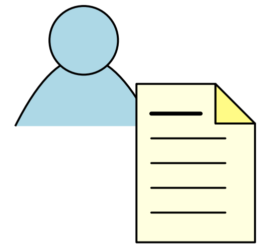
Secret Sharing

Imagine some sensitive information that is kept by a single agent

- A master encryption key
- Your bitcoin wallet
- Nuclear codes
- ...



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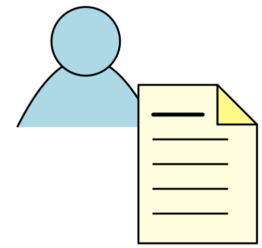
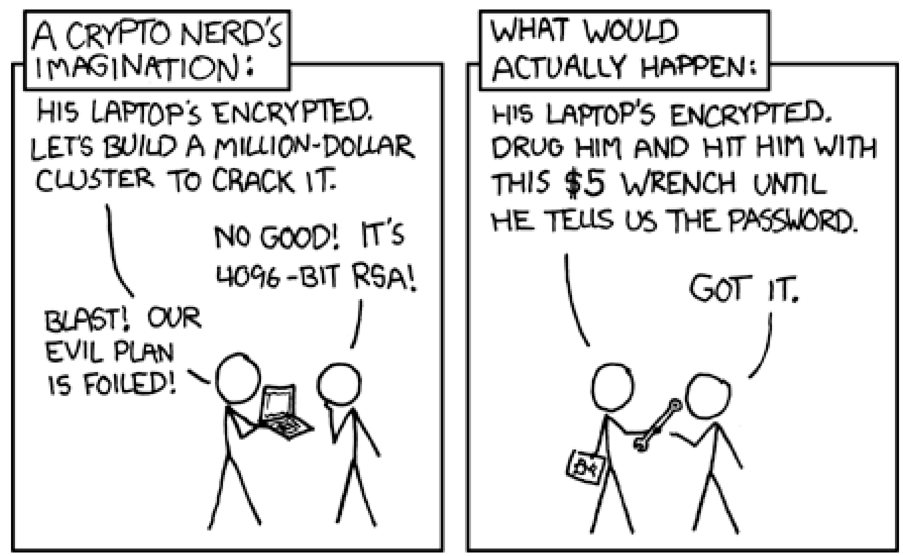


Single point of failure!



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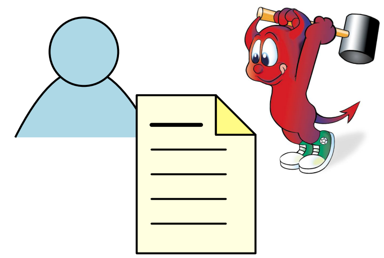

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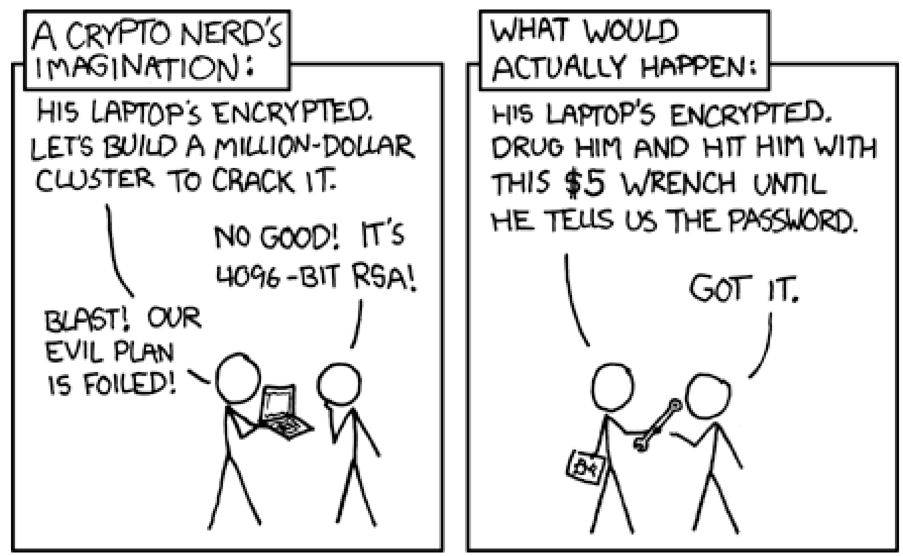
An attacker can compromise one machine and steal the sensitive information



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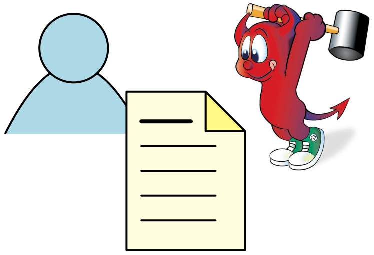

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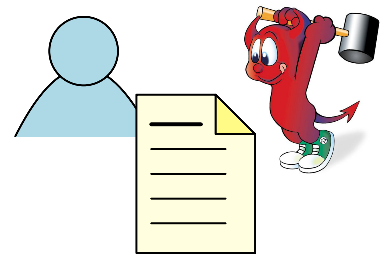

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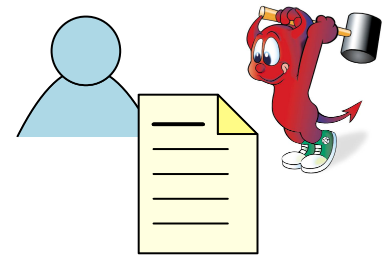

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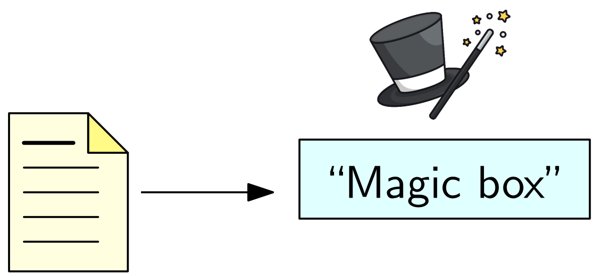

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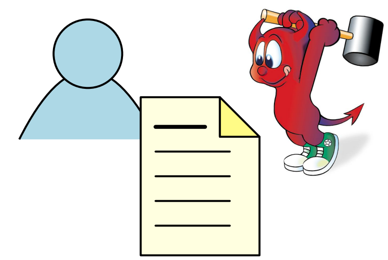

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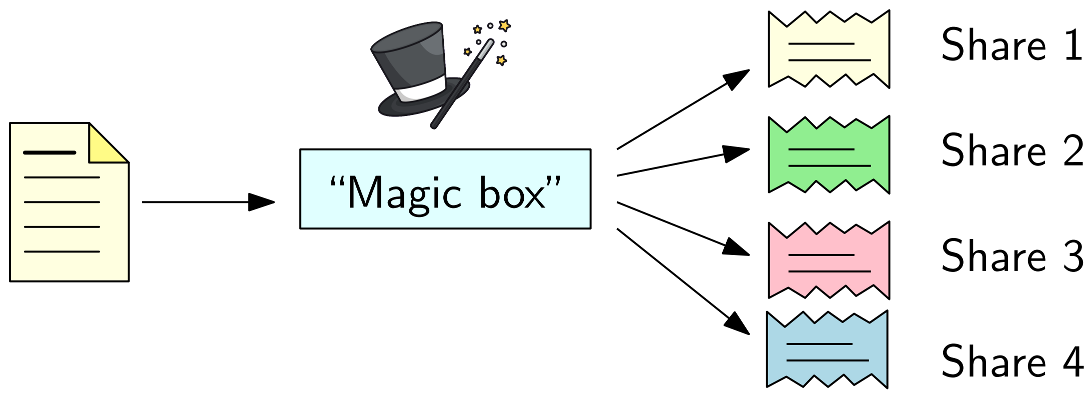

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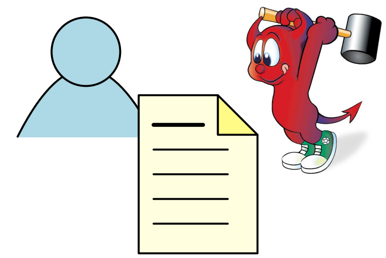

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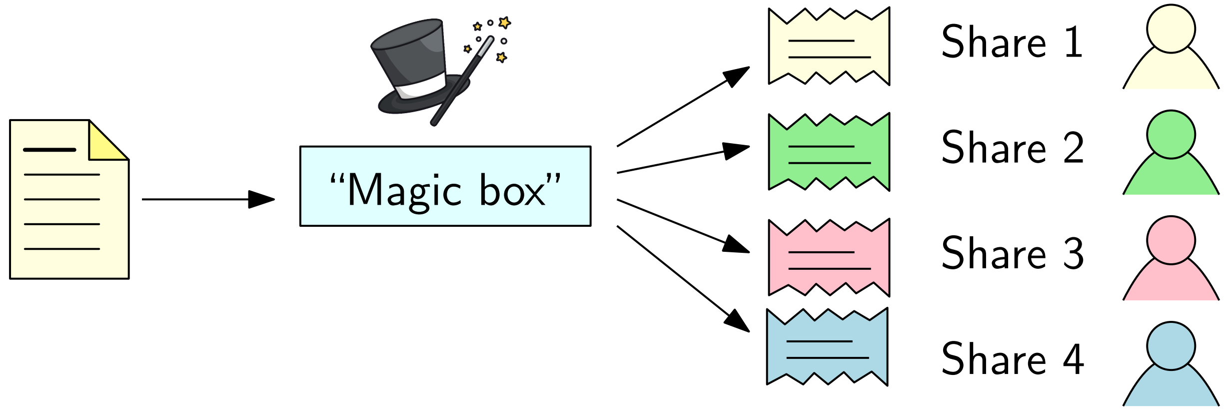

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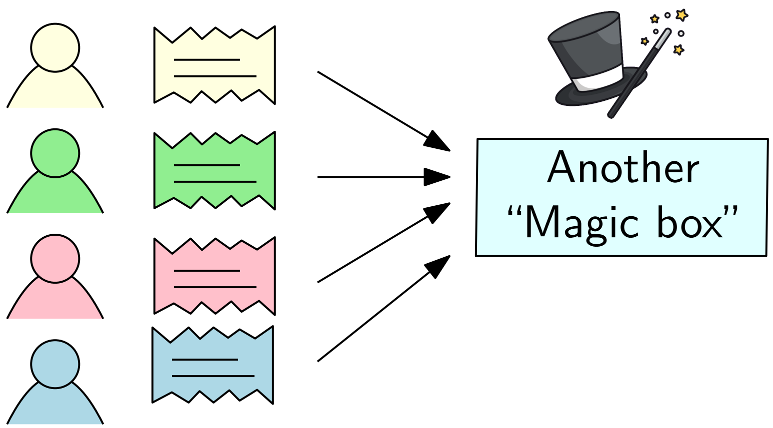

