Public-Key Setting

Why study private key cryptography at all?

- If two parties with to communicate, they can always generate two public/secret key pairs instead of a secret shared key
- Public-key schemes are slower
- Public-key schemes require longer keys
- Public-key schemes have larger ciphertext expansion
- Public-key schemes require stronger assumptions

These issues can be mitigated

Encrypting long messages

The public key encryption schemes we will see are defined for short messages

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$m = m_1 \,\|\, m_2 \,\|\, m_3 \,\|\, \ldots$

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Can we do better?

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- Asymptotically optimal ciphertext expansion: the only blocks that are expanded are the first (due to public-key encryption) and possibly the last (due to padding)

Key Encapsulation Mechanism

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We don't need the fully fledged formal definition of a public-key encryption scheme for such task

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Note: a public-key encryption scheme is *a possible way* to build a KEM

The KEM/DEM Paradigm

Unsurprisingly... we can use a KEM $\Pi = (\mathsf{Gen}, \mathsf{Encaps}, \mathsf{Decaps})$ and a private-key encryption scheme $\Pi' = (\mathsf{Gen}', \mathsf{Enc}', \mathsf{Dec}')$ to build a public-key encryption scheme $\Pi^{hy} = (\mathsf{Gen}^{hy}, \mathsf{Enc}^{hy}, \mathsf{Dec}^{hy})$.

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$\mathsf{Dec}^{hy}_{sk}(c'')$:

- Parse c'' as $c\|c'$ (return $\perp$ if this fails)
- $k \leftarrow \mathsf{Decaps}_{sk}(c)$
- $m \leftarrow \mathsf{Dec}'_k(c')$
- Return m

CPA-Security of KEMs

Let $\Pi = (\mathsf{Gen}, \mathsf{Encaps}, \mathsf{Decaps})$ be a KEM and let $\mathcal{A}$ be an algorithm. We denote the following experiment by $\mathsf{KEM}^{\mathsf{cpa}}_{\mathcal{A},\Pi}$

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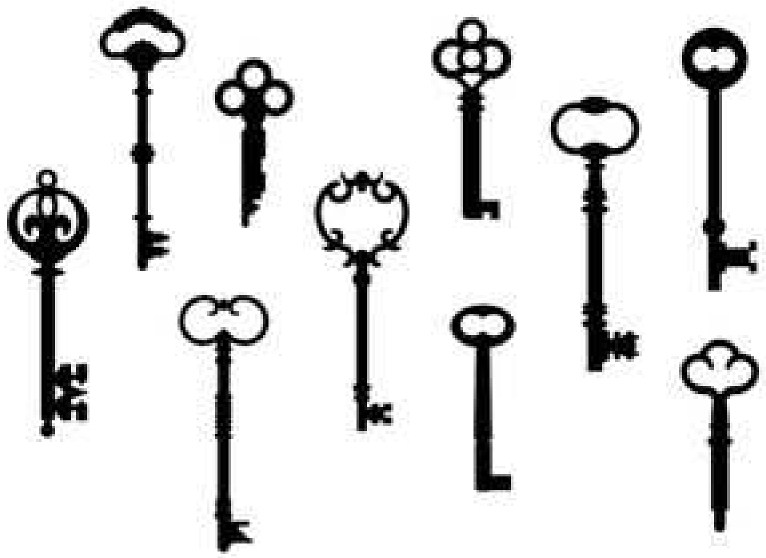
- A new key k is used for each encryption!



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High-level idea of the proof:

- Let k^* be a key chosen u.a.r. from $\{0,1\}^{\ell(n)}$
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From CPA-security of the KEM Π

(can't tell k and k^* apart)

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From EAV-security of the DEM Π'

Security Against Chosen Ciphertext Attacks

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The adversary can compute $c'' = c' \oplus \underbrace{00\ldots0}_{|c'|-1 \text{ times}} 1$, query the decryption oracle with c'' to obtain

$m'' = \text{Dec}_{sk}(\langle c, c'' \rangle)$, and recover $m = m'' \oplus 00\ldots01$

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- Pick x u.a.r. from $\{0, \ldots, q-1\}$
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- Send (G, q, g, h_A) to Bob

$\longrightarrow$

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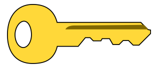

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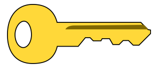

