# A little (Computational) Number Theory and Group Theory

Public key cryptographic constructions require some notions of number theory and group theory

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Differently from the pure mathematics approach, we will also be interested in **how quickly** we can solve various problems

In particular, we are interested in whether the problems at hand can be solved in **polynomial time** 

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74 241 176 81 206 92 108 31 42

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Encodes:  $74 \cdot 256^8 + 241 \cdot 256^7 + 176 \cdot 256^6 + 81 \cdot 256^5 + 206 \cdot 256^4 + 92 \cdot 256^3 + 108 \cdot 256^2 + 31 \cdot 256 + 42$ 

 $= 1\,382\,474\,571\,160\,304\,230\,186$ 

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Recall the difference between polynomial-time and pseudopolynomial-time algorithms

Running times are measured as a function of the input length

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- The running time is polynomial w.r.t. the **value** of the integer n
- It is not polynomial in the length of the input, i.e., the number of bits needed to represent n
- As a function of the input length  $\eta$ , the time complexity is  $\Theta(2^{\eta})$
- This is an **exponential-time** algorithm!

The grade-school algorithms for addition and multiplication (over big integers) run in polynomial-time

- Adding n and m requires time  $O(\log n + \log m)$
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• We cannot even write out the result in polynomial-time

**Proposition:** Let a be an integer and let N be a positive integer. There exist unique integers q, r for which a = qN + r and  $0 \le r < N$ .

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We can reduce intermediate values during computation of additions and products:

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- $(a \cdot b) \mod N = ((a \mod N) \cdot (b \mod N)) \mod N$

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 $(7236782 \cdot 23392301) \mod 100 = (82 \cdot 1) \mod 100 = 82$ 

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Recusion depth:  $O(\log b)$ 

The non-recursive part of each call involves a constant number of polynomial-time operations

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Two integers a, b are coprime if gcd(a, b) = 1

**Theorem:** b is invertible modulo N if and only if b and N are coprime

### Bézout's identity

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- Since XN + Yb = 1 we have  $0 + Yb = 1 \pmod{N} \implies Y$  is an inverse for b.

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The **order** of a group is the cardinality |G| of G. If G is a finite set, then the group is **finite**. If the operation  $\circ$  is commutative (i.e.,  $a \circ b = b \circ a$  for all  $a, b \in G$ ) then the group is **Abelian**.

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- $(\mathbb{Z}, +)$
- $(\mathbb{Z}, \cdot)$
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# or 2, ... $\mathbb{Q} \setminus \{0\}$

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- $(\{1, ..., N-1\}, \circ)$  where  $a \circ b = ab \mod N$
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In the following we will only consider finite Abelian groups!

Depending on the context, it might be convenient to write the group operation as + or as  $\cdot$ Keep in mind that it is still **not** a regular addition or multiplication, but the group operation instead!