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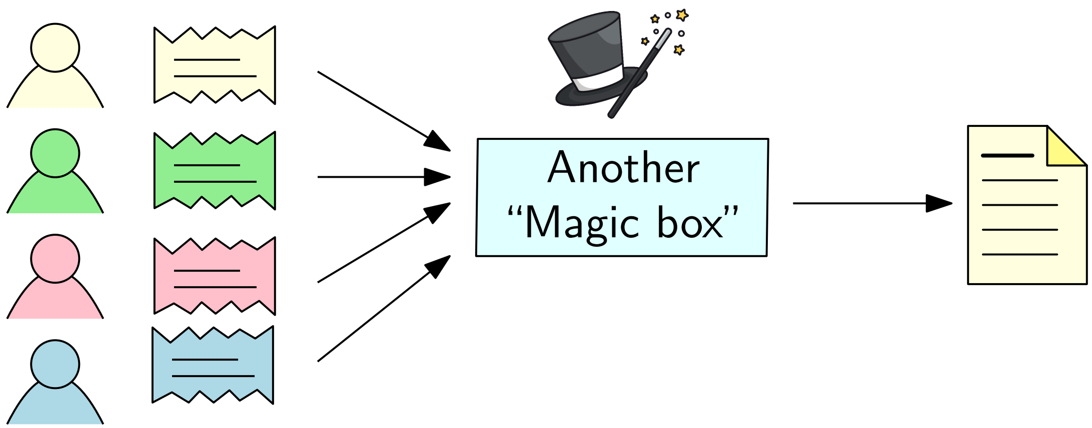

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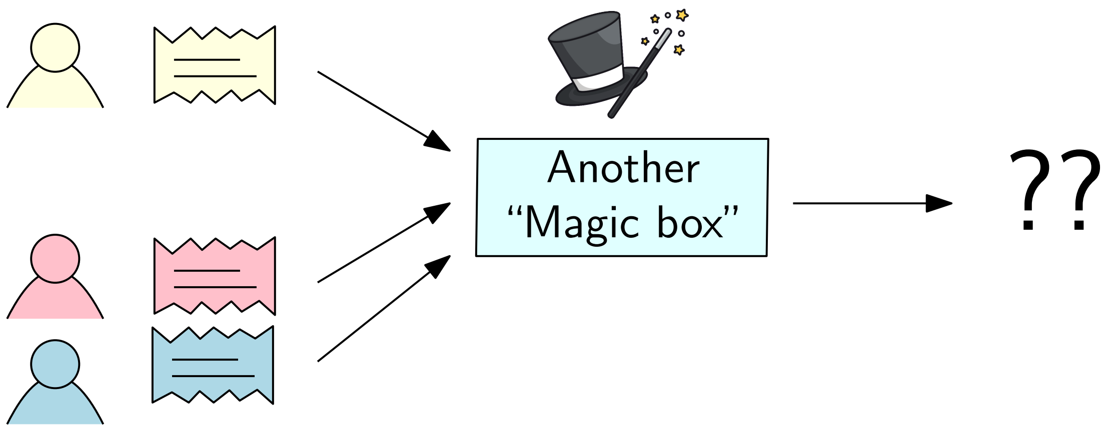

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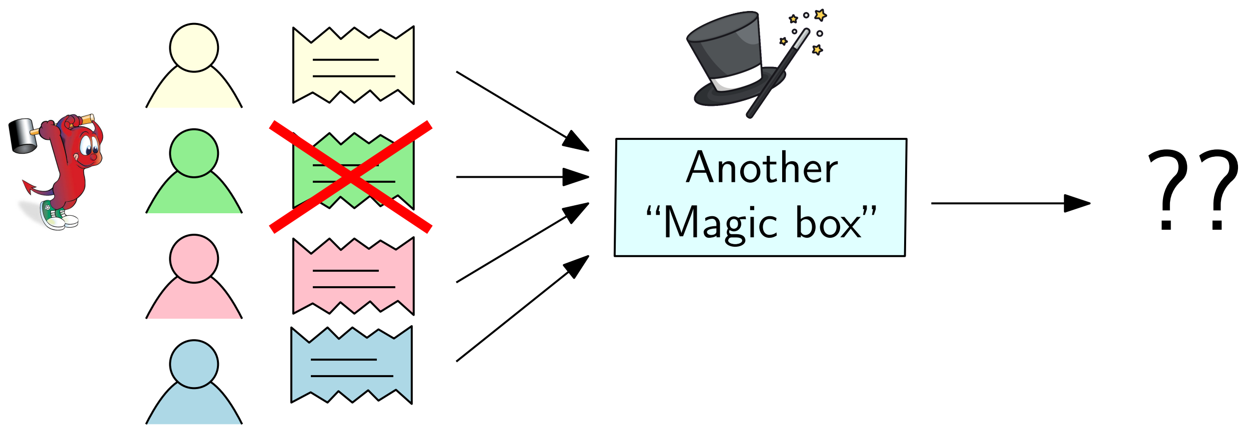

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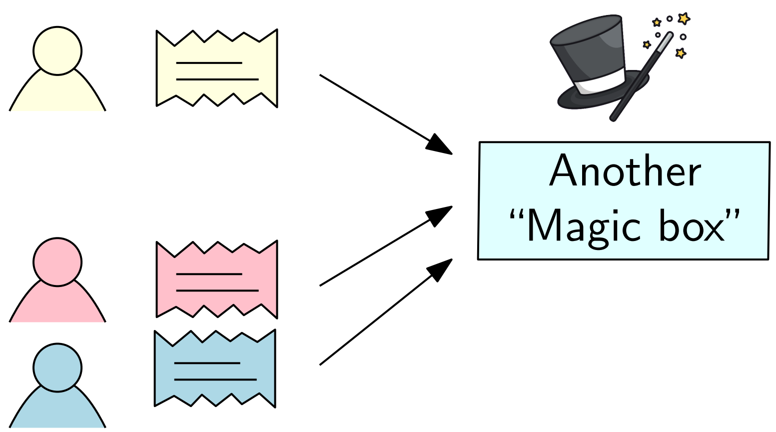

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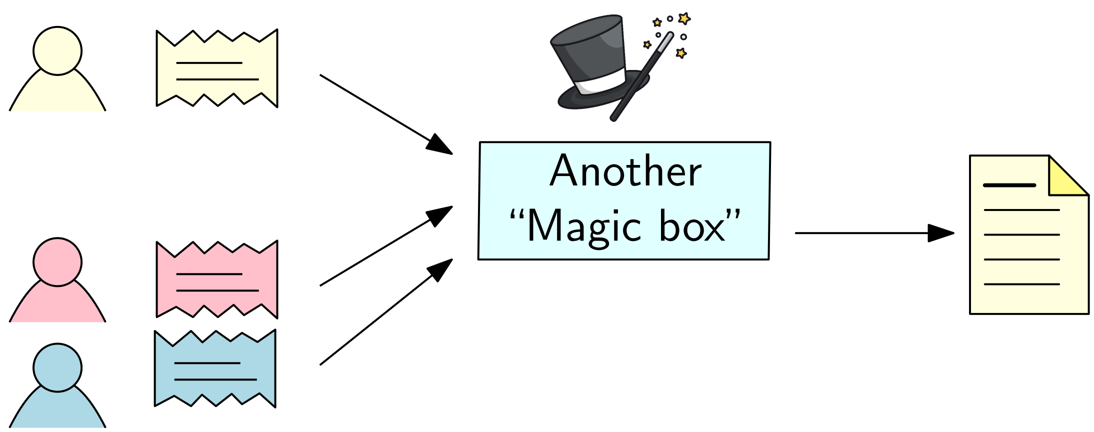

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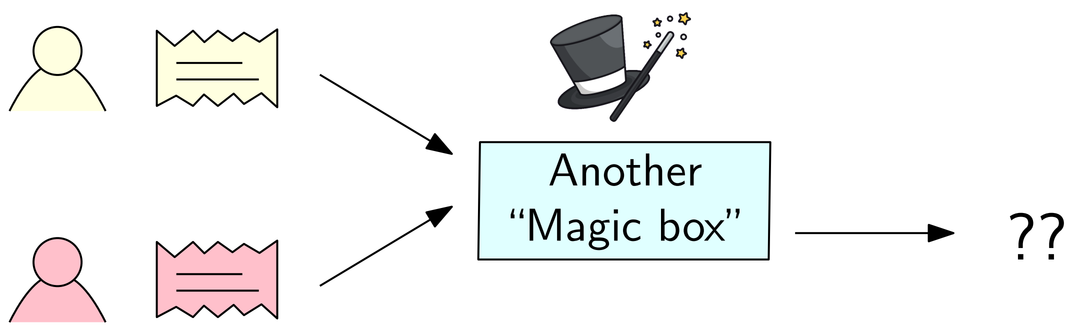

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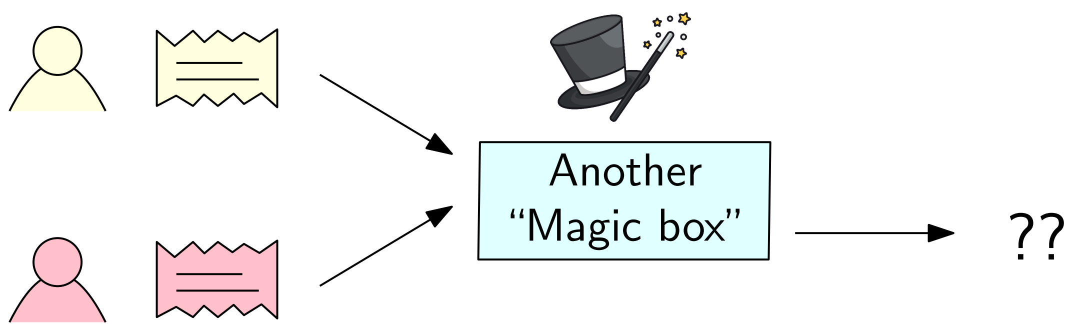

k**-out-of-**n **threshold secret-sharing scheme**

Access Structures and Qualifying Sets

Even more general:

- Let $\mathcal{A}$ be a set of n parties $a_1, \ldots, a_n$
- Let $\Gamma \subseteq 2^{\mathcal{A}}$ a collection of subsets of $\mathcal{A}$ such that:
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We can further assume that: $\forall a \in \mathcal{A}$ s,t, $\{a\} \notin \Gamma$ since otherwise we can simply send the secret to a and restrict ourselves to the access structure $\Gamma' = \{A \in \Gamma \mid a \notin A\}$

(this implies $\emptyset \notin \Gamma$)

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Example:

- $\mathcal{A} = \{\text{Alice}, \text{Bob}, \text{Charlie}, \text{Dan}\}$, $n = |\mathcal{A}| = 4$, $k = 2$
- $\Gamma = \{ \{\text{Alice}, \text{Bob}\}, \{\text{Alice}, \text{Charlie}\}, \{\text{Alice}, \text{Dan}\}, \{\text{Bob}, \text{Charlie}\}, \{\text{Bob}, \text{Dan}\}, \{\text{Charlie}, \text{Dan}\},$
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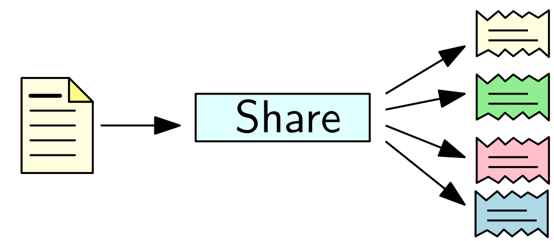

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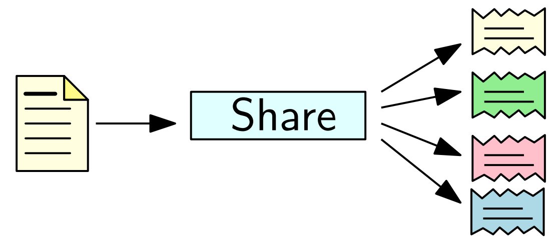
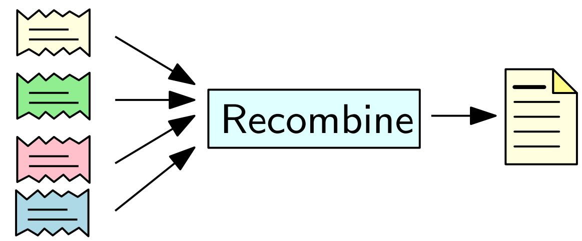


- Recombine(H): a deterministic algorithm that takes a set $H = \{s_a \mid a \in A\}$ containing a share for each party in some set $A \subseteq \mathcal{A}$ and outputs a secret $s \in \mathcal{S}$ if $A \in \Gamma$ and a failure symbol $\perp$ if $A \notin \Gamma$.