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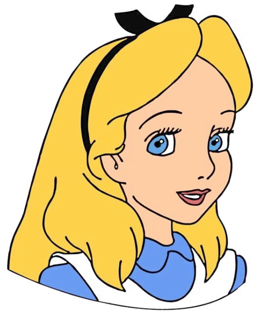

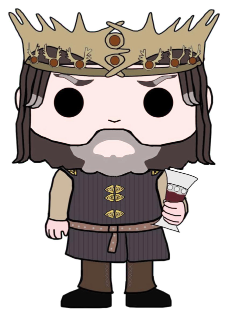

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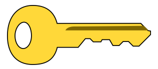
- Compute $k = h_A^y = (g^x)^y = g^{xy}$ 
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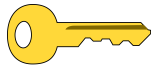
- Compute $k = h_B^x = (g^y)^x = g^{xy}$ 

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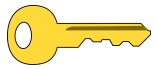
- Compute $k = h_A^y = (g^x)^y = g^{xy}$ 
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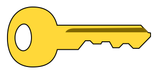
- Compute $k = h_B^x = (g^y)^x = g^{xy}$ 

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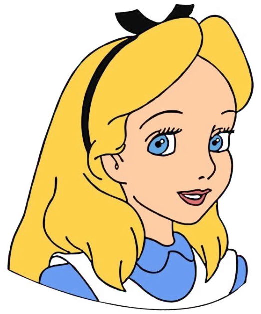

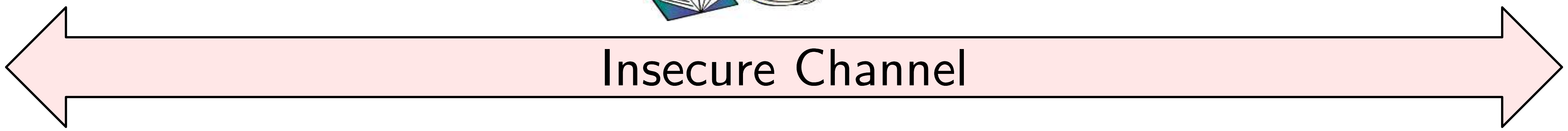

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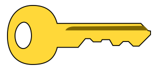
- Compute $k = h_A^y = (g^x)^y = g^{xy}$ 
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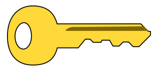
- Compute $k = h_B^x = (g^y)^x = g^{xy}$ 
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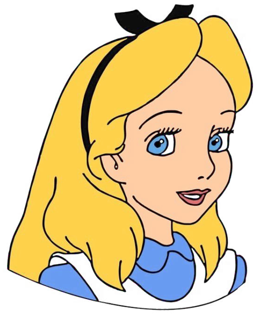

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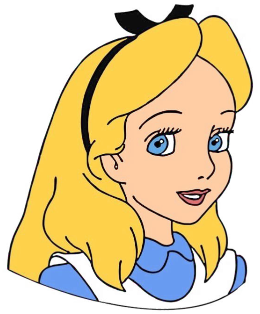

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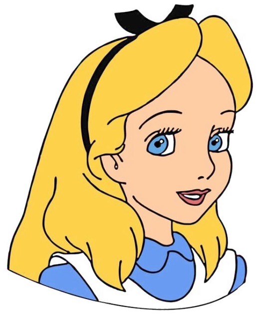

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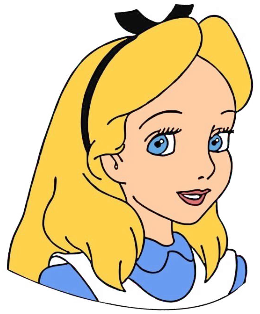

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We (essentially) have a public-key encryption scheme!

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El Gamal Encryption (more formally)

Gen(1^n):

- Run $\mathcal{G}(1^n)$, where $\mathcal{G}$ is a group generation algorithm, to obtain (G, q, g) where G is a group of order q and $g \in G$ is a generator
- Choose a uniform x u.a.r. from $\{0, \ldots, q-1\}$
- Compute $h = g^x$
- Output (pk, sk) where $pk = (G, q, g, h)$ and $sk = (G, q, g, x)$.

The message space $\mathcal{M}_{pk}$ is G.

Enc$_{pk}(m)$:

- Here $pk = (G, q, g, h)$ and $m \in G$
- Choose a uniform y u.a.r. from $\{0, \ldots, q-1\}$
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In practice the group G (and its order q) is fixed in advance along with a generator $g \in G$.

