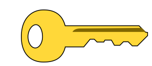
The Key Exchange Experiment

Fix a key exchange protocol Π and an attacker $\mathcal{A}$

We define the **key-exchange experiment** $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ as follows:

- The honest parties run Π using n as the security parameter.
- The interaction results in a transcript τ and in a shared key $k \in \{0,1\}^n$



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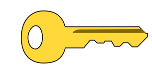

- A random bit b is chosen u.a.r. from $\{0,1\}$.
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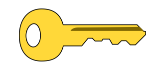

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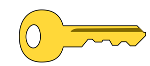

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The outcome of the experiment is defined to be 1 if $b' = b$ and 0 otherwise

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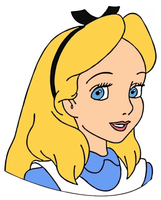

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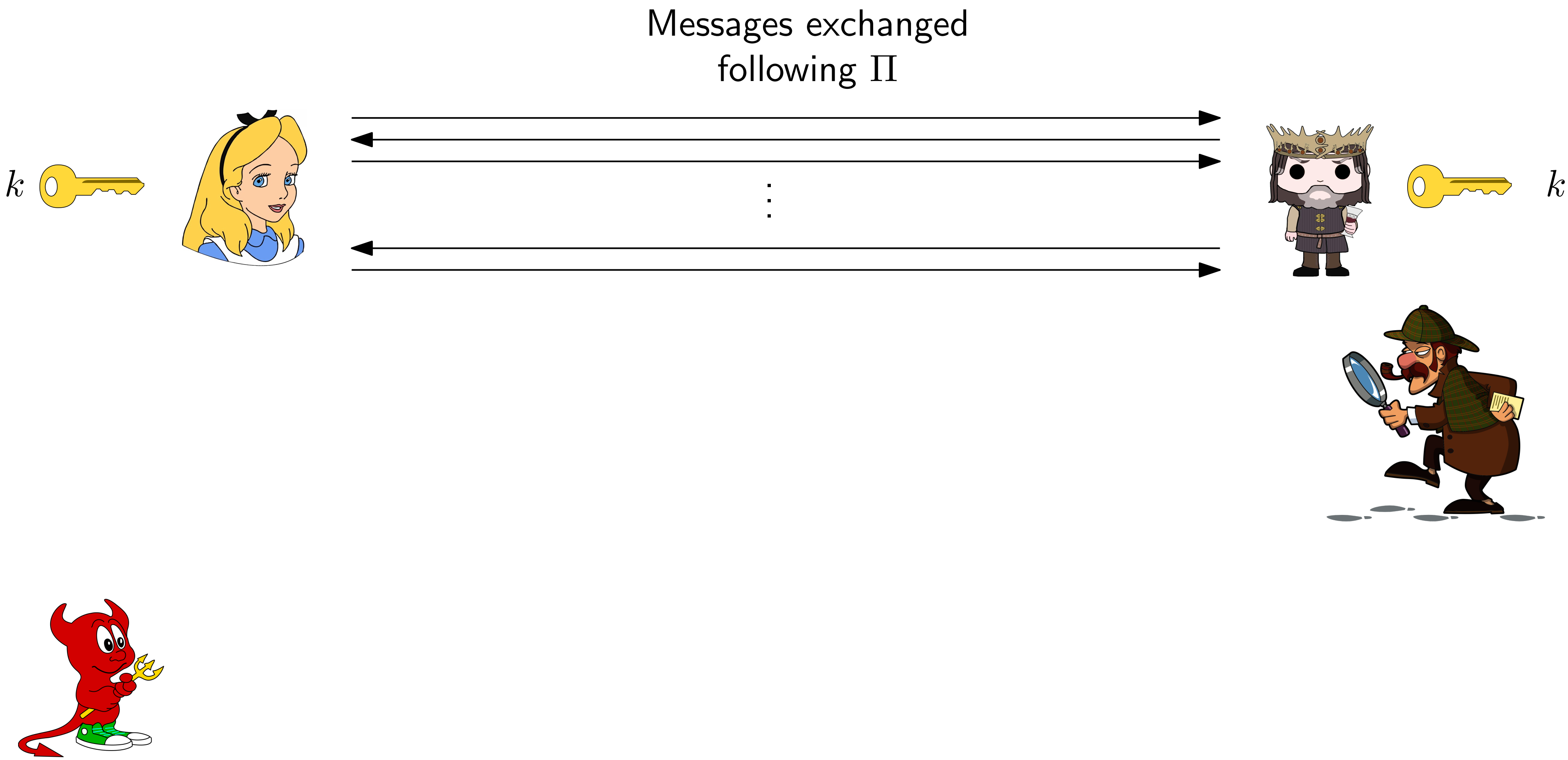