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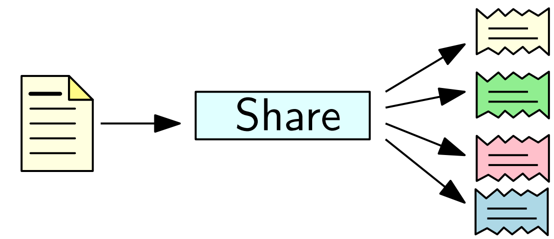

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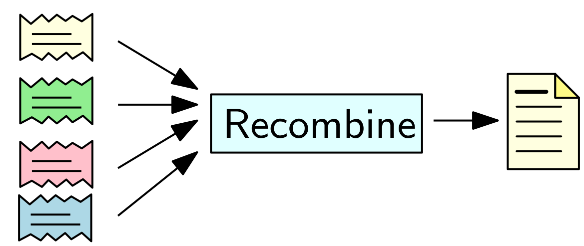
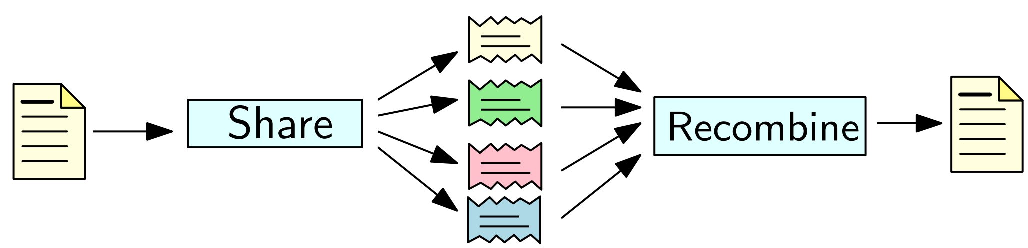


Correctness: If $H = \{s_a \mid a \in A\}$ for a set $A \in \Gamma$ and all s_a were output by Share(s, Γ), then Recombine$(H) = s$.



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Formalized similarly to perfect secrecy (there are multiple equivalent definitions):

A secret sharing scheme is secure if, for every $s, s' \in \mathcal{S}$, every access structure Γ, every $A \subset \mathcal{A}$ with $A \notin \Gamma$, and every vector of shares $\alpha = (\alpha_a)_{a \in A}$:

$$\Pr[(S_a)_{a \in A} = \alpha] = \Pr[(S'_a)_{a \in A} = \alpha],$$

where S_a (resp. S'_a) is a random variable representing the share given to the party $a \in A$ by $\mathsf{Share}(\Gamma, s)$ (resp. $\mathsf{Share}(\Gamma, s')$)

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Correctness: $s_a \oplus s_b = r \oplus (r \oplus s) = s$.

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- If $A = \{b\}$, then for an arbitrary $\alpha = (\alpha_b)$:

$$\Pr[S_b = \alpha_b] = \Pr[r \oplus s = \alpha_b]$$

2-out-of-2 threshold secret sharing: security

Let $s, s' \in \{0,1\}^\ell$ be two arbitrary secrets and consider S_a, S_b output by Share(s, Γ) (resp. S'_a, S'_b output by Share(s', Γ)).

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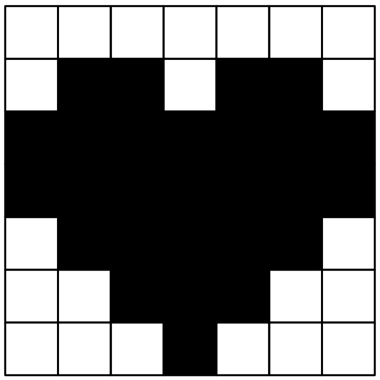
We have shown show that, regardless of s, $\Pr[S_a = \alpha]$ and $\Pr[S_b = \alpha]$ are constants

2-out-of-2 threshold secret sharing: a visual interpretation

Imagine that the secret s is the following image:



s

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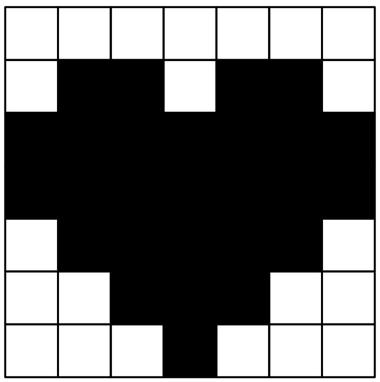
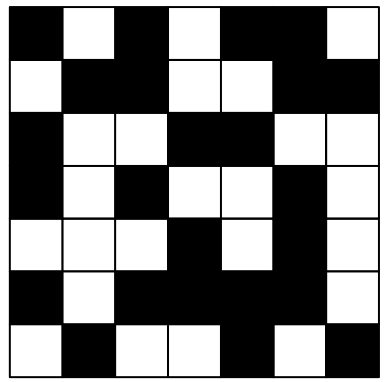


s

$=$



s_a

We generate the first share by coloring each pixel white or black u.a.r.

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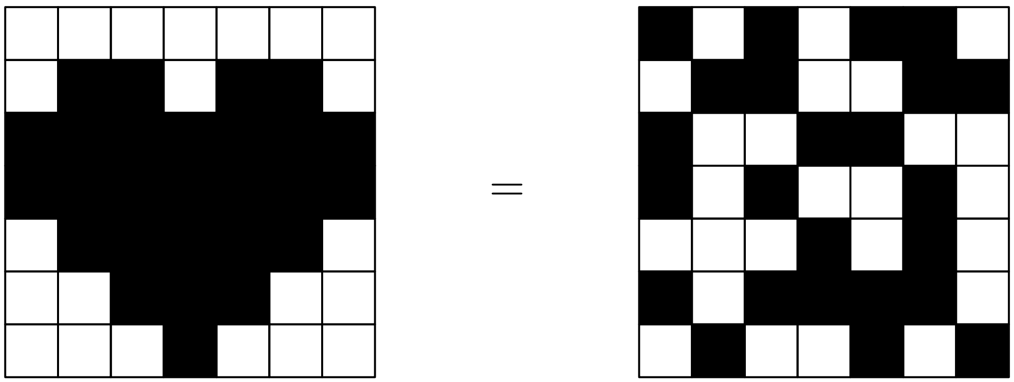

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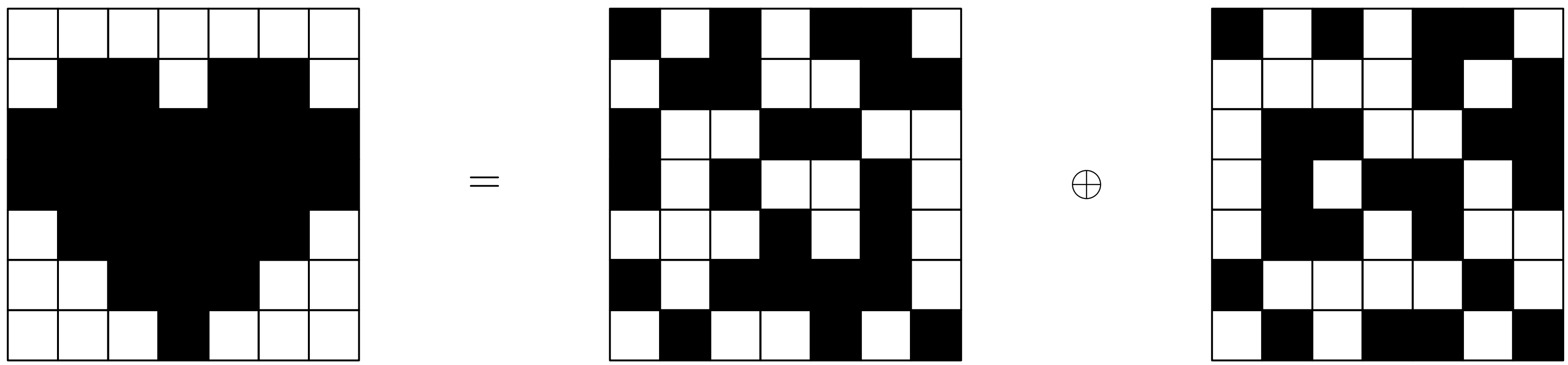

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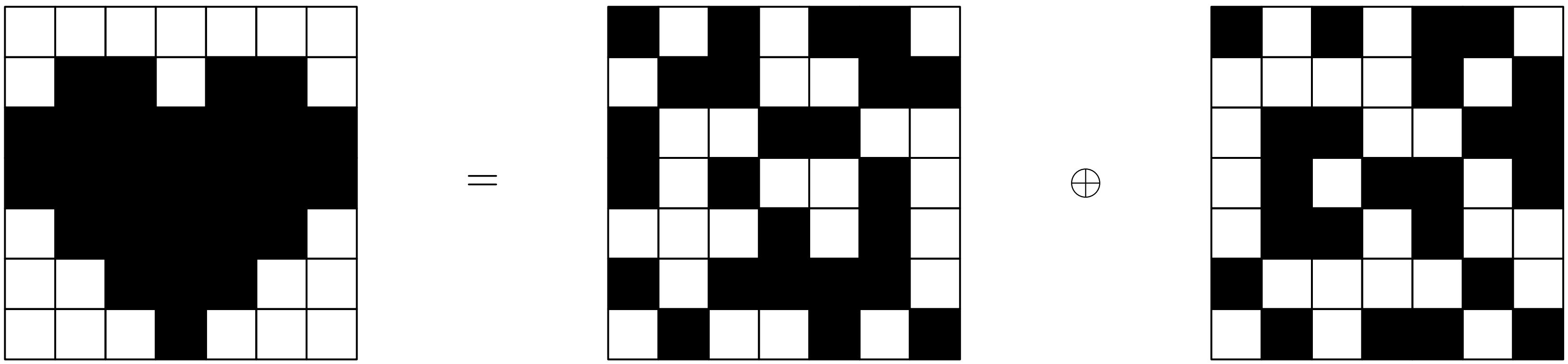