(just like in the Diffie-Hellman key exchange).

Security of El Gamal Encryption

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We show that a polynomial-time adversary that wins the $\mathsf{PubK}^{\mathsf{cpa}}_{\Pi,\mathcal{A}}(n)$ experiment can be used to solve the DDH problem with non-negligible advantage (in polynomial time).

Suppose towards a contradiction that there is some polynomial-time algorithm $\mathcal{A}$ such that $\Pr[\mathsf{PubK}^{\mathsf{cpa}}_{\Pi,\mathcal{A}}(n) = 1] = \frac{1}{2} + \varepsilon(n)$ for non-negligible $\varepsilon(n)$.

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- Notice that $\widetilde{\Pi}$ is **not** a correct public-key encryption scheme (decryption breaks).

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Let $\widetilde{\Pi}$ be a modified version of Π in which encryption is done by selecting $y, z \in \{0, 1, \ldots, q-1\}$ u.a.r., and returning $\langle g^y, g^z \rangle$.

- Notice that $\widetilde{\Pi}$ is **not** a correct public-key encryption scheme (decryption breaks).
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Security of El Gamal Encryption

If the DDH problem is hard relative to G, then the El Gamal encryption scheme Π is CPA-secure.

Proof:

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- The ciphertext $\langle g^y, g^z \rangle$ is completely independent from m!

Security of El Gamal Encryption (cont.)

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As an alternative, we can use our public-key encryption scheme as a KEM for Hybrid Encryption

El Gamal Key Encapsulation Mechanism

El Gamal as a KEM:

- Gen: as before
- Encaps: pick a random group element $\widetilde{k} \in G$ and encrypt it as $\langle g^y, h^y \cdot \widetilde{k} \rangle$. Return $(k, \langle g^y, h^y \cdot \widetilde{k} \rangle)$
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Use c_1^x as the key!

DDH-Based Key Encapsulation Mechanism

$\mathsf{Gen}(1^n)$:

- Run $\mathcal{G}(1^n)$, where $\mathcal{G}$ is a group generation algorithm, to obtain (G, q, g) where G is a group of order q and $g \in G$ is a generator
- Choose a uniform x u.a.r. from $\{0, \ldots, q-1\}$
- Compute $h = g^x$
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- Here $pk = (G, q, g, h, H)$
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$$\sum_{k \in \{0,1\}^{\ell(n)}} \left| \Pr[H(g) = k] - 2^{-\ell(n)} \right| \leq \varepsilon(n),$$

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If the DDH problem is hard relative to G, and H is chosen as above, then the DDH-based KEM is CPA-secure.

(Plain) RSA Encryption

Reminder: e-th roots

Let $N = pq$ where p and q are distinct odd primes

The order of Z_N^* is $\phi(N) = (p-1) \cdot (q-1)$

- Trivial to compute if we know p and q
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Pick $e \in \mathbb{Z}_N^*$ such that $\gcd(e, \phi(N)) = 1$.

- $f_e(x) = x^e$ is a permutation of $\mathbb{Z}_N^*$
- Let d be the inverse of e modulo $\phi(N)$. Then $f_d(x) = x^d$ is the inverse of f_e.
- $(x^e)^d = (x^d)^e = x$

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- "Hard" to compute if we know N but not p and q (can be shown to be equivalent to factoring N)

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- $f_e(x) = x^e$ is a permutation of $\mathbb{Z}_N^*$
- Let d be the inverse of e modulo $\phi(N)$. Then $f_d(x) = x^d$ is the inverse of f_e.
- $(x^e)^d = (x^d)^e = x$

Since $(x^e)^d = x$ we can think of x^d as the e-th root of x

- We define $x^{1/e} = x^d$

Reminder: the RSA problem

Let GenRSA be a polynomial-time algorithm that, on input 1^n , outputs a triple (N, e, d) where:

- $N = pq$, and p and q are n-bit primes
- $ed = 1 \pmod{\phi(N)}$

The algorithm is allowed to fail with negligible probability.

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For an algorithm $\mathcal{A}$, define $\text{RSA-inv}_{\mathcal{A},\text{GenRSA}}(n)$ as:

- Run $\text{GenRSA}(1^n)$ to obtain (N, e, d).
- Choose $y \in \mathbb{Z}_N^*$ u.a.r.
- Send N, e and y to $\mathcal{A}$
- $\mathcal{A}$ outputs $x \in \mathbb{Z}_N^*$
- The outcome of the experiment is 1 if $x^e = y$. Otherwise the outcome is 0.