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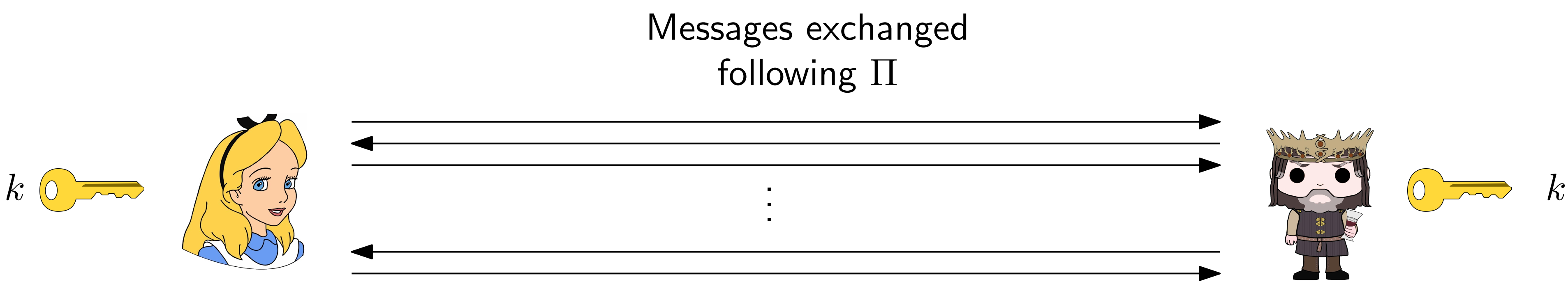
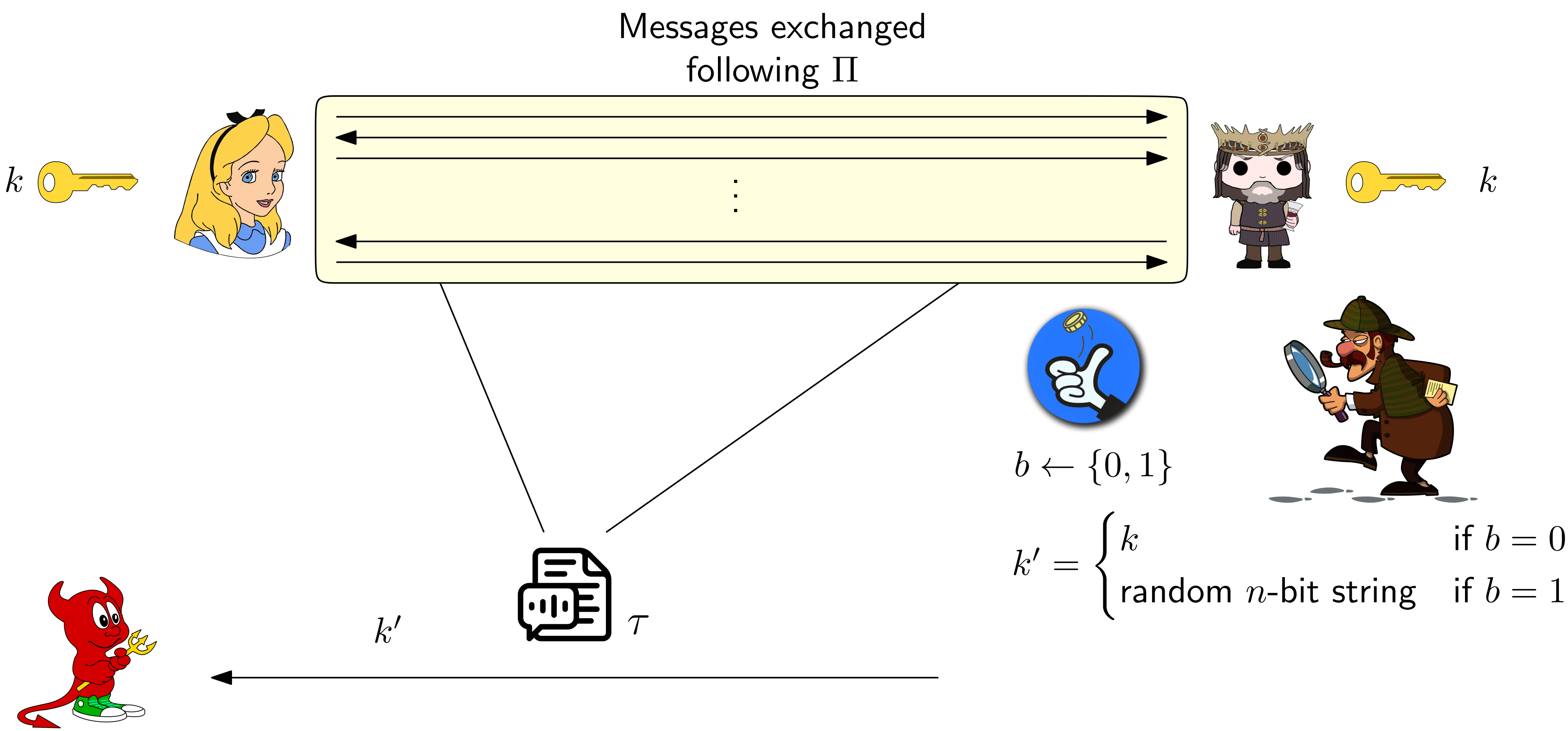


$b \leftarrow \{0,1\}$

$$k' = \begin{cases} k & \text{if } b = 0 \\ \text{random } n\text{-bit string} & \text{if } b = 1 \end{cases}$$

The Key Exchange Experiment



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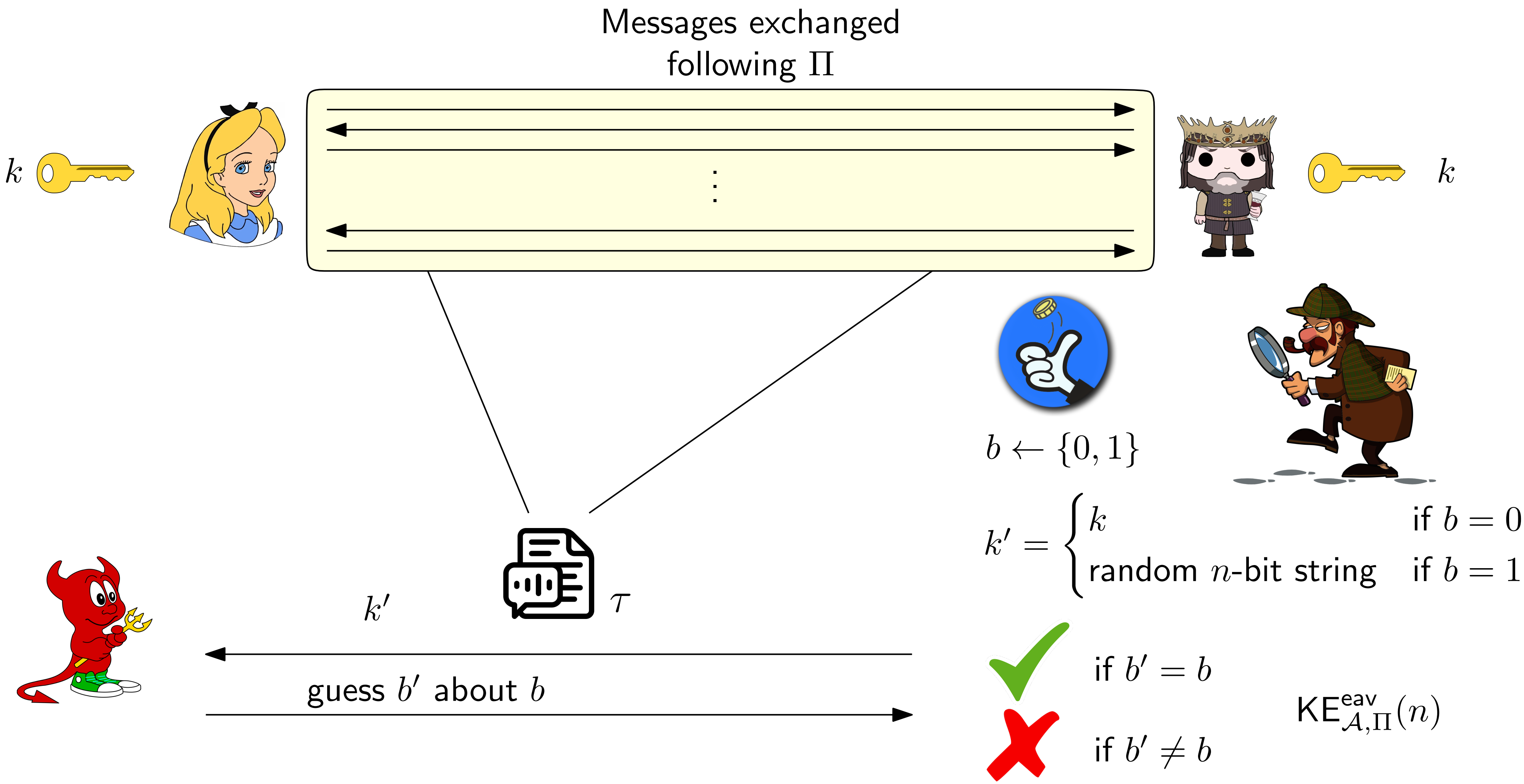

Key Exchange: Formal Security Definition

Definition: A key-exchange protocol Π is **secure in the presence of an eavesdropper** if for all probabilistic polynomial-time adversaries $\mathcal{A}$ there is a negligible function ε such that

$$\Pr[\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \varepsilon(n).$$

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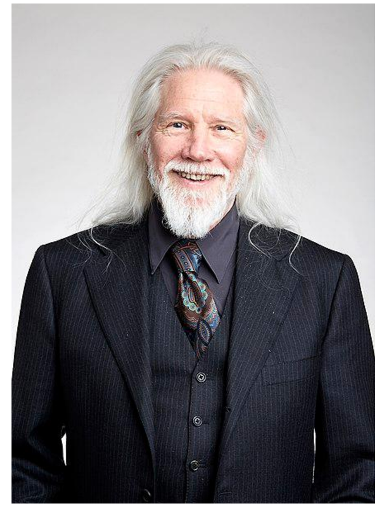
Notice that being unable to compute k from the transcript τ is not a strong enough security guarantee

- The requirement we impose is **stronger**. Namely k must look just like a random string.
- This is necessary since we are going to use k for private-key cryptography.