Additive notation

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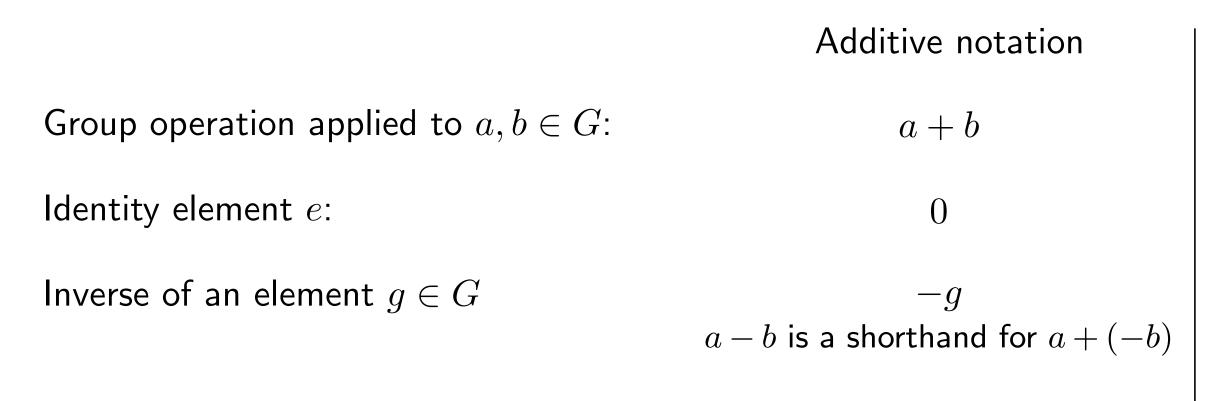
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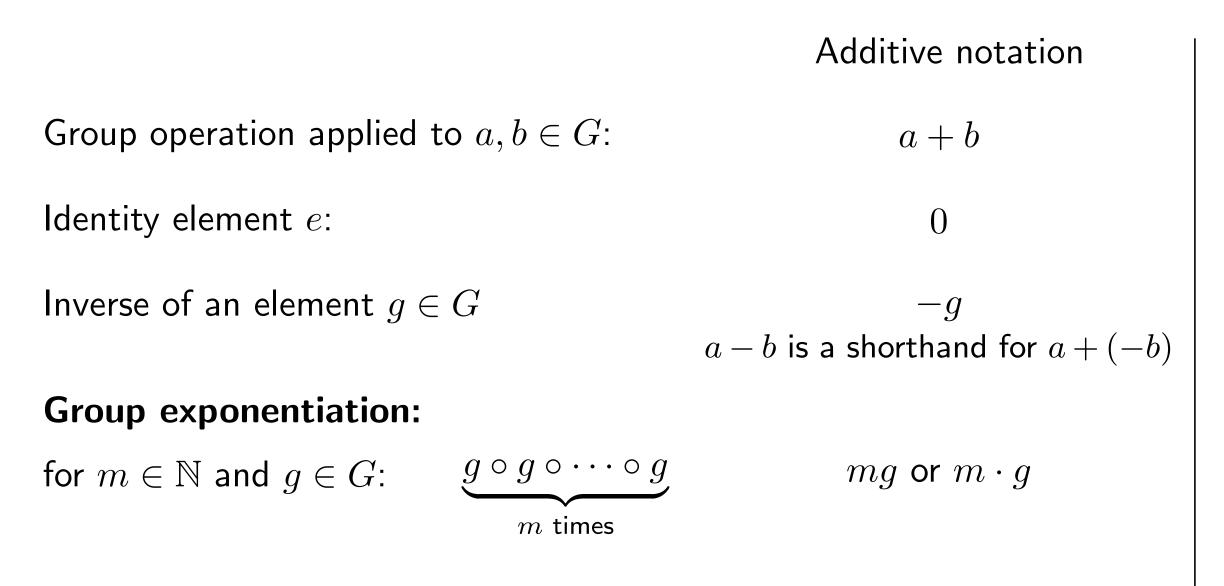
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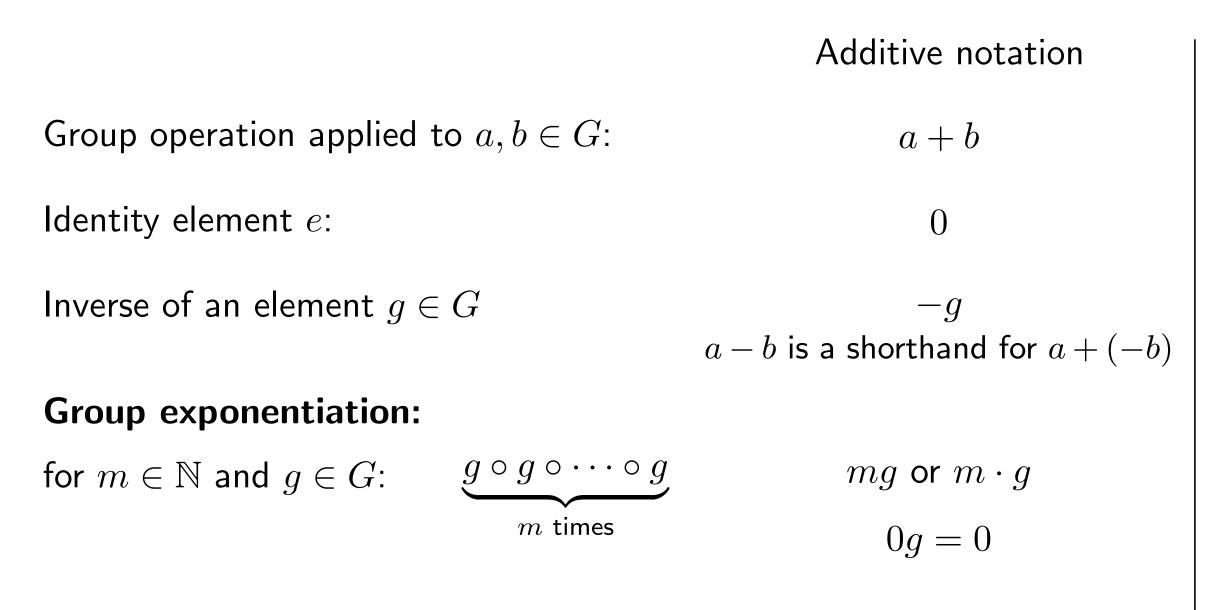
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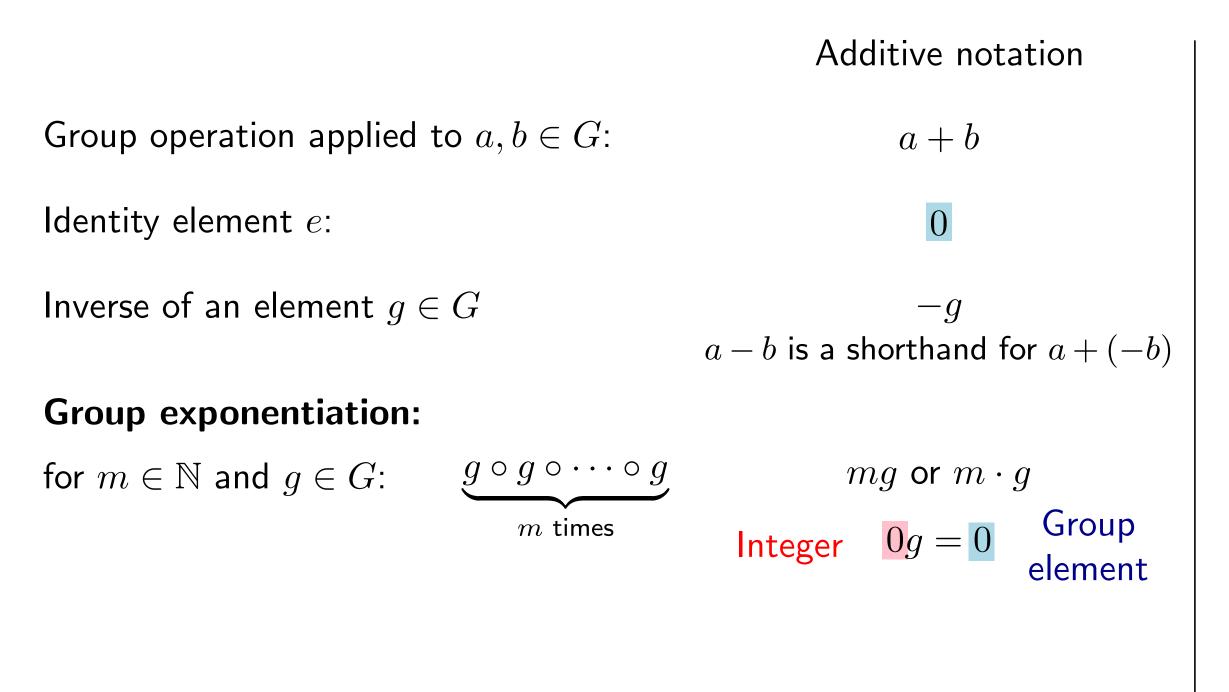
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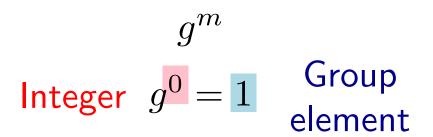