Reminder: the RSA assumption

Definition: The RSA problem is hard relative to GenRSA if for all probabilistic polynomial-time algorithms $\mathcal{A}$ there exists a negligible function ε such that

$$\Pr[\mathsf{RSA\text{-}inv}_{\mathcal{A},\mathsf{GenRSA}}(n) = 1] \leq \varepsilon(n).$$

The RSA assumption: there exists a GenRSA algorithm relative to which the RSA problem is hard.

RSA-Based Public Key Encryption

We can define a public-key encryption scheme (for short messages) based on the RSA assumption

Gen(1^n):

- $(N, e, d) \leftarrow \mathsf{GenRSA}(1^n)$
- Return (pk, sk) where $pk = \langle N, e \rangle$ and $sk = \langle N, d \rangle$

The message space $\mathcal{M}_{pk}$ is Z_N^*.

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Plain RSA

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RSA-Based Public Key Encryption: Example

Say that we run $\texttt{GenRSA}(1^5)$ and it returns $(N, e, d) = (391, 3, 235)$

- We are going to work in the group $\mathbb{Z}^*_{391}$
- The public key pk is $(391, 3)$
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To encrypt $m = 158 \in \mathbb{Z}^*_{391}$:

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To decrypt $c = 295$:

- Compute $m = 295^{235} \bmod 391$

RSA-Based Public Key Encryption: Example

We reduce the result modulo 295 after every product

$$295^{235} \bmod 391 \quad = (295^{117} \bmod 391)^2 \cdot 295 \bmod 391$$

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Is plain RSA secure (under the RSA assumption)?

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Plain RSA should never be used!

Attacks on Plain RSA: Better than bruteforce

Suppose that plain-RSA is used to encrypt a random group element $m \in \mathbb{Z}_N^*$ with $N \approx 2^\eta$

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Let $P_\alpha(\eta)$ denote the probability that a η-bit message m can be written as $m = m_1 \cdot m_2$ with $m_1, m_2 \leq 2^{\alpha\eta}$.

$$\lim_{\eta\to\infty} \inf P_\alpha(\eta) \geq \ln(2\alpha)$$

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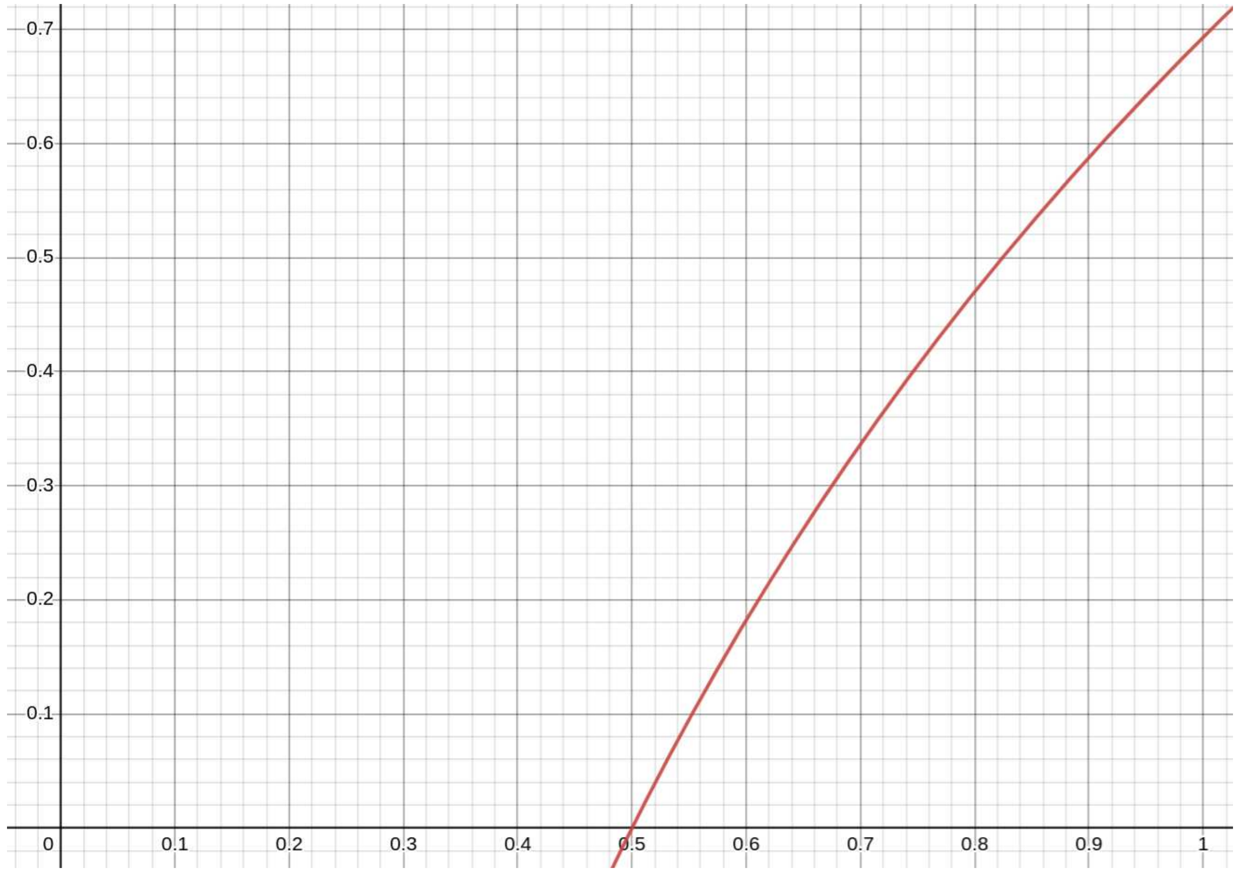
For any constant $\alpha > \frac{1}{2}$, $P_\alpha(\eta)$ is a positive constant!