Diffie-Hellman Key Exchange

Let $\mathcal{G}$ be a probabilistic polynomial-time algorithm that, on input 1^n , outputs a description of a cyclic group G, its order q, where q is a n-bit integer, and a generator $g \in G$.

- Alice runs $\mathcal{G}(1^n)$ to obtain (G, q, g).



Whitfield Diffie

Martin Hellman

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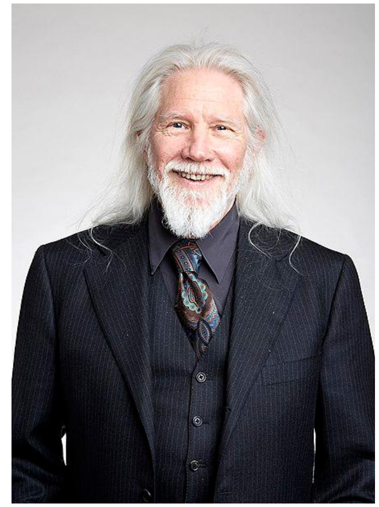

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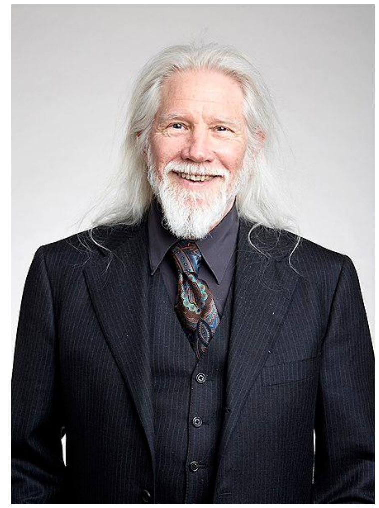

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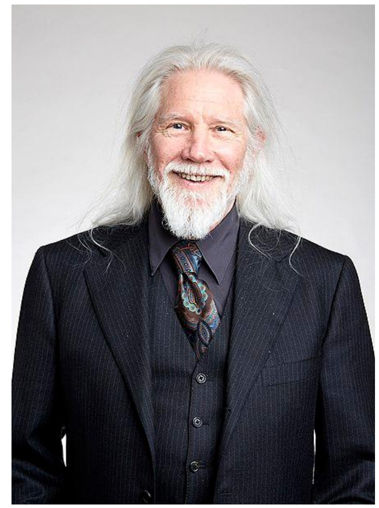

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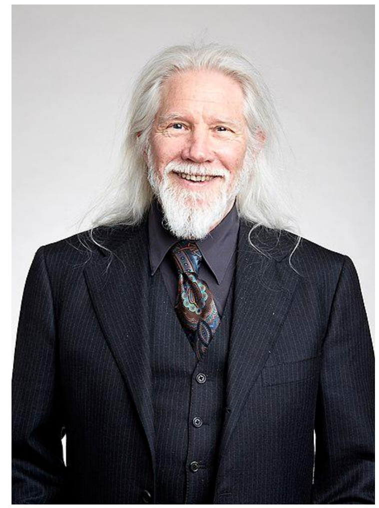

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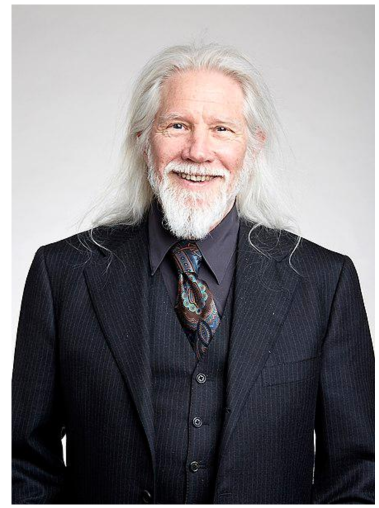

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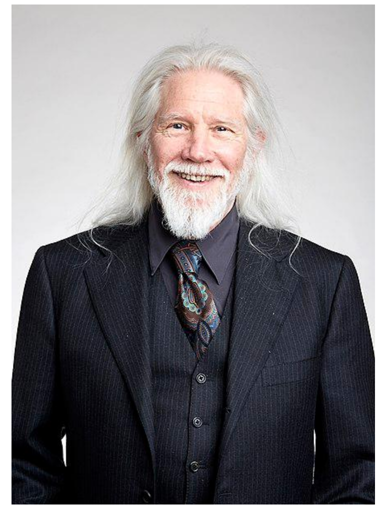
Notice that $k = h_A^y = (g^x)^y = g^{xy} = (g^y)^x = h_B^x$



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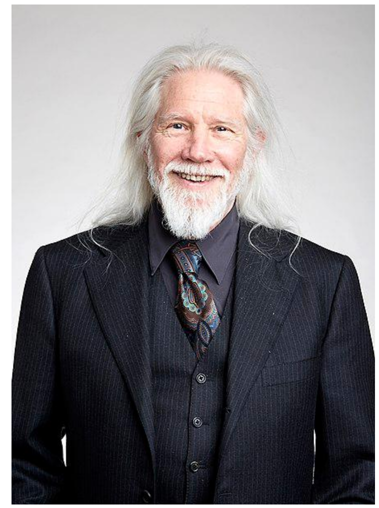

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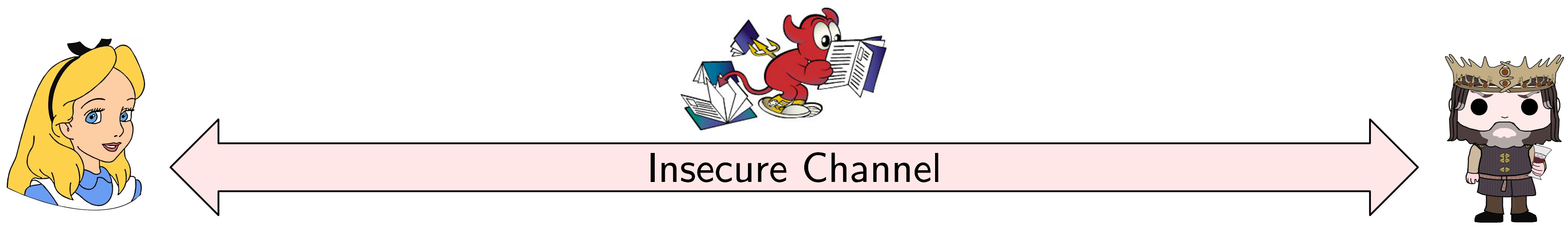
In practice the group G and the generator g are fixed and already known by Alice and Bob

Alice only needs to send h^A to Bob

Diffie-Hellman Key Exchange



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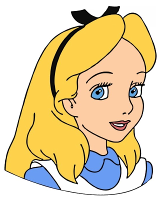

Insecure Channel

Suppose that Alice and Bob agreed to use a group G of order q and a generator $g \in G$.