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Physical visual 2-out-of-2 threshold secret sharing scheme: subdivide each pixel in 4 subpixels

$\oplus \longrightarrow$ overlay the two images

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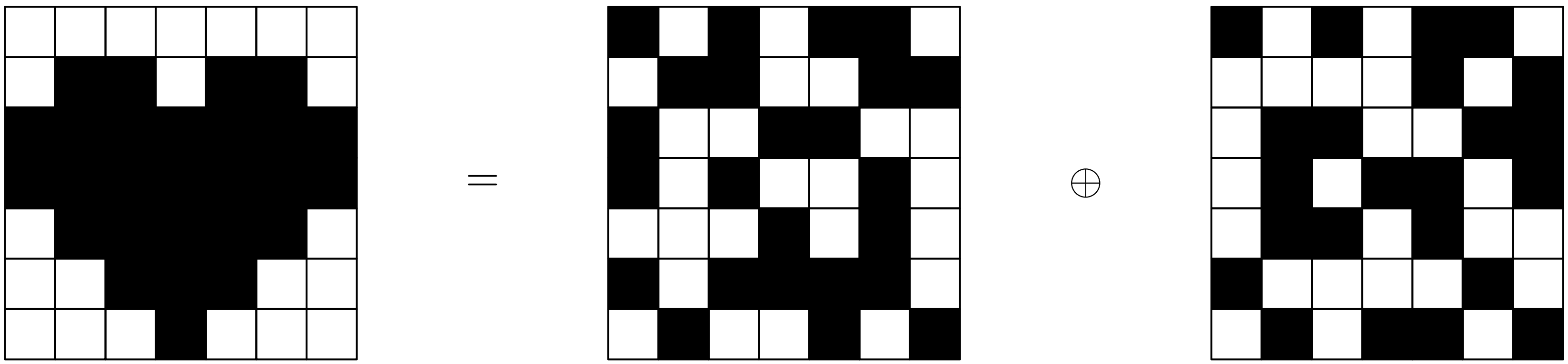

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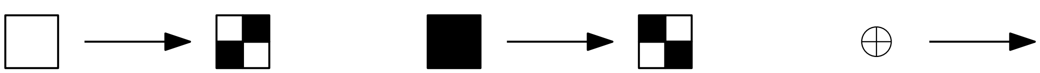

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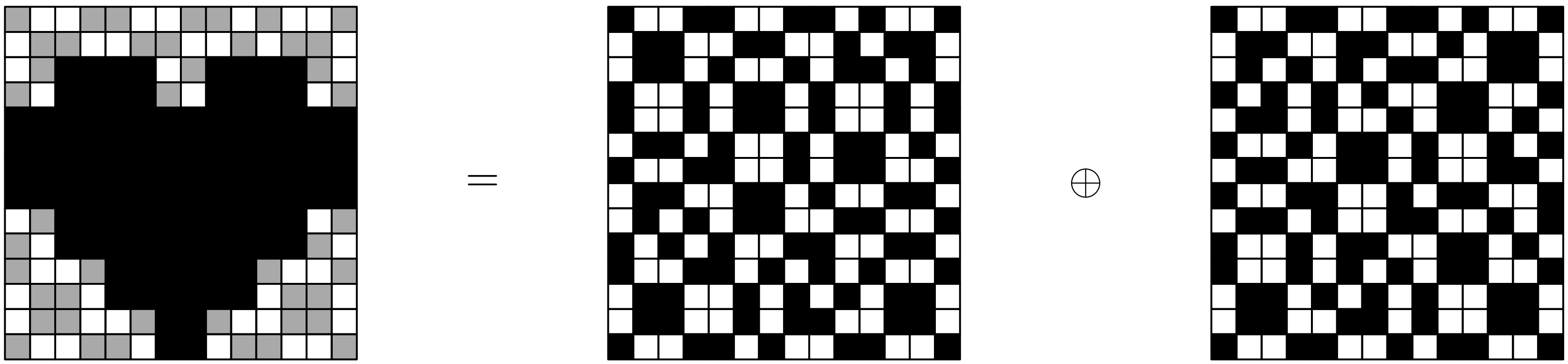

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n-out-of-n threshold secret sharing

The above idea generalizes easily to $n \geq 2$ parties:

Consider any $\mathcal{A} = \{1, 2 \ldots, n\}$ with $|A| = n \geq 2$ and the access structure $\Gamma = \{\mathcal{A}\}$

Let the space of secrets be $\mathcal{S} = \{0, 1\}^\ell$

Index the parties with integers.
Makes notation easier.

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- Let $r_1, \cdots r_{n-1}$ be $n-1$ strings chosen independent and u.a.r. from $\{0, 1\}^\ell$.
- Return $(s_1, s_2, \ldots, s_n)$ where $s_i = r_i$ for $i < n$ and $s_n = r_1 \oplus r_2 \oplus \cdots \oplus r_{n-1} \oplus s$

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Recombine(H):

- If $|H| < n$ return $\perp$.
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Secret sharing with arbitrary access structures

Let Γ be an access structure (for an arbitrary number of parties n)

A qualifying set $B \in \Gamma$ is minimal if there is no qualifying set $B' \in \Gamma$ such that $B' \subset B$.

Let $m(\Gamma) = \{B_1, B_2, \dots\}$ denote the set of all minimal qualifying sets in Γ

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- $\mathcal{A} = \{X, Y, W, Z\}$
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If we think of a each party $a \in A$ as a Boolean variable, we can define the following Boolean formula in **disjunctive** normal form:

$$\bigvee_{B_i \in m(\Gamma)} \left(\bigwedge_{b \in B_i} b \right)$$

Each set B_i is a **clause** (conjunction of variables)

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A set A of parties induces a truth assignment in which a is true iff $a \in A$

The truth assignment satisfies the formula if and only if A is a qualifying set

Ito–Nishizeki–Saito Secret Sharing

Share:

We can read the DNF formula as a set of instructions to build the shares s_a, $a \in \mathcal{A}$

- Each clause B_i corresponds to an "inner" $|B_i|$-out-of-$|B_i|$ threshold secret sharing scheme

  Each agent $b \in B_i$ gets a share $s_b^{(i)}$

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Recombine & Correctness:

If A is a qualifying set, then there is some clause consisting only of variables in A.

The parties involved in the clause can recover s using the Recombine step of the corresponding k-out-of-k threshold secret sharing scheme

Shamir Secret Sharing

The previous secret sharing scheme can produce shares that are much larger than the secret s

One notable example where this happens is the k-out-of-n case

- If $k = \frac{n}{2}$ there are $\binom{n}{n/2} = \Omega(2^n/\sqrt{n})$ minimal qualifying sets
- The shares are exponentially longer than the secret!

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Shamir proposed a secret k-out-of-n threshold secret-sharing scheme in which all the shares have (approximately) the same length as the secret

The scheme uses Lagrange interpolating polynomials

Lagrange interpolating polynomials

Consider a set $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ of k points in $\mathbb{R}^2$ with distinct x_is.

We want to build a polynomial f that "passes through" all the points (i.e., $f(x_i) = y_i$ for $i = 1, \ldots, k$)

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Consider the polynomial:

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What happens when ℓ_1 is evaluated at the points $x_1, x_2, \ldots, x_k$?

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- If $x = x_i$ for $i \neq 1$ then the product includes $(x - x_i) = 0 \implies \ell_1(x_i) = 0$

Lagrange interpolating polynomials

Consider a set $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ of k points in $\mathbb{R}^2$ with distinct x_is.

We want to build a polynomial f that "passes through" all the points (i.e., $f(x_i) = y_i$ for $i = 1, \ldots, k$)

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The collection of polynomials $\ell_1(x), \ldots, \ell_k(x)$ is called a **Lagrange basis**

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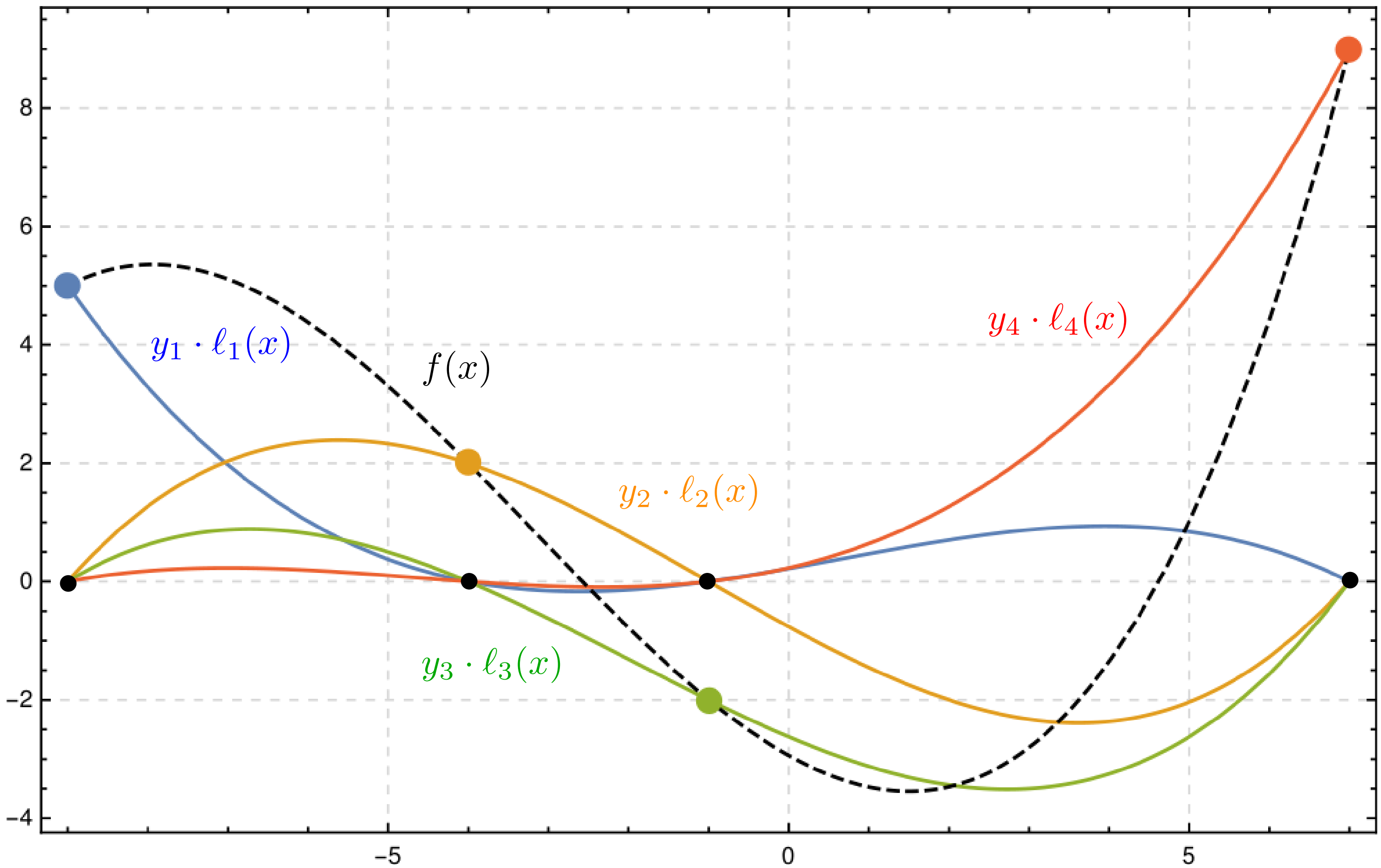
- Each ℓ_j is the product of $k-1$ terms $(x - x_i)$ (and some constants), therefore ℓ_j has degree $k-1$
- $f(x)$ is a sum of polynomials of degree $k-1$, therefore $f(x)$ has degree $k-1$

Lagrange interpolating polynomials



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Lagrange interpolating polynomials with coefficient over $\mathbb{Z}_p$

We will need to choose an interpolating polynomial **uniformly at random** to obtain a secure secret-sharing scheme

- Unclear how to do that over the reals
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- $(F, \oplus)$ is an Abelian group, we call its identity element 0
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Good news:

- The fundamental theorem of algebra can be extended to *univariate* polynomials *over a finite field*
- $(\mathbb{Z}_p, +, \cdot)$ is a finite filed

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Theorem: Let $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a set of k points in $\mathbb{Z}_p \times \mathbb{Z}_p$ with distinct x_is. There is a unique polynomial $f(x)$ of degree at most $k-1$ with coefficients in $\mathbb{Z}_p$ such that $f(x_i) = y_i \pmod{p}$ for all $i = 1, \ldots, k$.