Dan Boneh, Antoine Joux & Phong Q. Nguyen.
Why Textbook ElGamal and RSA Encryption Are Insecure

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Table 1. Experimental probabilities of splitting into two factors.

Bit-length m	m_1	m_2	Probability
40	20	20	18%
	21	21	32%
	22	22	39%
	20	25	50%
64	32	32	18%
	33	33	29%
	34	34	35%
	30	36	40%

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$c = m^e = (m_1 \cdot m_2)^e = m_1^e \cdot m_2^e \qquad \implies \qquad m_1^e = c \cdot (m_2^e)^{-1}$

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Input: c

Time

- $L \leftarrow$ empty list
- For $\widetilde{m}_2 = 1, 2, \ldots, 2^{\alpha\eta}$: (guess m_2) $\approx 2^{\alpha n}$
  - Compute $\widetilde{x} = c \cdot (\widetilde{m}_2^e)^{-1}$ (compute what the value of x would be if the guess was correct)
  - Append $(\widetilde{m}_2, \widetilde{x})$ to L

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$$c = m^e = (m_1 \cdot m_2)^e = m_1^e \cdot m_2^e \implies m_1^e = c \cdot (m_2^e)^{-1}$$

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Success probability: $P_\alpha(\eta)$

Time: $\approx \alpha n \cdot 2^{\alpha n}$

Attacks on Plain RSA: short messages and small exponent

Suppose that $m \leq \sqrt[e]{N}$ (here operations are performed over the reals)

- This happens when e is small (recall that $e = 3$ is a common choice)...
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No modular reduction!

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- m can be recovered from c by computing $\sqrt[e]{c}$ **over the reals**!

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- This happens when e is small (recall that $e = 3$ is a common choice)...
- ...and m is short
- For example when N has 2048 bits, $e = 3$, and m has ≤ 85 bytes

Then $m^e \leq N$ and hence $c = m^e \bmod N = m^e$

No modular reduction!

- m can be recovered from c by computing $\sqrt[e]{c}$ **over the reals**!
- This requires polynomial-time!

Attacks on Plain RSA: partially-known messages

Theorem: Let $p(x)$ be a polynomial of degree e. All x such that $p(x) = 0 \bmod N$ and $|x| \leq N^{1/e}$ can be found in time poly$(\log N, e)$.

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If $2^k \leq N^{1/e}$ and e is small, the above theorem allows us to list all candidates for m_2 in polynomial-time

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Chinese Remainder Theorem: If the n_i are pairwise coprime then the system

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ \vdots \\ x \equiv a_k \pmod{n_k} \end{cases}$$

has a unique solution x^* s.t. $0 \leq x^* < \prod_{i=1}^{k} n_i$. Moreover, all solutions are congruent modulo $\prod_{i=1}^{k} n_i$.

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To recover m we just need to compute $\sqrt[e]{c^*}$ (over the reals!)