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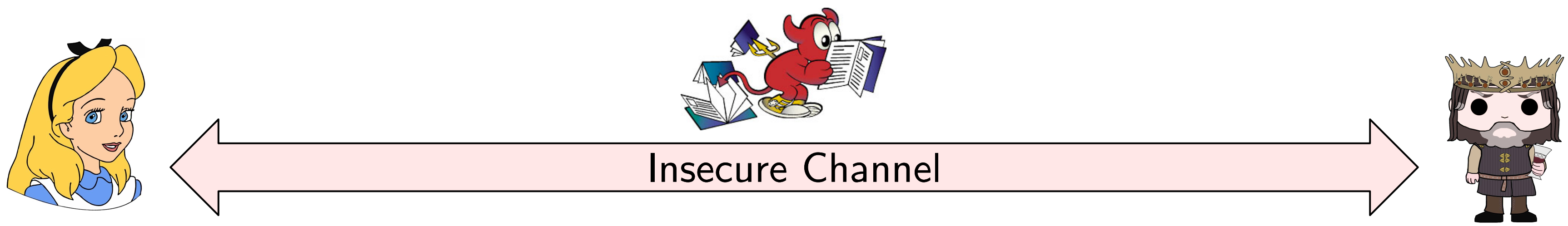
Diffie-Hellman Key Exchange



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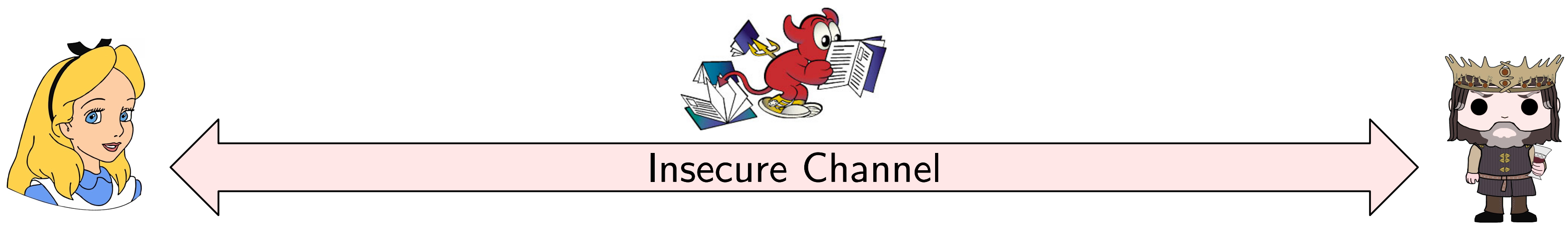
- Compute $h_B = g^y$
- Send h_B to Alice

Diffie-Hellman Key Exchange



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- Pick x u.a.r. from $\{0, \dots, q-1\}$
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- Send h_A to Bob
- Compute $h_B^x = (g^y)^x = g^{xy}$

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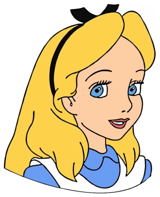

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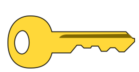
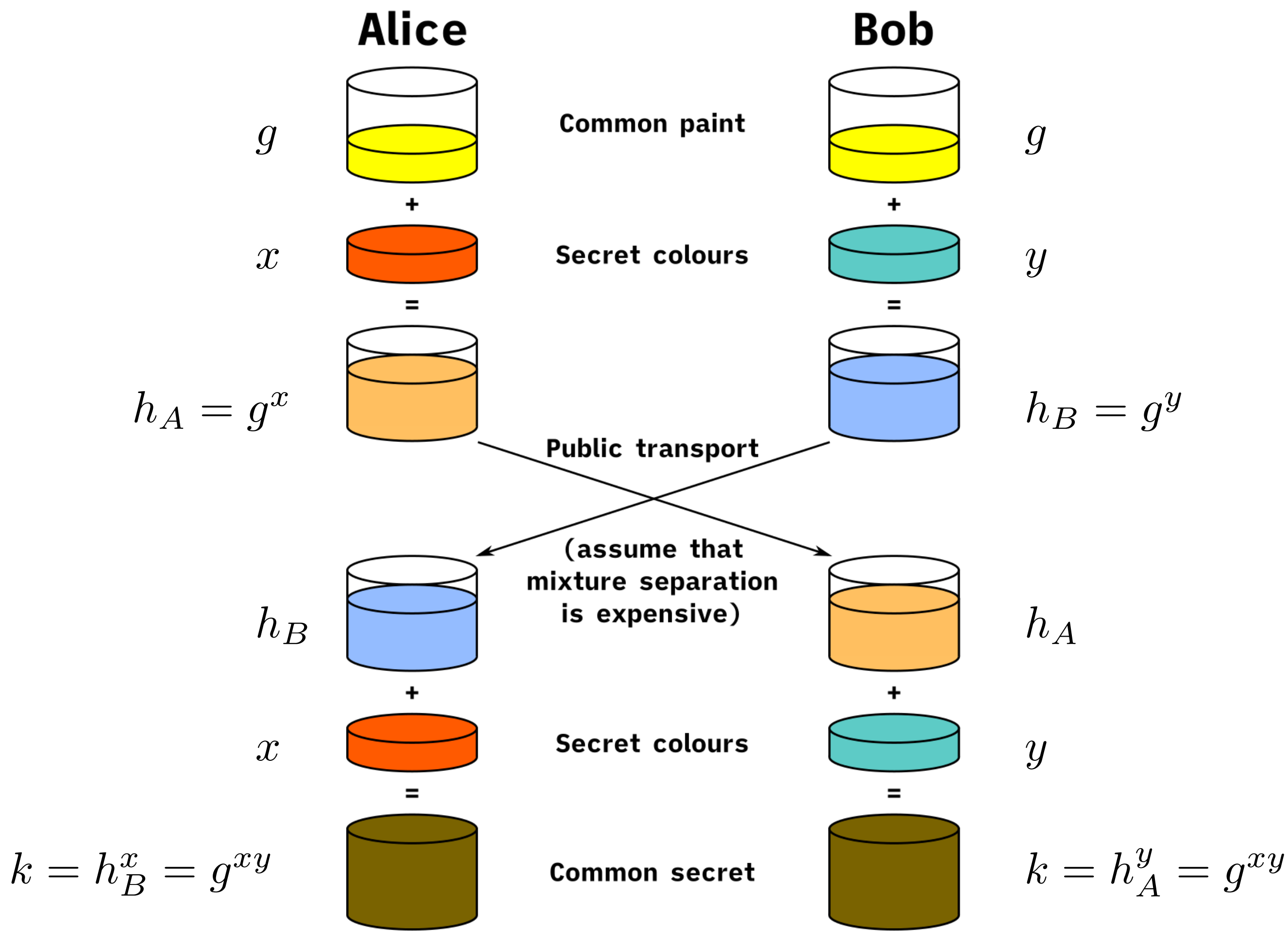


The shared secret key is $k = g^{xy}$

Diffie-Hellman Key Exchange



Diffie-Hellman Key Exchange: Example

Suppose that Alice and Bob agreed to use the group $G = \mathbb{Z}_{11}^*$ (of order $q = 10$) and the generator $g = 2$

This is just for the sake of the example! Recall that Z_{11}^* is **not** a good choice!