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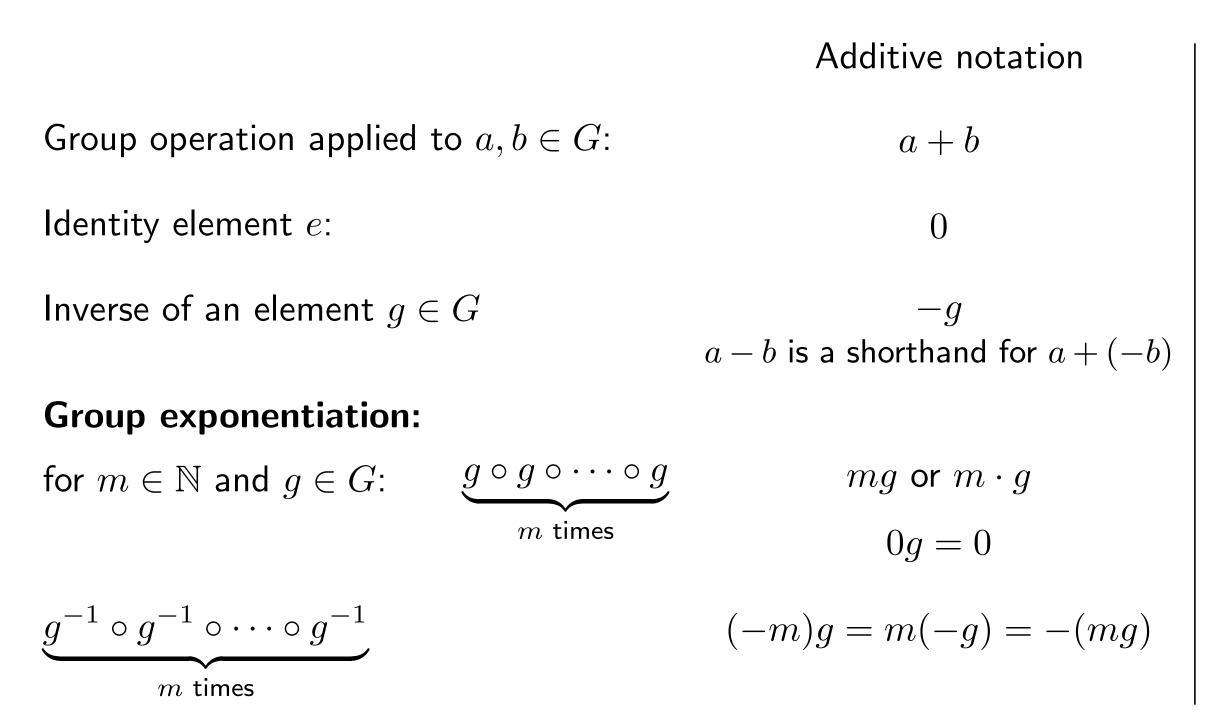
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 $g^{-m} = (g^{-1})^m = (g^m)^{-1}$ 