The construction and the proof of uniqueness are identical to the previous ones (where $-x$ and x^{-1} denote the additive and multiplicative inverses of x in $\mathbb{Z}_p$).

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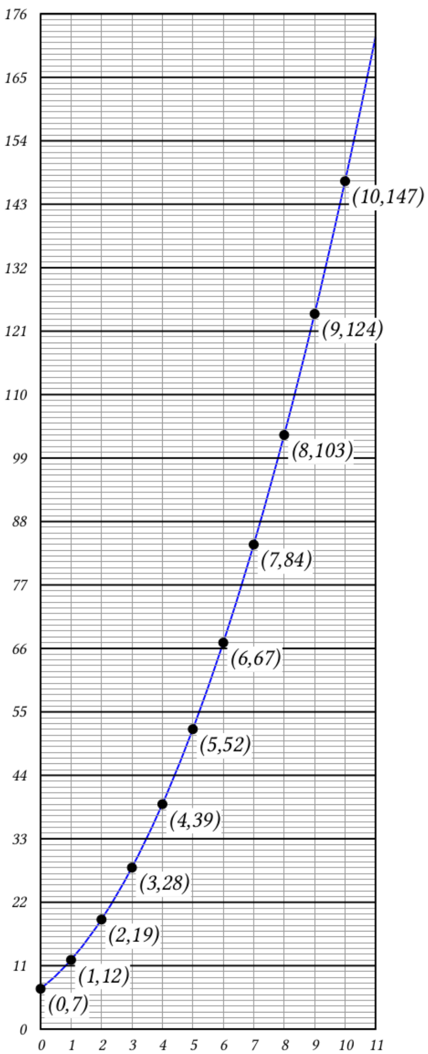

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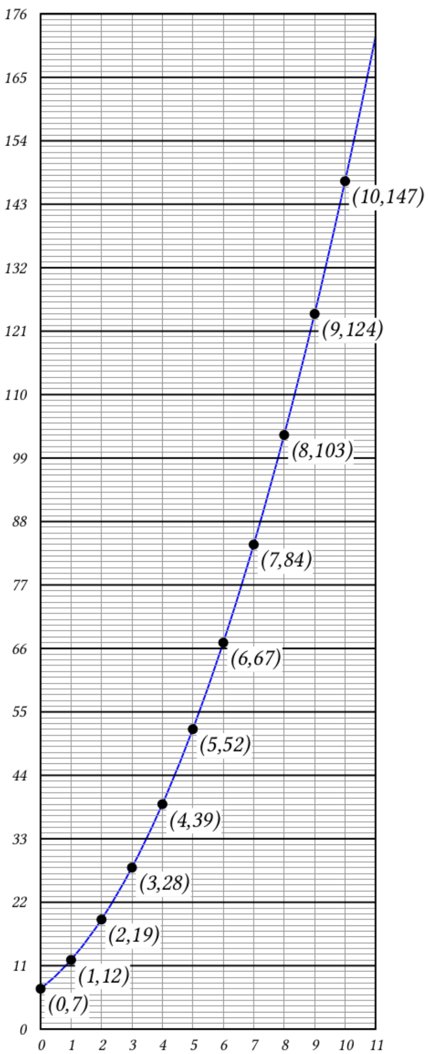

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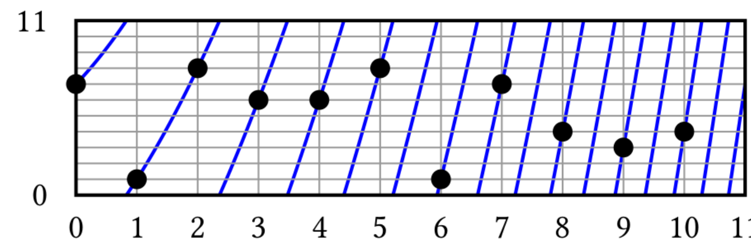

Back to Shamir Secret Sharing

The set of parties is $\mathcal{A} = \{1, 2, \ldots, n\}$

The space of secrets $\mathcal{S}$ is $\mathbb{Z}_p$ for some prime number p

If the secret s is a binary number with t bits, we can pick a prime $p > \max\{s, n\}$ with $\Theta(t + \log n)$ bits.

The Shamir k-out-of-n threshold secret sharing scheme is as follows:

Share(s): (we omit the access structure, which is determined by k and n)

- Choose $k-1$ coefficients $\beta_1, \ldots, \beta_{k-1}$ independently and u.a.r. from $\mathbb{Z}_p$
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Recombine($\{s_i \mid i \in A\}$) (A is a qualifying set)

- Compute the (unique) interpolating polynomial f (with coefficient in $\mathbb{Z}_p$) of degree $k-1$ such that $f(i) = s_i$
- Return $f(0)$

Shamir Secret Sharing: Example

Consider a set of $n = 5$ parties that want to share a secret $s = 8$ using Sharmir's 3-out-of-5 threshold secret sharing scheme

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- $\ell_1(x) = (x-2)(1-2)^{-1} \cdot (x-4)(1-4)^{-1} = (x-2)10 \cdot (x-4)7 = 4x^2 + 9x + 10$
- $\ell_3(x) = (x-1)(4-1)^{-1} \cdot (x-2)(4-2)^{-1} = (x-1)4 \cdot (x-2)6 = 2x^2 + 5x + 4$

Shamir Secret Sharing: Example

Consider a set of $n = 5$ parties that want to share a secret $s = 8$ using Sharmir's 3-out-of-5 threshold secret sharing scheme

We will work in the field $\mathbb{Z}_{11}$

Sharing:

- We pick two random coefficients, e.g., $\beta_1 = 4$, $\beta_2 = 7$
- The polynomial $f(x) = s + \beta_1 x + \beta_2 x^2$ is $8 + 4x + 7x^2$
- The five shares are: $s_1 = (1, f(1)) = (1, 8)$ $s_2 = (2, f(2)) = (2, 0)$ $s_3 = (3, f(3)) = (3, 6)$ $s_4 = (4, f(4)) = (4, 4)$ $s_5 = (5, f(5)) = (5, 8)$

Reconstructing the secret from the shares s_1, s_2, and s_4:

$f(0)$

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Shamir Secret Sharing: Security

Let $A \subset \mathcal{A}$ be a non-qualifying set, and consider any vector $\alpha = (\alpha_i)_{i \in A}$.

Let $\eta(\alpha, s)$ be the number of polynomials g (with coefficients in $\mathbb{Z}_p$) of degree $k-1$ such that $g(i) = \alpha_i \pmod p$ for $i \in A$, and $g(0) = s \pmod p$.

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Theorem: Let $\{(x_1, y_1), \ldots, (x_h, y_h)\}$ be a set of $h \le k$ points in $\mathbb{Z}_p \times \mathbb{Z}_p$ with distinct x_is, where $p \ge k$. The number of polynomials g of degree $k-1$ with coefficients in $\mathbb{Z}_p$ that such that $y_i = g(x_i) \pmod p$ for all $i = 1, \ldots, h$ is exactly p^{k-h}.

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$\square$

Two-Party Computation

Alice and Bob want to jointly compute a function $f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$

- Alice knows the inputs $x_1, \ldots, x_m$
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Example: In the "movie selection" scenario, Alice and Bob wanted to compute $f(x_1, y_1) = x_1 \wedge y_1$

We actually consider a stronger variant: Alice wants to learn $f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ while Bob learns nothing

- If we can solve this variant, then we can solve the above case (Alice sends the final output to Bob)
- This allows us to solve the more general case in which Alice learns $f_A(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ and Bob learns $f_B(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$

Two-Party Computation: The Honest but Curious Model

We will design a **Two-Party computation protocol** that solves this problem for functions f that can be computed in polynomial-time in the **honest but curious model**.

Honest but curious model: Alice and Bob obey the protocol, but they try to gather as much information as they can (each of them wants to break the privacy of the other party)