- Alice picks $x = 4$
  (chosen u.a.r. from $\{0, \ldots, 10\}$)

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- Bob computes $h_B = 2^3 = 8$

- Alice sends h_A to Bob

h_A

h_B

- Bob sends h_B to Alice

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• Alice computes $h_A = 2^4 = 5$		• Bob computes $h_B = 2^3 = 8$
• Alice sends h_A to Bob	$\xrightarrow{h_A}$	
	$\xleftarrow{h_B}$	• Bob sends h_B to Alice
• Alice computes $k = 8^4 = 4$		• Bob computes $k = 5^3 = 4$

The shared secret key is $k = 4$

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$\xrightarrow{h_A}$

$\xleftarrow{h_B}$

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- Bob computes $k = 5^3 = 4$

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Diffie-Hellman Key Exchange: Security

The adversary sees (or knows already) G, q, g

The adversary sees $h_A = g^x$ and $h_B = g^y$

The shared key is $k = g^{xy}$

What kind of assumption do we need to guarantee security of the Diffie-Hellman Key Exchange?

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What kind of assumption do we need to guarantee security of the Diffie-Hellman Key Exchange?

Wait... what were the assumptions again?

A reminder...

Reminder: The Discrete Logarithm Assumption

For a group-generation algorithm $\mathcal{G}$ and an algorithm $\mathcal{A}$, define the experiment $\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n)$ as:

- Run $\mathcal{G}(1^n)$ to obtain (G, q, g), where G is a cyclic group of order q (where q is a n-bit integer), and g is a generator of G.
- Choose a uniform $h \in G$.
- G, q, g and h are given to $\mathcal{A}$
- $\mathcal{A}$ outputs $x \in \{0, \ldots q-1\}$
- The outcome of the experiment is 1 if $g^x = h$. Otherwise the outcome is 0.

Definition The discrete-logarithm problem is hard relative to $\mathcal{G}$ if for all probabilistic polynomial-time algorithms $\mathcal{A}$ there exists a negligible function ε

$$\Pr[\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n) = 1] \leq \varepsilon(n).$$

The discrete logarithm assumption: there exists a group-generation algorithm $\mathcal{G}$ for which the discrete-logarithm problem is hard

Reminder: The CHD Problem and Assumption

Given $g, h_1, h_2 \in G$, define: $\mathsf{DH}_g(h_1, h_2) = g^{\log_g h_1 \cdot \log_g h_2}$

The **Computational Diffie-Hellman (CDH) problem** is that of computing $\mathsf{DH}_g(h_1, h_2)$ given a group G, a generator g, and two elements h_1, and h_2 chosen u.a.r. from G

Definition The CDH problem is hard relative to $\mathcal{G}$ if for all probabilistic polynomial-time algorithms $\mathcal{A}$ there exists a negligible function ε such that

$$\Pr[\mathcal{A}(G, q, g, h_1, h_2) = \mathsf{DH}_g(h_1, h_2)] = \varepsilon(n),$$

where the probabilities are taken over the experiment in which $\mathcal{G}(1^n)$ outputs (G, q, g), and uniform $h_1, h_2 \in G$ are chosen.

The CDH assumption: there exists a group-generation algorithm $\mathcal{G}$ for which the CDH problem is hard

Reminder: the DDH Problem and Assumption

Definition The DDH problem is hard relative to $\mathcal{G}$ if for all probabilistic polynomial-time algorithms $\mathcal{A}$ there exists a negligible function ε such that

$$\Big| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] \Big| \leq \varepsilon(n),$$

where the probabilities are taken over the experiment in which $\mathcal{G}(1^n)$ outputs (G, q, g), and then uniform $x, y, z \in \{0, 1, \ldots, q-1\}$ are chosen (therefore g^x and g^y are uniformly distributed in G).

The DDH assumption: there exists a group-generation algorithm $\mathcal{G}$ for which the DDH problem is hard

Diffie-Hellman Key Exchange: Security (DLog)

The adversary sees (or knows already) G, q, g

The adversary sees $h_A = g^x$ and $h_B = g^y$

The shared key is $k = g^{xy}$

If the Discrete-Logarithm assumption does not hold, then computing discrete logarithms is easy for all groups output by poly-time group generation algorithms

In particular, it is easy for our group G

Diffie-Hellman Key Exchange: Security (DLog)

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In particular, it is easy for our group G

- The adversary knows $h_A = g^x$
- It can recover $x = \log_g h_A$ with non-negligible probability
- Once x is known, it can compute $h_B^x = g^{xy} = k$

Diffie-Hellman Key Exchange: Security (DLog)

The adversary sees (or knows already) G, q, g

The adversary sees $h_A = g^x$ and $h_B = g^y$

The shared key is $k = g^{xy}$

If the Discrete-Logarithm assumption does not hold, then computing discrete logarithms is easy for all groups output by poly-time group generation algorithms

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The Discrete-Logarithm assumption is a **necessary condition**

Diffie-Hellman Key Exchange: Security (CDH)

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If the CDH assumption does not hold, then computing $\mathsf{DH}(\cdot, \cdot)$ is easy for all groups output by poly-time group generation algorithms

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- The adversary knows $h_A = g^x$ and $h_B = g^y$
- It can compute $\mathsf{DH}_g(h_A, h_B) = g^{\log_g h_A \cdot \log_g h_B} = g^{xy} = k$ with non negligible probability

Diffie-Hellman Key Exchange: Security (CDH)

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Diffie-Hellman Key Exchange: Security (DDH)

The adversary sees (or knows already) G, q, g

The adversary sees $h_A = g^x$ and $h_B = g^y$

The shared key is $k = g^{xy}$

If the DDH assumption does not hold then, given g, g^x, and g^y, it possible to distinguish $k = g^{ab}$ from a group element chosen uniformly at random (for all groups output by poly-time group generation algorithms).

In particular, this holds for our group G.