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If b < 0 then compute  $h = g^{-1}$  and then  $h^{|b|}$ . For  $b \ge 0$ :

Divide and conquer:

- If b = 0 return 1
- If b is even: recursively compute  $x=g^{b/2}$  and return  $x\cdot x$
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If the group operation can be computed in polynomial-time, then group exponentiation can be performed in polynomial-time

# The group $\mathbb{Z}_N$ under addition modulo N

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- Existence of inverses: The inverse of g is  $-g \mod N$  (recall that  $-g \mod N$  is an integer between 0 and N - 1).

Let  $\mathbb{Z}_N^* = \{0 < x < N \mid gcd(x, N) = 1\}$ . The set  $\mathbb{Z}_N^*$  is an Abelian group under multiplication modulo N.

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**Consequence:** If p is a prime number then  $\{1, 2, \ldots, p-1\}$  is an Abelian group under multiplication modulo p.

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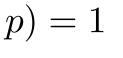
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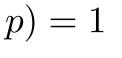
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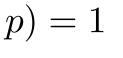
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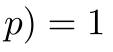
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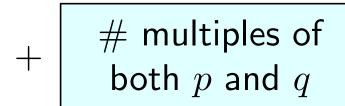
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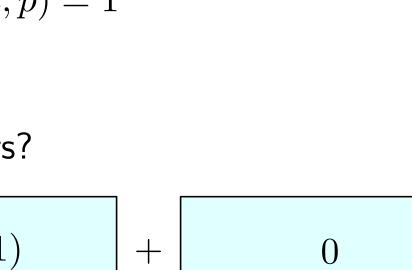
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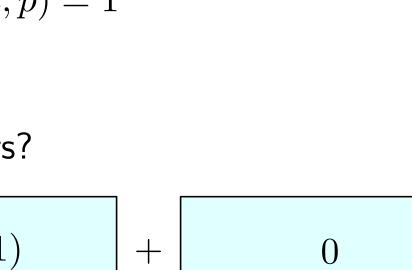
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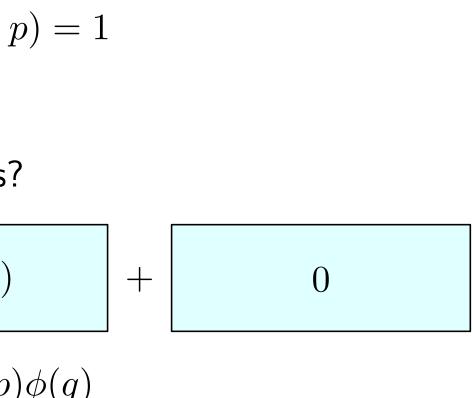
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Multiplying both sides by  $(g_1g_2\ldots g_m)^{-1}$ 

$$(g_1g_2\dots g_m)^{-1}(g_1g_2\dots g_m) = g^m(g_1g_2\dots g_m)(g_1g_2\dots g_m)$$

