probabilistic polynomial-time distinguishers D, there is a negligible function ε such that:

$$\left|\ \Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]\ \right| \leq \varepsilon(n)$$

Probability over the randomness of the distinguisher and the choice of k

Probability over the randomness of the distinguisher and the uniform choice of $f \in \mathsf{Perm}_n$

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Probability over the randomness of the distinguisher and the uniform choice of $f \in \mathsf{Perm}_n$

Intuitition: a keyed permutation is pseudorandom permutation if no polynomial-time algorithm can distinguish it from a random permutation

Pseudorandom permutations

Recall that (asymptotically) almost no function in Func_n is a permutation

Nevertheless:

- As soon as $\ell_{in}(n) \geq n$, a PRP is indistinguishable (in polynomial time, with non-negligible gap) from PRF

- Since a PRF is indistinguishable from a random function, this implies that PRPs with $\ell_{in}(n) \geq n$ are also indistinguishable from random functions!

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Sometimes we need even even “stronger” functions than pseudorandom permutation

The adversary might be able to exploit the fact that a pseudorandom permutation is invertible to gain a non-negligible advantage

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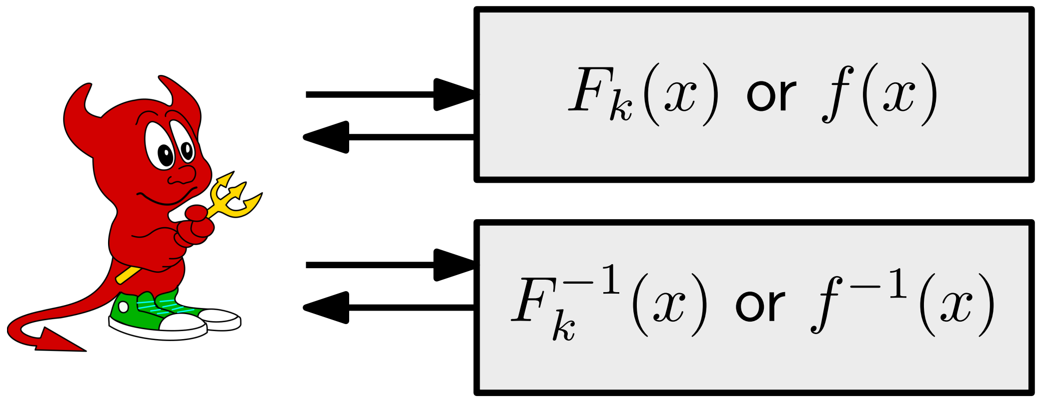
We define **strong** pseudorandom permutations that are indistinguishable from random permutation even if the adversary has oracle access to **both** the permutation and its inverse



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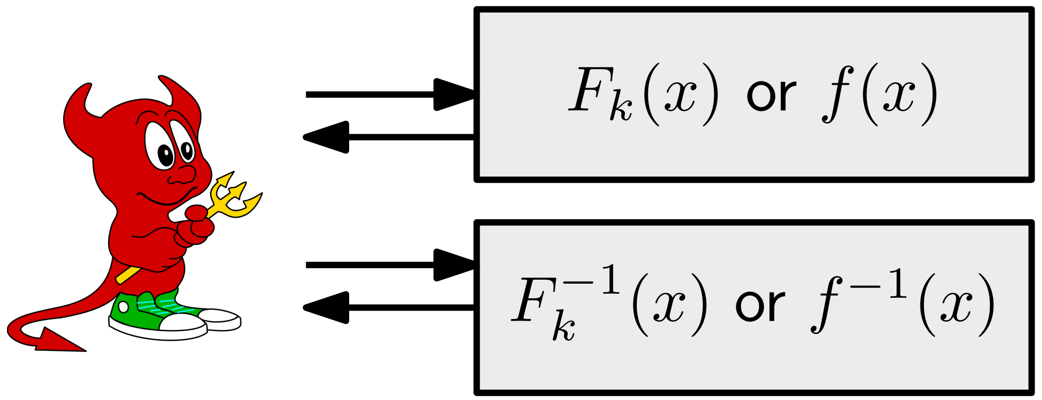

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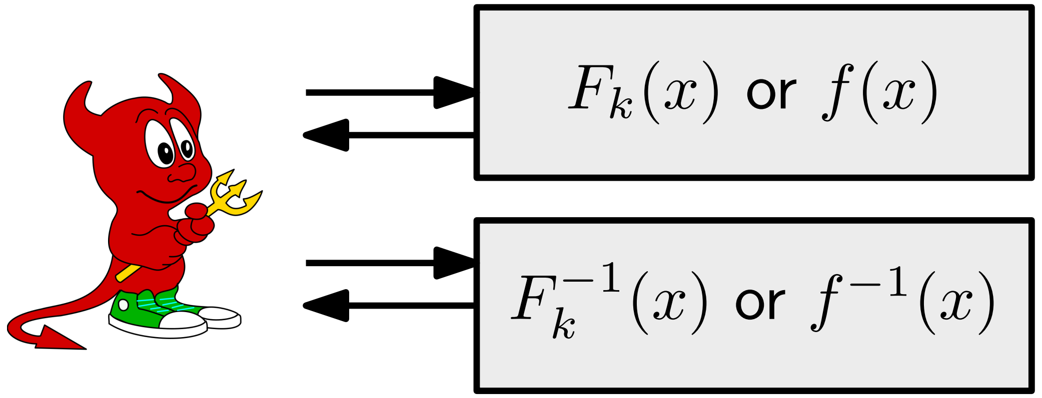