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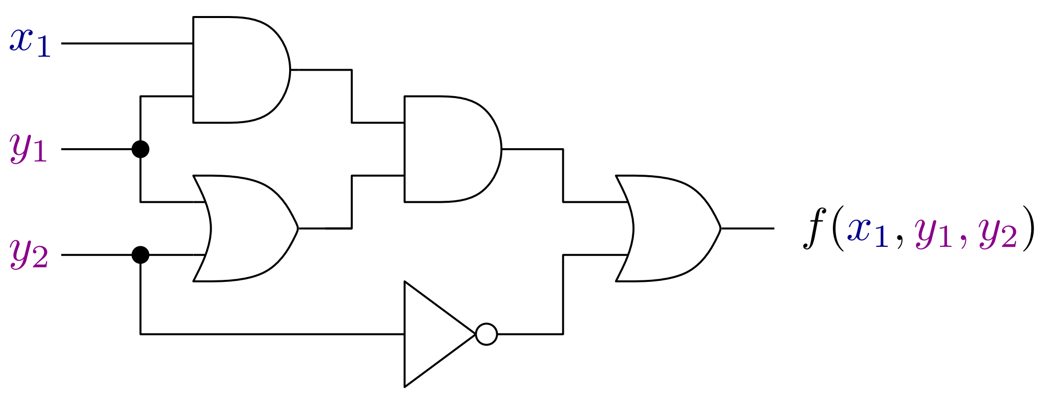
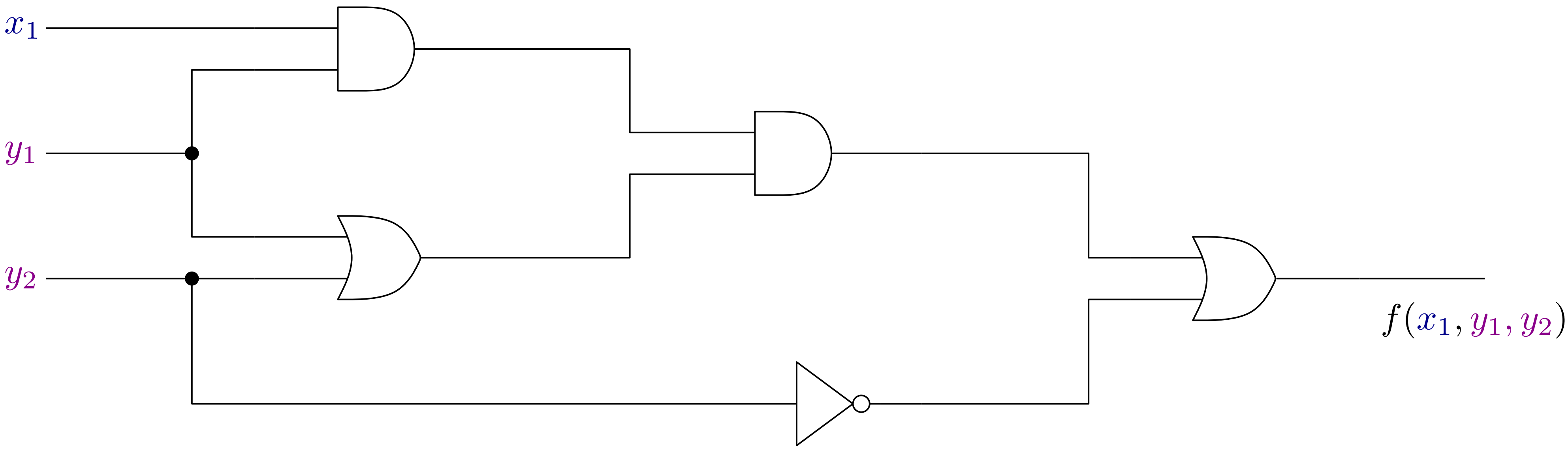
The protocol will be based on evaluating a (polynomial-size) **Boolean circuit** that computes f

For simplicity, think of Boolean circuits with a single output (the protocol extends to multiple outputs in a straightforward way)



Yao's Garbled Circuits: Building the Circuit

Alice replaces each logic gate with an explicit description of its truth table



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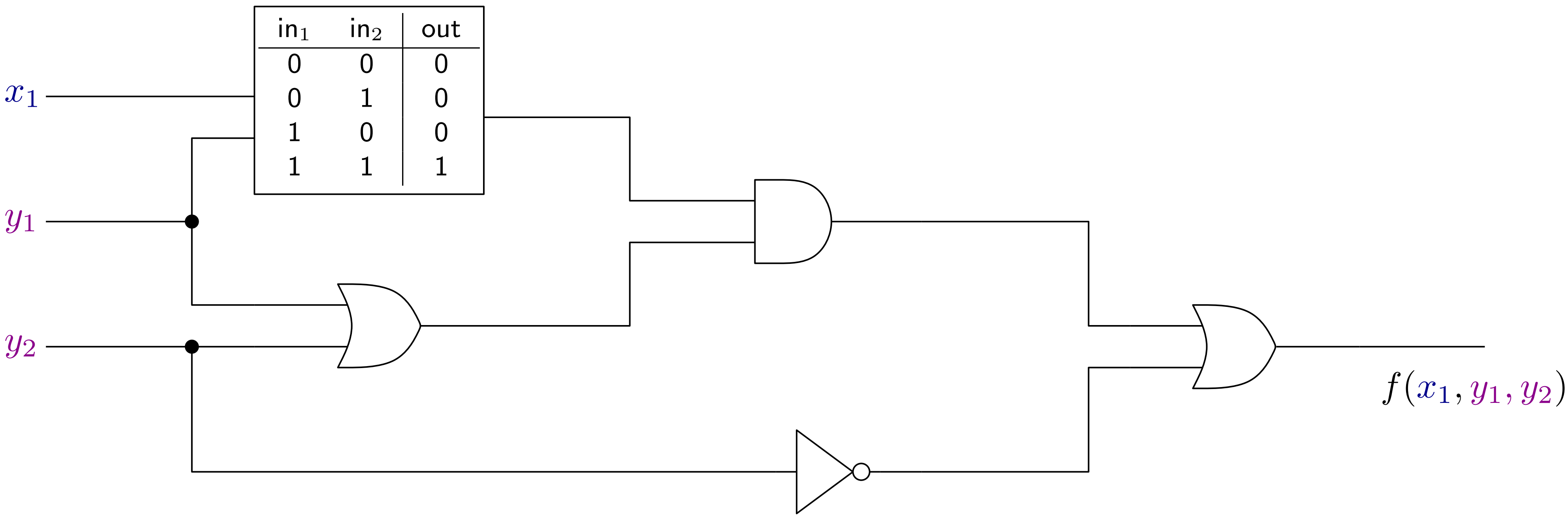

in_1	in_2	out
0	0	0
0	1	0
1	0	0
1	1	1

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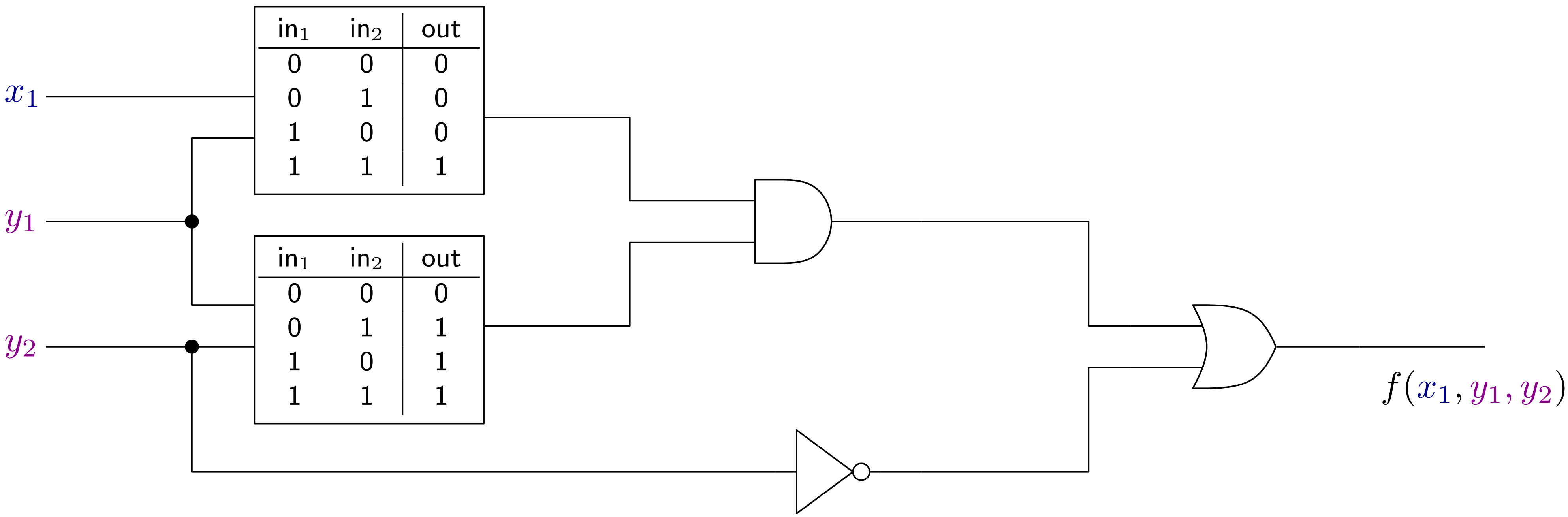

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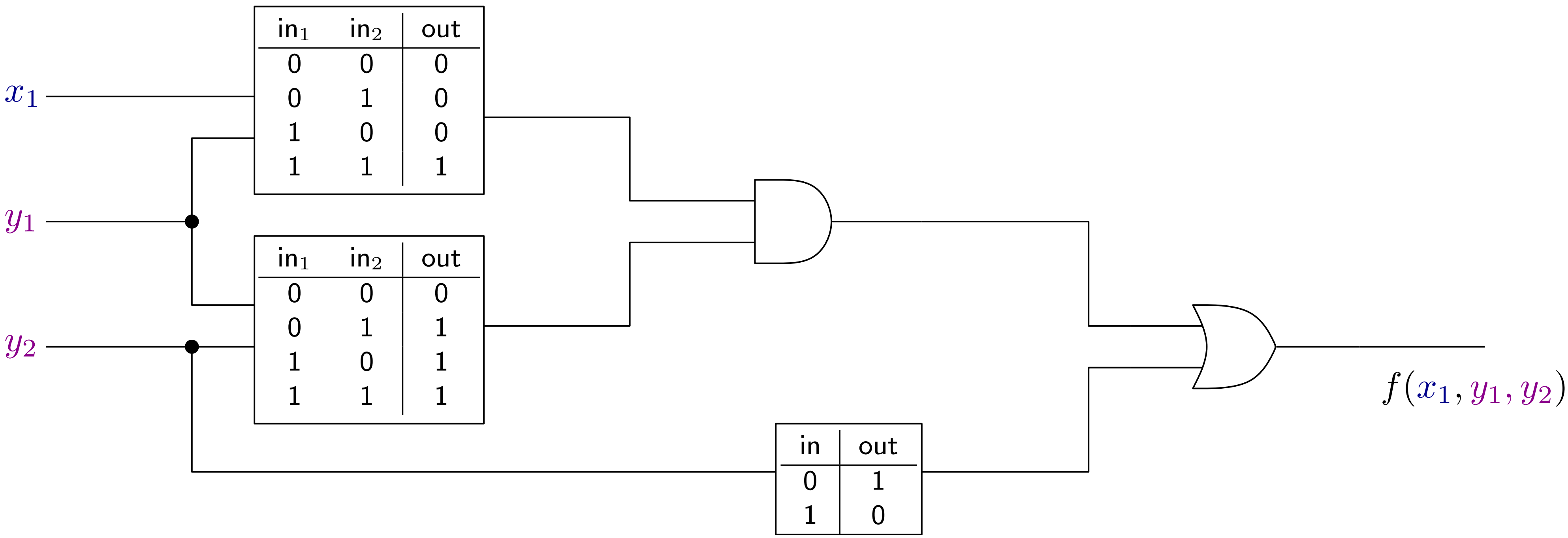