 $(g_m)^{-1}$ 

**Theorem:** Let G be a finite group of order m and let  $g \in G$ . Then  $g^m = 1$ .

Proof in the Abelian case:

Let 
$$G = \{g_1, g_2, \dots, g_m\}.$$

Since  $gg_i = gg_j \implies g^{-1}gg_i = g^{-1}gg_j \implies g_i = g_j$  we have  $g_i \neq g_j \implies gg_i \neq gg_j$ Then:

$$g_1g_2...g_m = (gg_1)(gg_2)...(gg_m) = g^m(g_1g_2...g_m)$$

(Each side of the equation contains only distinct elements, since the order of G in m, all elements are multiplied)

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$$1 = (g_1 g_2 \dots g_m)^{-1} (g_1 g_2 \dots g_m) = g^m (g_1 g_2 \dots g_m) (g_1 g_2 \dots g_m)^{-1} = g^m$$

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**Theorem:** Let G be a finite group of order m and let  $g \in G$ . Then  $g^m = 1$ .

In  $\mathbb{Z}_N$  (under addition modulo N):

• For all 
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In  $\mathbb{Z}_N^*$  (under multiplication modulo N):

- For all  $a \in \mathbb{Z}_N^*$ , we have  $a^{\phi(N)} = 1$
- For all  $a \in \mathbb{Z}_p^*$  where p is prime, we have  $a^{p-1} = 1$

### $\pmod{N}$ .

**Theorem:** Let G be a finite group of order m and let  $g \in G$ . Then  $g^m = 1$ .

**Corollary:** Let G be a finite group of order m > 1 and let  $g \in G$ . For any integer x,  $g^x = g^{x \mod m}$ .

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Proof: Write x as qm + r with  $r \in \{0, ..., m-1\}$ .  $g^x = g^{qm+r} = (g^m)^q \cdot g^r = 1^q \cdot g^r = g^r$ .

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**Corollary:** Let G be a finite group of order m > 1. Let e > 0 be an integer, and define the function  $f: G \to G$  as  $f_e(g) = g^e$ . If gcd(e, m) = 1 then

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Proof: We just need to show 2) since this implies that  $f_e$  injective and surjective, i.e., a bijection.

$$f_d(f_e(g)) = (g^e)^d = g^{ed} = g^{ed \mod m} = g^{1 \mod m} = g.$$

$$(g^r \cdot g^r) = 1^q \cdot g^r = g^r.$$

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Use the tools from number theory and group theory to...

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- Use the "hardness" of this problem to build secure public-key schemes

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We will be interested in working with prime numbers

The security parameter n will be related to the number of bits of the prime numbers

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- Repeat up to t times:
  - Choose a number p u.a.r. among all n-bit numbers Pick r u.a.r. in  $\{0,1\}^{n-1}$  and let  $p \leftarrow 1 || r$ .
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The algorithm has a polynomial running time and fails with negligible proability!

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Negligible

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In practice randomized algorithm are used, since they are faster and fail with negligible probability.

- The Miller-Rabin primality test is a probabilistic polynomial-time algorithm with one-sided error
- If n is prime, the Miller-Rabin primality test reports n as prime with certainty
- If n is composite, the Miller-Rabin primality test might report n as prime, but only with negligible probability.

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- Two *n*-bit integers  $x_1, x_2$  are chosen u.a.r., and  $N = x_1 \cdot x_2$  is computed
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With probability  $1 - (\frac{1}{2})^2 = \frac{3}{4}$  at least one of  $x_1$  and  $x_2$  is even  $\implies N$  is even  $\implies \mathcal{A}$  wins the experiment.

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Let GenModulus be a polynomial-time algorithm that, on input  $1^n$ , outputs a triple (N, p, q) where N = pq, and p and q are n-bit primes, except with probability negligible in n.



# The Factoring Assumption

We can now revise the previous experiment. For an algorithm  $\mathcal{A}$ , define Factor<sub> $\mathcal{A}$ ,GenModulus</sub>(n) as:

- Run GenModulus $(1^n)$  to obtain (N, p, q).
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**The factoring assumption:** there exists a GenModulus algorithm relative to which the factoring problem is hard.