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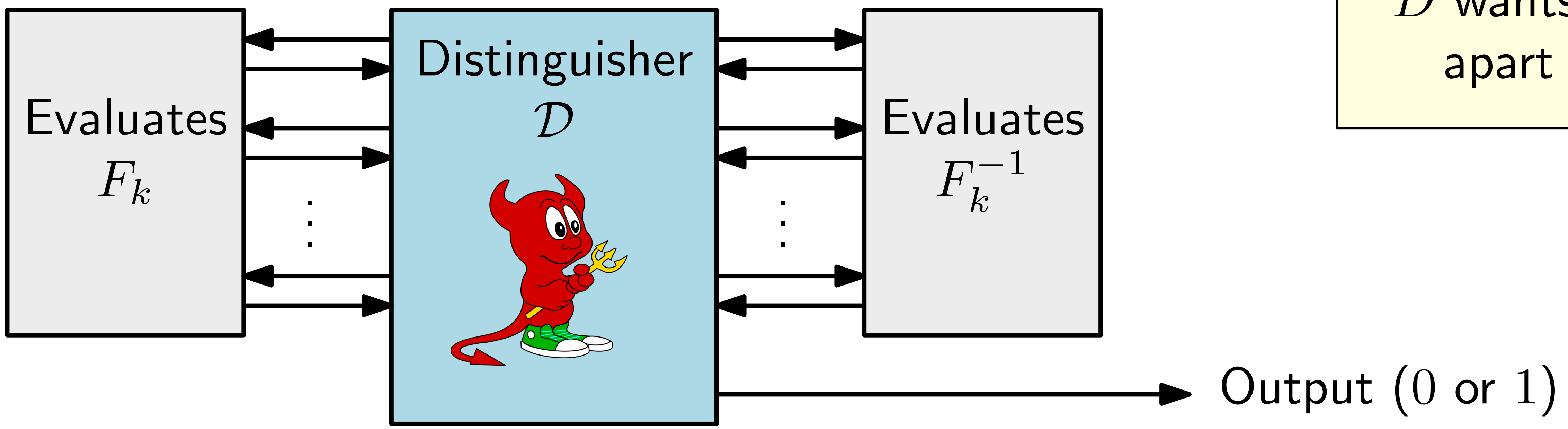
Strong pesudorandom permutations

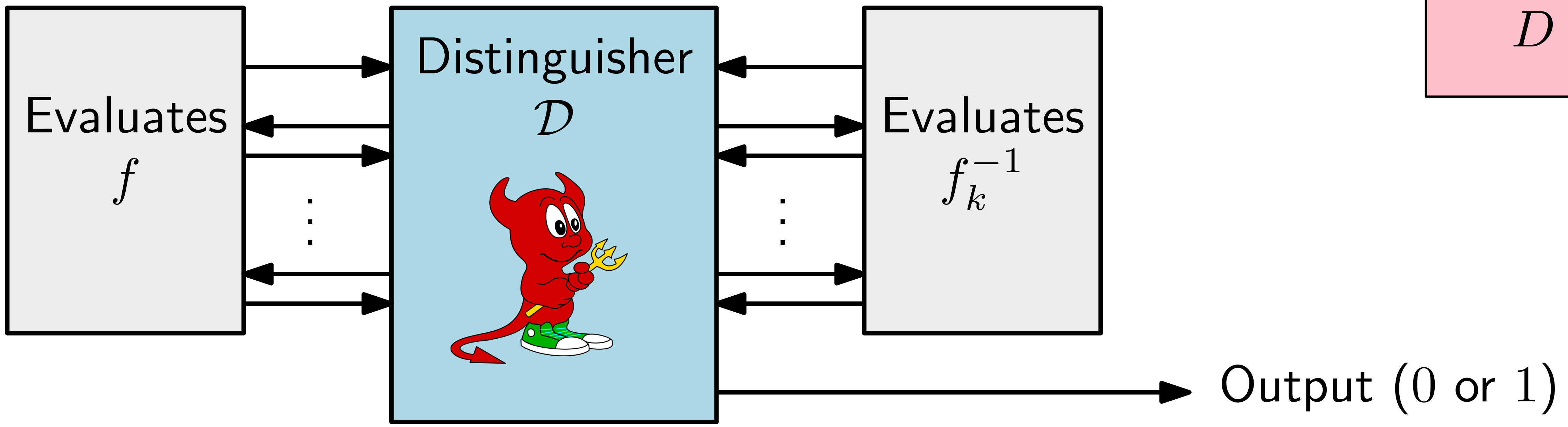
"World 1": k is chosen u.a.r. in $\{0,1\}^n$



D wants to tell "World 0" apart from "World 1"

$D^{F_k(\cdot),F_k^{-1}(\cdot)}(1^n)$

"World 0": f is chosen u.a.r. in Perm_n



Denotes the kind of oracle D is interacting with

$D^{f(\cdot),f^{-1}(\cdot)}(1^n)$