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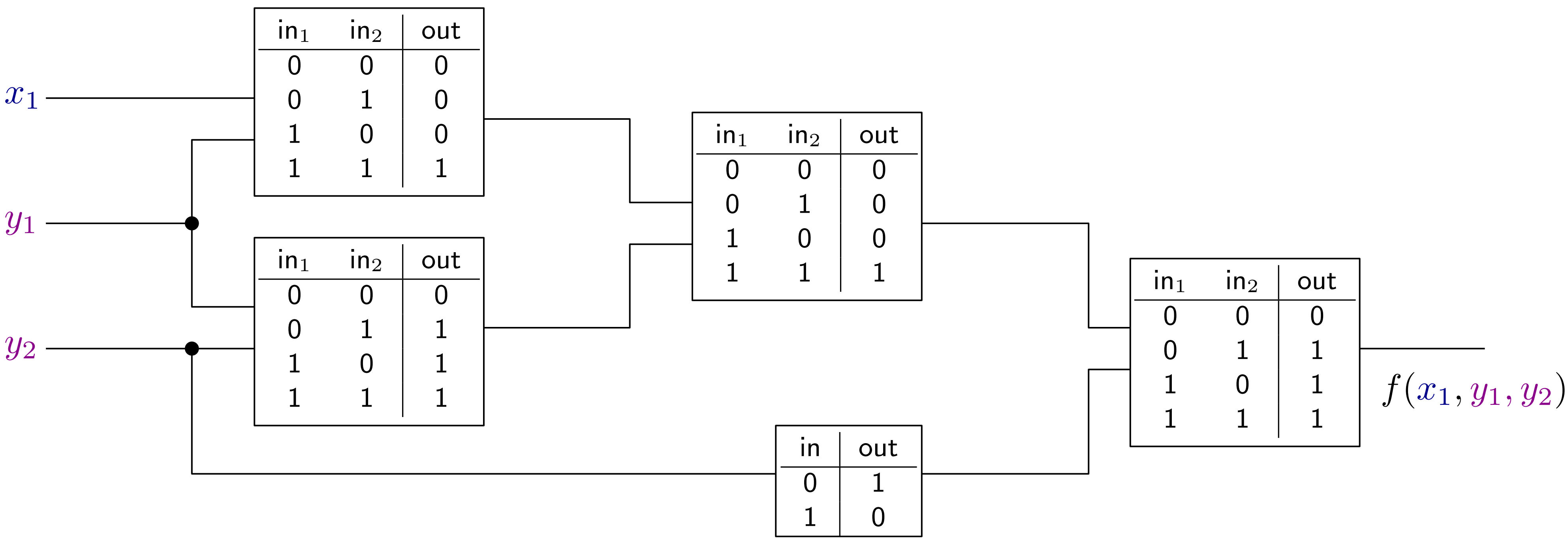

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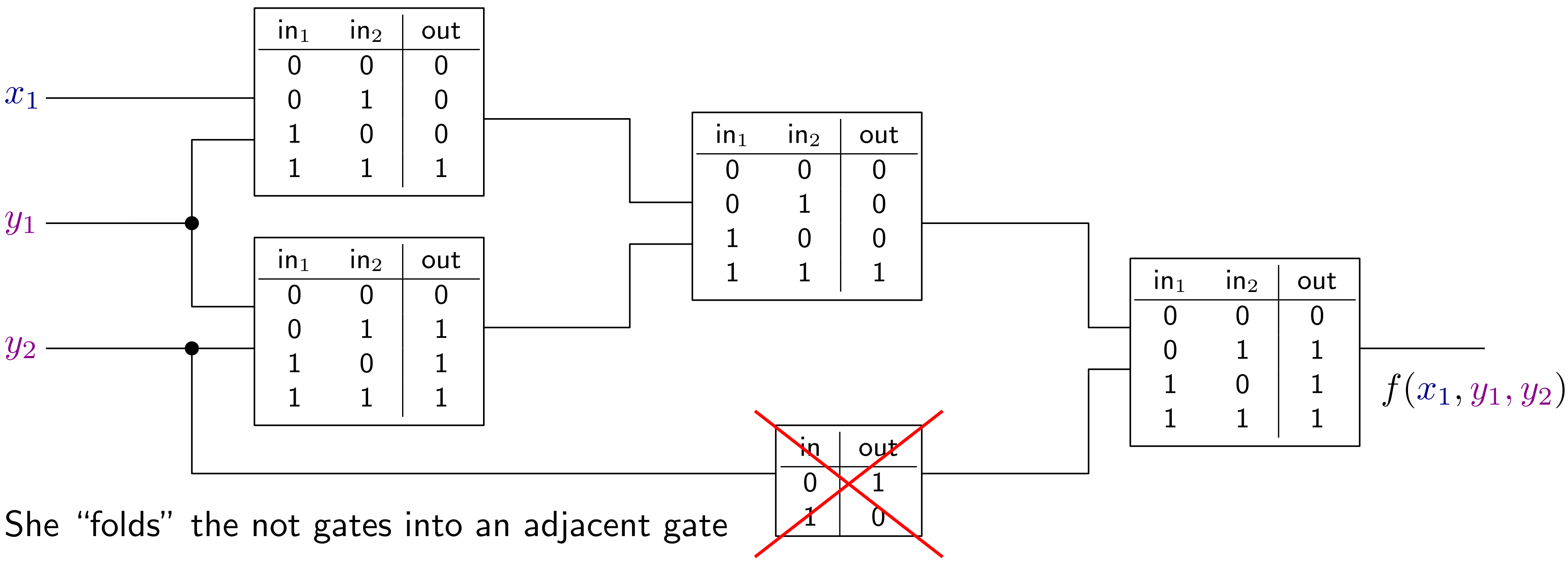

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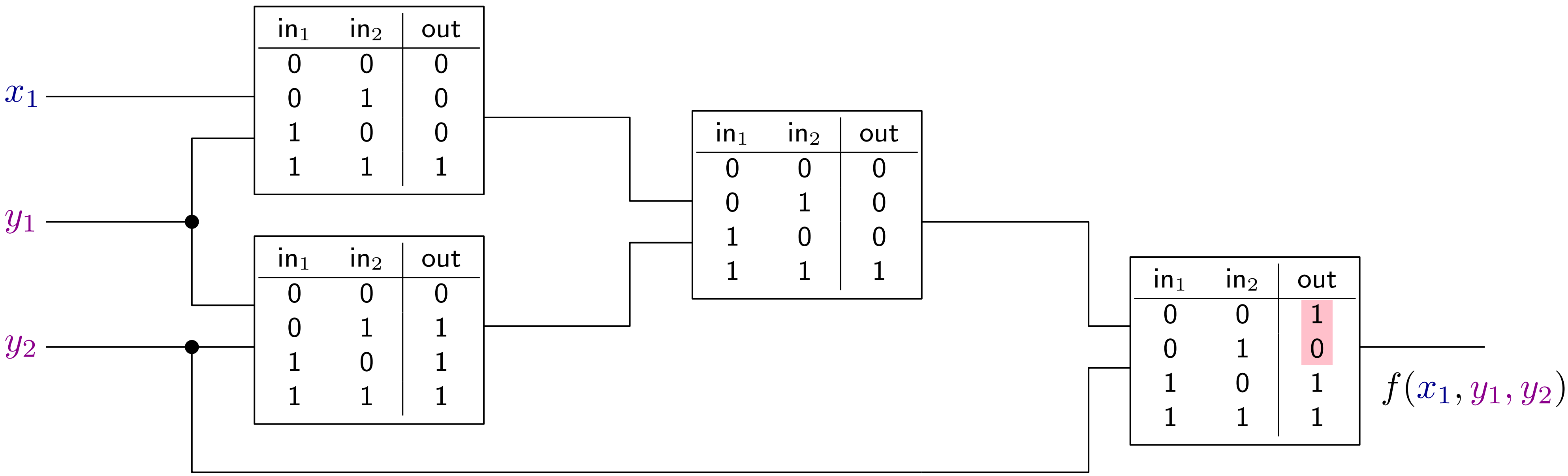

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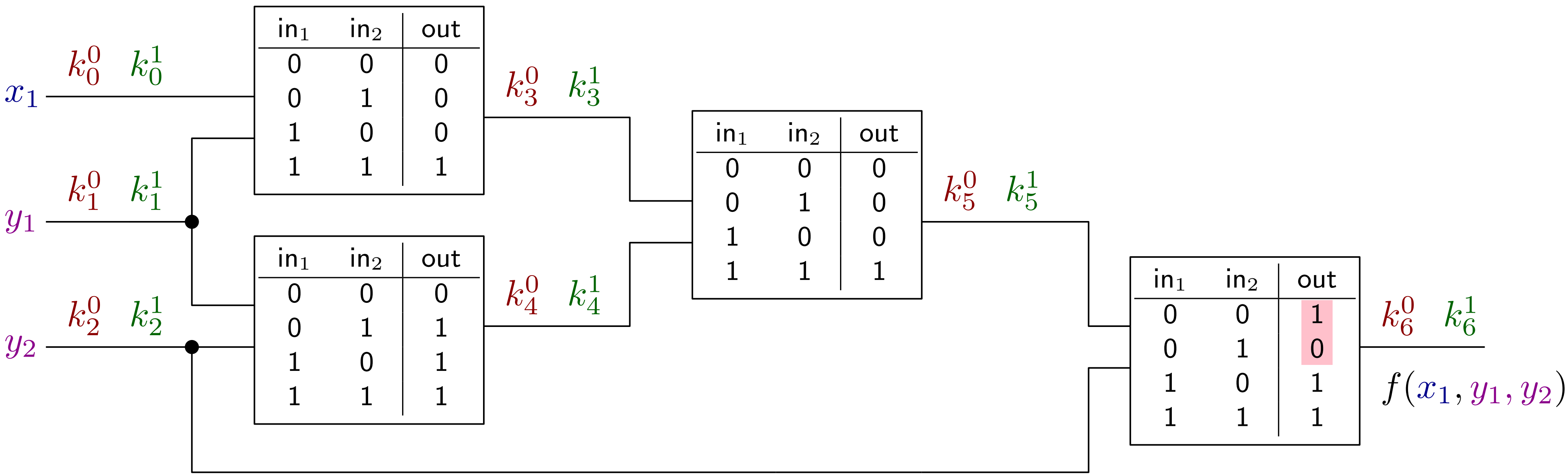

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Each wire carries either a logic 0 or a logic 1

She assigns two random symmetric keys k_i^0 and k_i^1 to the generic i-th wire to represent 0 and 1, resp.