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- The adversary knows $h_A = g^x$ and $h_B = g^y$
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- It can use this to win the $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ experiment

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The DDH assumption is a **necessary condition**

Diffie-Hellman Key Exchange: Security (Roadmap)

We will argue that the DDH assumption is also a **sufficient condition**

We first need to address a technical subtlety

- Intuitively, the DDH assumption guarantees that it is hard to distinguish $k = g^{xy}$ from a **random group element**
- The $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ experiment asks to distinguish k from a **random** n**-bit string**

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Solution (roadmap):

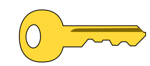
- We revise the $\mathsf{KE}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ experiment to take into account groups
- We prove security w.r.t. the revised experiment
- We argue that the shared key k can be turned into a (essentially random) n-bit string

Diffie-Hellman Key Exchange: Revised Experiment

Fix a group G, a key exchange protocol Π, and an attacker $\mathcal{A}$

We define the **key-exchange experiment** $\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ as follows:

- The honest parties run Π using n as the security parameter.
- The interaction results in a transcript τ and in a shared key $k \in G$



- A random bit b is chosen u.a.r. from $\{0, 1\}$.
  - If $b = 0$ then $k' \leftarrow k$.
  - Otherwise (when $b = 1$) k' is chosen as a random element from G.
- $\mathcal{A}$ is given k' and the transcript τ

- $\mathcal{A}$ outputs a bit $b' \in \{0, 1\}$

The outcome of the experiment is defined to be 1 if $b' = b$ and 0 otherwise

Diffie-Hellman Key Exchange: Security Proof

Theorem: If the DDH problem is hard relative to G, then the Diffie–Hellman key-exchange protocol Π is secure in the presence of an eavesdropper with respect to the modified experiment $\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$

Proof:

We show that a probabilistic polynomial-time adversary $\mathcal{A}$ that wins the experiment $\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$ with non-negligible advantage, can be used to design a probabilistic polynomial-time algorithm that solves the DDH problem with non-negligible gap.

- Suppose towards a contradiction that there exists $\mathcal{A}$ such that $\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] = \frac{1}{2} + \varepsilon(n)$ for non-negligible $\varepsilon(n)$.

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Diffie-Hellman Key Exchange: Security Proof (cont.)

$$\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 1] = \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]$$

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$$\tfrac{1}{2} + \varepsilon(n) = \Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] = \tfrac{1}{2}\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 1]$$

Diffie-Hellman Key Exchange: Security Proof (cont.)

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$$= \frac{1}{2} - \frac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \frac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]$$

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= \; & \tfrac{1}{2} - \tfrac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \tfrac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \\
= \; & \tfrac{1}{2} + \tfrac{1}{2} \cdot (-\Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1])
\end{aligned}$$

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& \leq \tfrac{1}{2} + \tfrac{1}{2} \cdot \Big| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \Big|
\end{aligned}$$

Diffie-Hellman Key Exchange: Security Proof (cont.)

$$\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 1] = \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]$$

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$$\begin{aligned}
\tfrac{1}{2} + \varepsilon(n) = \Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1] &= \tfrac{1}{2}\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 1] \\
&= \tfrac{1}{2} - \tfrac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \tfrac{1}{2} \cdot \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \\
&= \tfrac{1}{2} + \tfrac{1}{2} \cdot (-\Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]) \\
&\leq \tfrac{1}{2} + \tfrac{1}{2} \cdot \Big| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \Big|
\end{aligned}$$

$$\Big| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \Big| \geq 2\varepsilon(n)$$

Diffie-Hellman Key Exchange: Security Proof (cont.)

$$\Pr[\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n) = 1 \mid b = 1] \;\; = \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]$$

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& = \tfrac{1}{2} + \tfrac{1}{2} \cdot (-\Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] + \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1]) \\
& \leq \tfrac{1}{2} + \tfrac{1}{2} \cdot \left| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \right|
\end{aligned}$$

$$\left| \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^{xy}) = 1] - \Pr[\mathcal{A}(G, q, g, g^x, g^y, g^z) = 1] \right| \geq 2\varepsilon(n)$$

Not negligible

$\square$

Turning a random group element into a random binary string

We have shown the (conditional) security of the Diffie-Hellman protocol with respect to the modified experiment $\widehat{\mathsf{KE}}^{\mathsf{eav}}_{\mathcal{A},\Pi}(n)$

The Diffie-Hellman key exchange returns a group element $k = g^{xy}$ that is indistinguishable from a random one to any polynomial-time adversary

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Key derivation! Use a hash function H and set $k^* = H(k)$

This is secure if we model H as a random oracle

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So far we have considered eavesdropping adversaries

What about active adversaries?

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- And adversary interacts with both Alice and Bob
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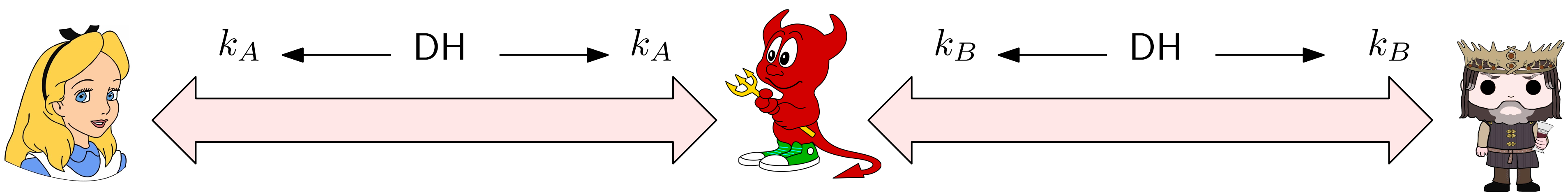

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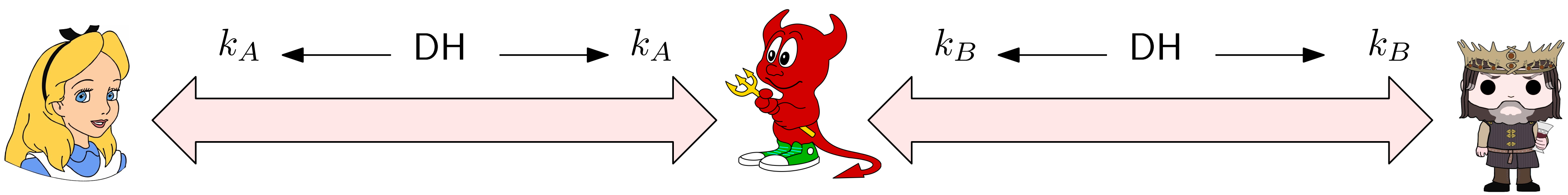

- At the end of the protocol Alice and Bob have two different keys k_A, k_B and both of these keys are known to the Adversary
- When a message is sent from A to B (or vice-versa) the adversary can decrypt it with k_A, read and possibily alter the plaintext, and re-encrypt it with k_B

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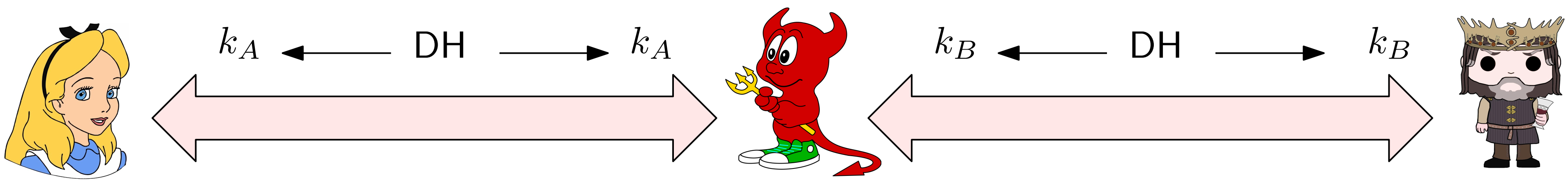