**Recall:** this is just an assumption. We don't currently know whether the factoring problem is hard.

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Pick  $e \in \mathbb{Z}_N^*$  such that  $gcd(e, \phi(N)) = 1$ .

- By the corollary of Fermat's little theorem,  $f_e(x) = x^e$  is a permutation of  $\mathbb{Z}_N^*$
- Let d be the inverse of e modulo  $\phi(N)$ . Then  $f_d(x) = x^d$  is the inverse of  $f_e$ .

$$(x^e)^d = (x^d)^e = x$$

(All the operations are performed modulo N)

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- If p and q are not known:
  - Computing  $\phi(N)$  is as hard as factoring N
  - We don't know how to compute d without knowing  $\phi(N)$
  - ???

# The RSA problem

**Informally:** given a random  $y \in \mathbb{Z}_N^*$ , computing  $y^{1/e}$  is hard

Let GenRSA be a polynomial-time algorithm that, on input  $1^n$ , outputs a triple (N, e, d) where:

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A possible implementation:

- Generate two n-bit primes p, q chosen u.a.r.
- $\bullet \ N \leftarrow p \cdot q$
- $\phi(N) \leftarrow (p-1) \cdot (q-1)$
- Pick some e with  $\gcd(e,\phi(N))=1$
- $d \leftarrow e^{-1} \pmod{\phi(N)}$
- Output (N, e, d)

# The RSA problem

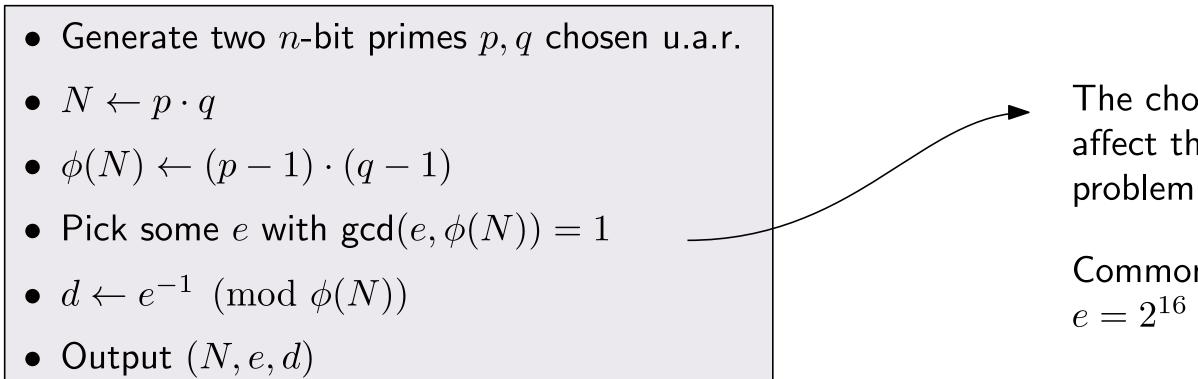
**Informally:** given a random  $y \in \mathbb{Z}_N^*$ , computing  $y^{1/e}$  is hard

Let GenRSA be a polynomial-time algorithm that, on input  $1^n$ , outputs a triple (N, e, d) where:

- N = pq, for two n-bit primes p and q
- $ed = 1 \pmod{\phi(N)}$

The algorithm is allowed to fail with negligible probability.

A possible implementation:



The choice of e is not believed to affect the hardness of the RSA problem

Common choices: e = 3 or  $e = 2^{16} + 1$  for efficiency reasons

# The RSA assumption

For an algorithm  $\mathcal{A}$ , define the experiment RSA-inv<sub> $\mathcal{A}$ ,GenRSA</sub>(n) as:

- Run GenRSA( $1^n$ ) to obtain (N, e, d).
- Choose  $y \in \mathbb{Z}_N^*$  u.a.r.
- Send N, e and y to  $\mathcal{A}$
- $\mathcal{A}$  outputs  $x \in \mathbb{Z}_N^*$
- The outcome of the experiment is 1 if x is the e-th root of y, i.e., if  $x^e = y$  (or equivalently  $y^{1/e} = y^d = x$ ). Otherwise the outcome is 0.

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**The RSA assumption:** there exists a GenRSA algorithm relative to which the RSA problem is hard.

### The RSA assumption and the factoring assumption

**The RSA assumption:** there exists a GenRSA algorithm relative to which the RSA problem is hard.

**The factoring assumption:** there exists a GenModulus algorithm relative to which the factoring problem is hard.