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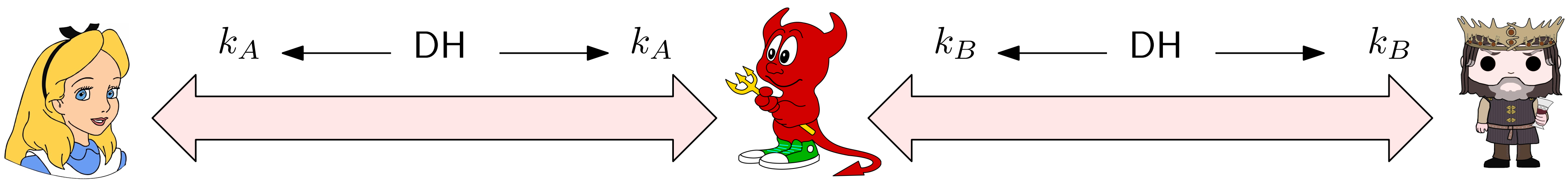

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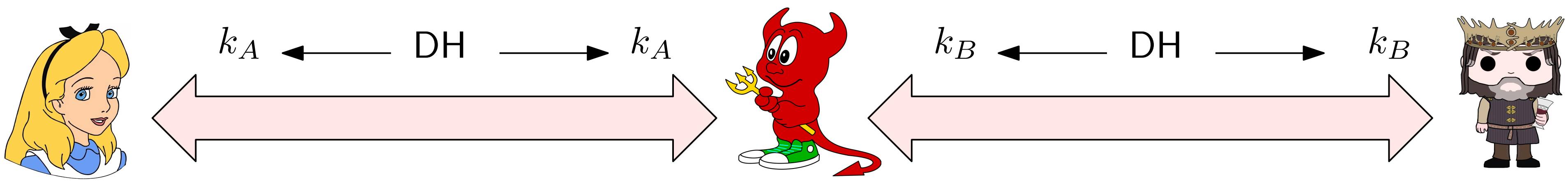

- In general we want authenticated key exchange protocols
- Can be achieved with public-key cryptography and digital signatures/certificates
- Modern key-exchange protocols (e.g., TLS) provide authentication

Private-Key Setting

Two (or more) parties who wish to communicate secretly need to share a uniform secret key k in advance

The same key can be used for both sending and receiving

- Each party can both send an receive
- If multiple parties share the same key, there is no way to distinguish them

The key must be kept secret!

- If an attacker gets to know k we lose all the security guarantees

Public-Key Setting

One party generates a pair of keys:

- A **public key** pk
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The secret key must be kept. . . secret

- It is only used by the party that generated it

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  - In a system with N users, there are $\Theta(N)$ key pairs, instead of $\Theta(N^2)$ private-keys
- **Open systems:** Two parties with no prior relationship can find each others' public keys*
  - The recipient does not even need to know who the sender is

*Requires a trusted third party

Public-Key Setting

Why study private key cryptography at all?

- If two parties wish to communicate, they can always generate two public/secret key pairs instead of a secret shared key

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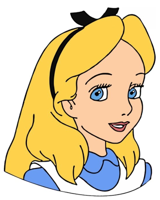
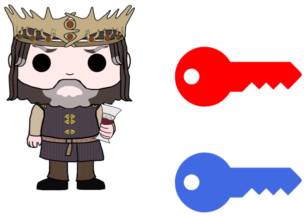
These issues can be mitigated

Public-Key Encryption





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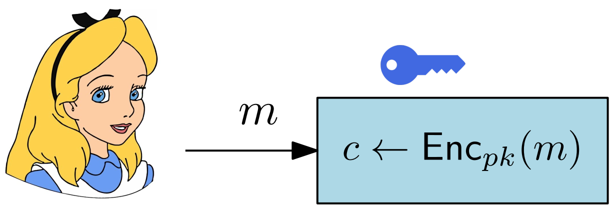

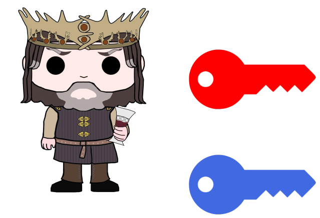

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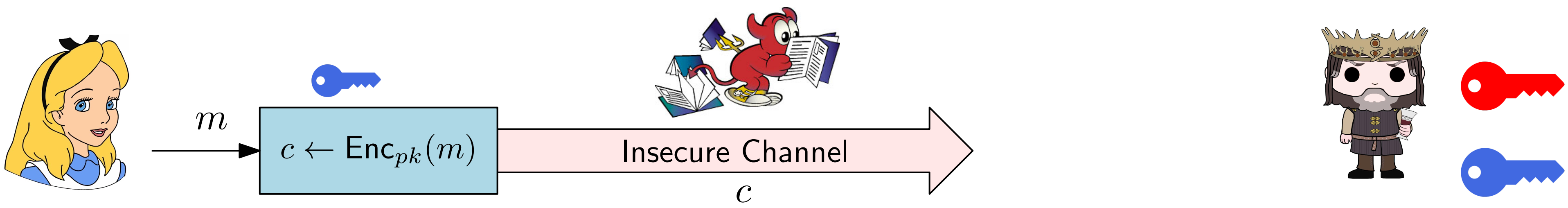

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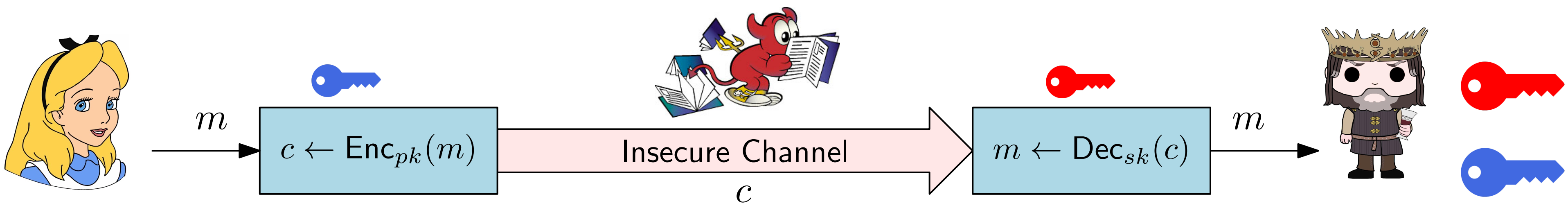

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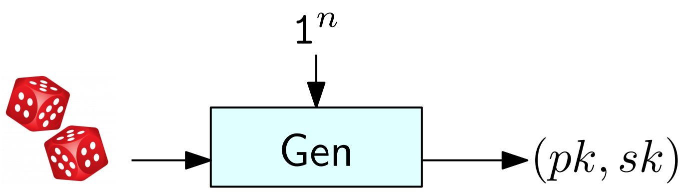
Bob **decrypts** the ciphertext c using his secret key sk

$$m \leftarrow \mathsf{Dec}_{sk}(c)$$

Public-key Encryption Schemes: Definition

A **public-key encryption scheme** consists of three algorithms:

- Gen is a randomized algorithm that takes 1^n as input and outputs a pair of keys (pk, sk), each of length at least n. The public key defines a message space $\mathcal{M}_{pk}$



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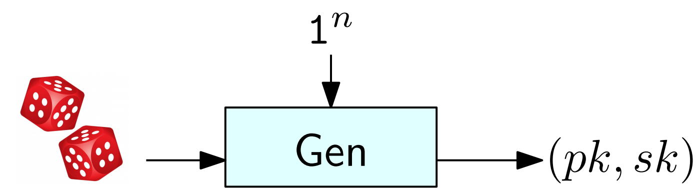

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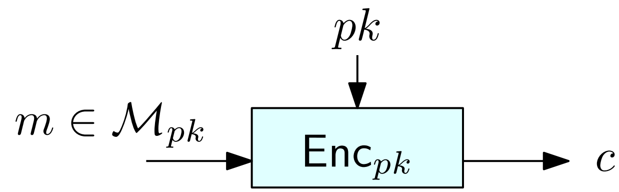

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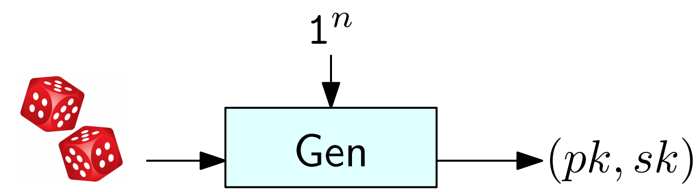

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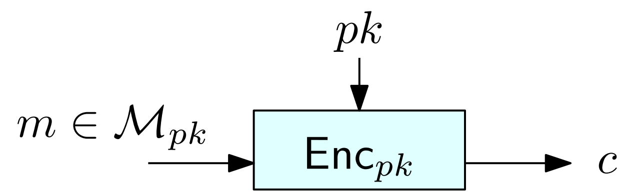

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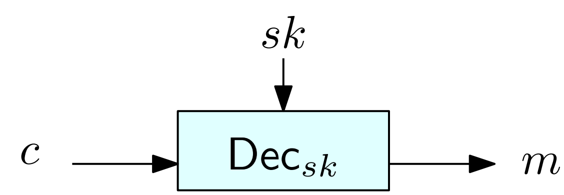

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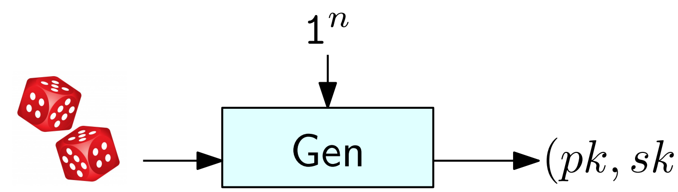

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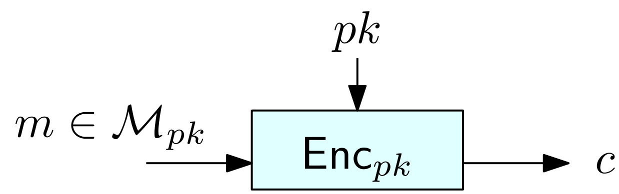

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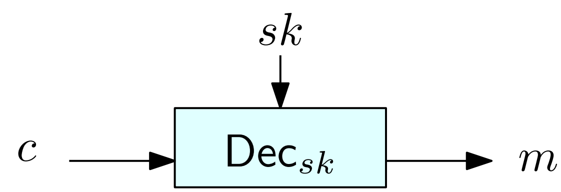

Correctness: We must have $\text{Dec}_{sk}(\,\text{Enc}_{pk}(m)\,) = m$ for any $m \in \mathcal{M}_{pk}$, except for negligible probability (over the randomness of Gen and Enc).

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Definition: *A public key encryption scheme* Π *is* **CPA-secure** *if, for every probabilistic polynomial-time adversary* $\mathcal{A}$, *there is a negligible function* ε *such that:*

$$\Pr[\mathit{PubK}^{\mathit{cpa}}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \varepsilon(n)$$

Some consequences

Any public-key encryption scheme that is CPA-secure is also CPA-secure for multiple encryptions

(just like the private-key setting)

No deterministic public-key encryption scheme can be CPA-secure

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- The adversary enumerates all possible messages $m' \in \mathcal{M}_{pk}$:
  - For each m', the adversary enumerates all R-bit binary strings r
    - The adversary runs $\mathsf{Enc}_{pk}(m')$ using r as the random bits to obtain a ciphertext c'
    - If $c = c'$: Claim that the plaintext m is exactly m'. Stop.

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    - If $c = c'$: Claim that the plaintext m is exactly m'. Stop.

Notice that:

- This algorithm must eventually stop (when $m' = m$ and r are the random bits used by $c \leftarrow \text{Enc}_{pk}(m)$)

Some consequences

There is no such thing as a perfectly secret public key encryption scheme

We show that an adversary can always decrypt a ciphertext $c \leftarrow \mathsf{Enc}_{pk}(m)$ using a brute-force attack.

(obviously, this adversary won't run in polynomial-time)

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This contradicts correctness.

Security Against Chosen-Ciphertext Attacks

Chosen-Ciphertext attacks are also a concern in the public-key setting

In fact, they can even be easier to execute

Think of the following scenario:

- The adversary intercepts the encrypted body of an email from Alice to Bob
- The adversary sends the encrypted body to Bob from his address
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- If the adversary is lucky, the reply email will contain a full decryption of the original message!

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It is possible to define CCA-security also for public-key encryption schemes

- In the private-key setting we have seen a stronger notion of security: **authenticated encryption**
- The definition of authenticated encryption does not immediately extend to the public-key setting (recall that anybody with the public key can encrypt messages)
- We will see another mechanism to guarantee authentication (digital signature schemes)

Definition of CCA-Security

Experiment $\mathsf{PubK}^{\mathsf{cca}}_{\mathcal{A},\Pi}(n)$:

- A random key pair $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$ is generated and pk is sent to $\mathcal{A}$
- $\mathcal{A}$ interacts with a decryption oracle
- $\mathcal{A}$ chooses two distinct messages $m_0, m_1 \in \mathcal{M}$
- A uniform random bit $b \in \{0, 1\}$ is generated
- The *challenge ciphertext* c is computed by $\mathsf{Enc}_{pk}(m_b)$, and given to $\mathcal{A}$
- $\mathcal{A}$ interacts with a decryption oracle but cannot request a decryption of c
- $\mathcal{A}$ outputs a guess $b' \in \{0, 1\}$ about b
- The *outcome of the experiment* is defined to be 1 if $b' = b$, and 0 otherwise

Definition: *A public key encryption scheme Π is* **CCA-secure** *if, for every probabilistic polynomial-time adversary $\mathcal{A}$, there is a negligible function ε such that:*

$$\Pr[\mathit{PubK}^{\mathit{cca}}_{\mathcal{A},\Pi}(n) = 1] \leq \frac{1}{2} + \varepsilon(n)$$

Public-Key Setting vs. Private-Key Setting

	Private-Key Setting	Public-Key Setting
Secrecy	Private-Key Encryption Schemes	
Integrity	Message Authentication Codes	

Public-Key Setting vs. Private-Key Setting

	Private-Key Setting	Public-Key Setting
Secrecy	Private-Key Encryption Schemes	Public-Key Encryption Schemes
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Public-Key Setting vs. Private-Key Setting

	Private-Key Setting	Public-Key Setting
Secrecy	Private-Key Encryption Schemes	Public-Key Encryption Schemes
Integrity	Message Authentication Codes	Digital Signature Schemes