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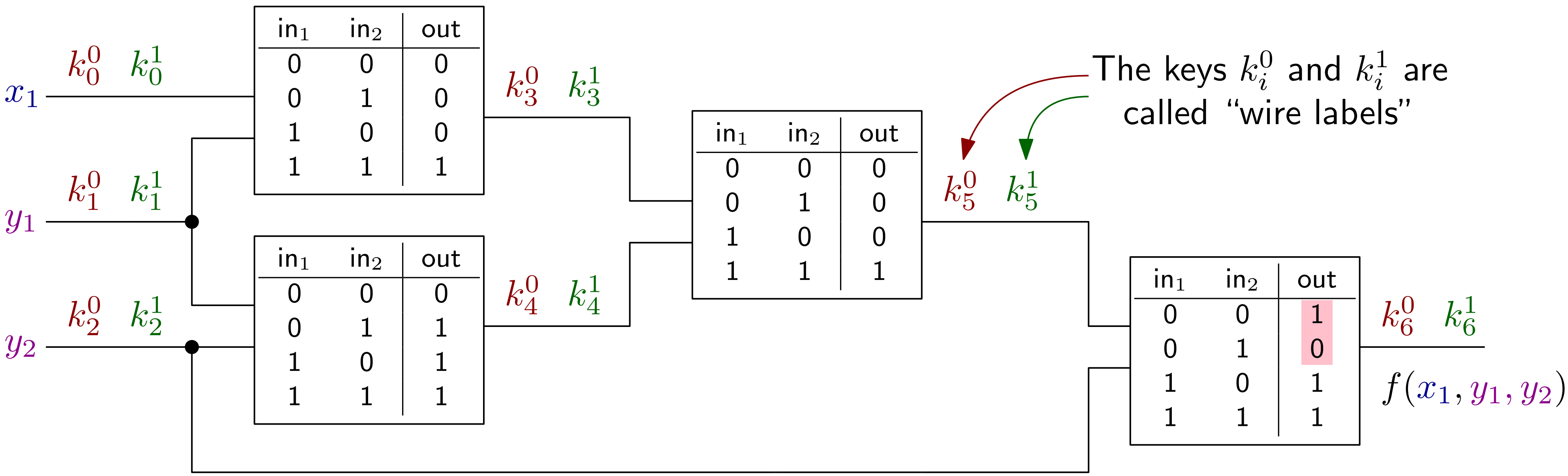

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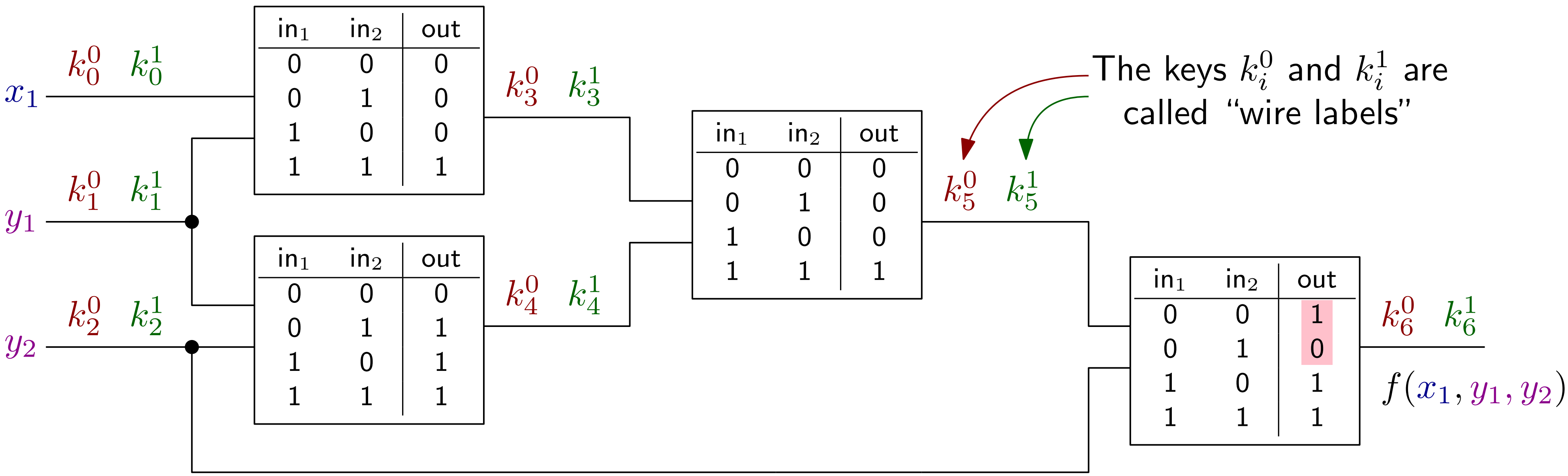

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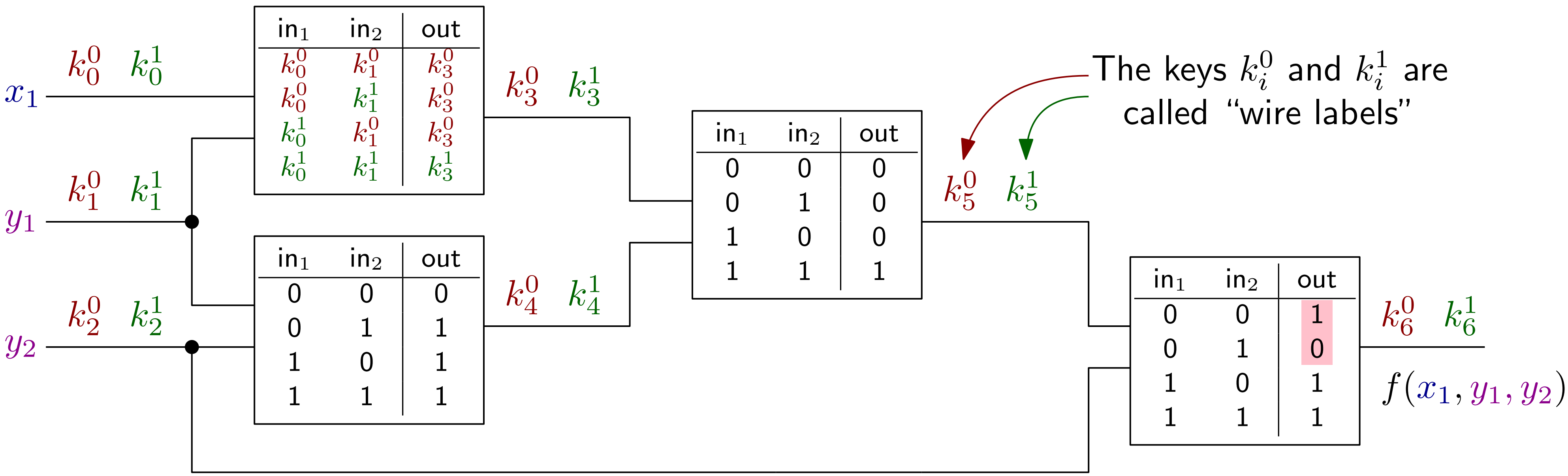

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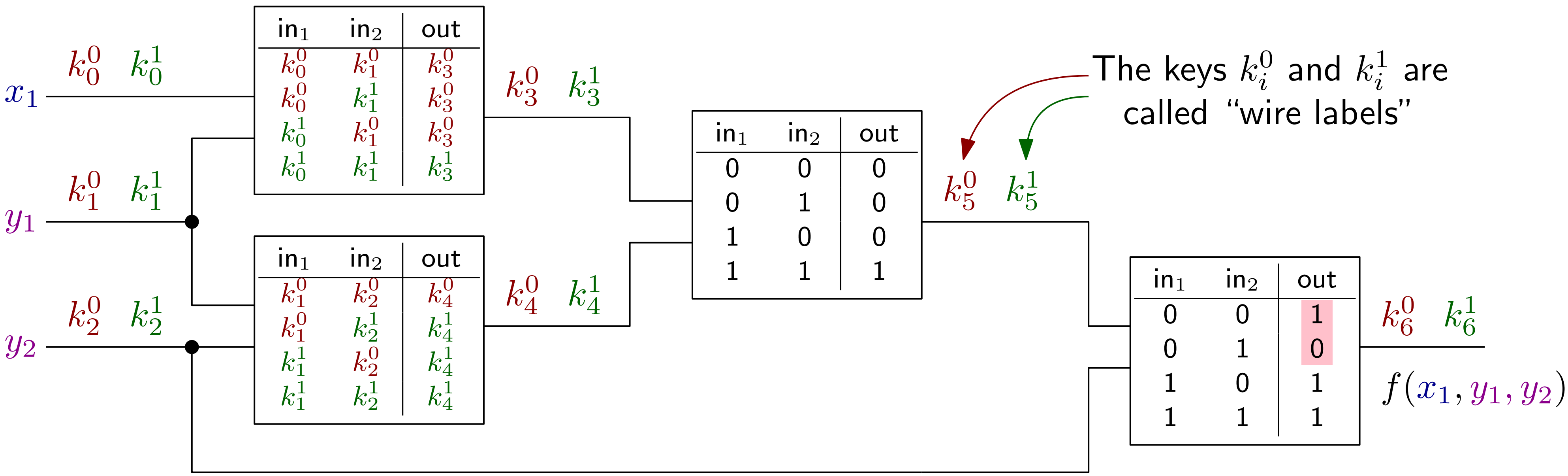

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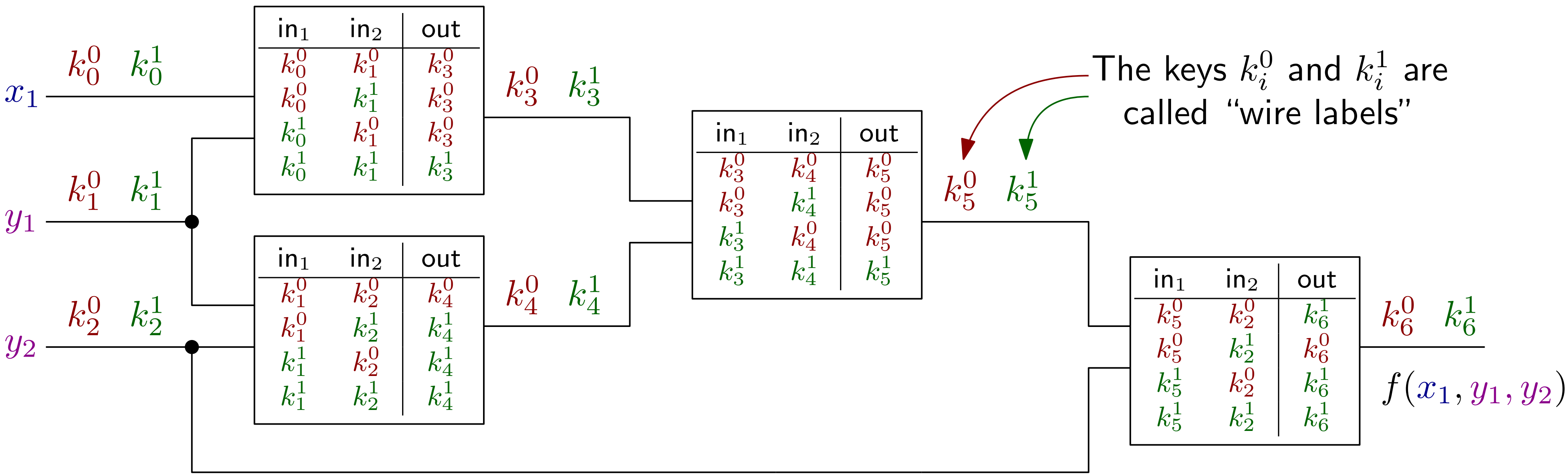

in$_1$	in$_2$	out
k_0^0	k_1^0	k_3^0
k_0^0	k_1^1	k_3^0
k_0^1	k_1^0	k_3^0
k_0^1	k_1^1	k_3^1

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in$_1$	in$_2$	out
k_3^0	k_4^0	k_5^0
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k_3^1	k_4^1	k_5^1

in$_1$	in$_2$	out
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k_5^0	k_2^1	k_6^0
k_5^1	k_2^0	k_6^1
k_5^1	k_2^1	k_6^1

She "folds" the not gates into an adjacent gate

Each wire carries either a logic 0 or a logic 1

She assigns two random symmetric keys k_i^0 and k_i^1 to the generic i-th wire to represent 0 and 1, resp.

She rewrites the truth tables in terms of the input and output wire labels

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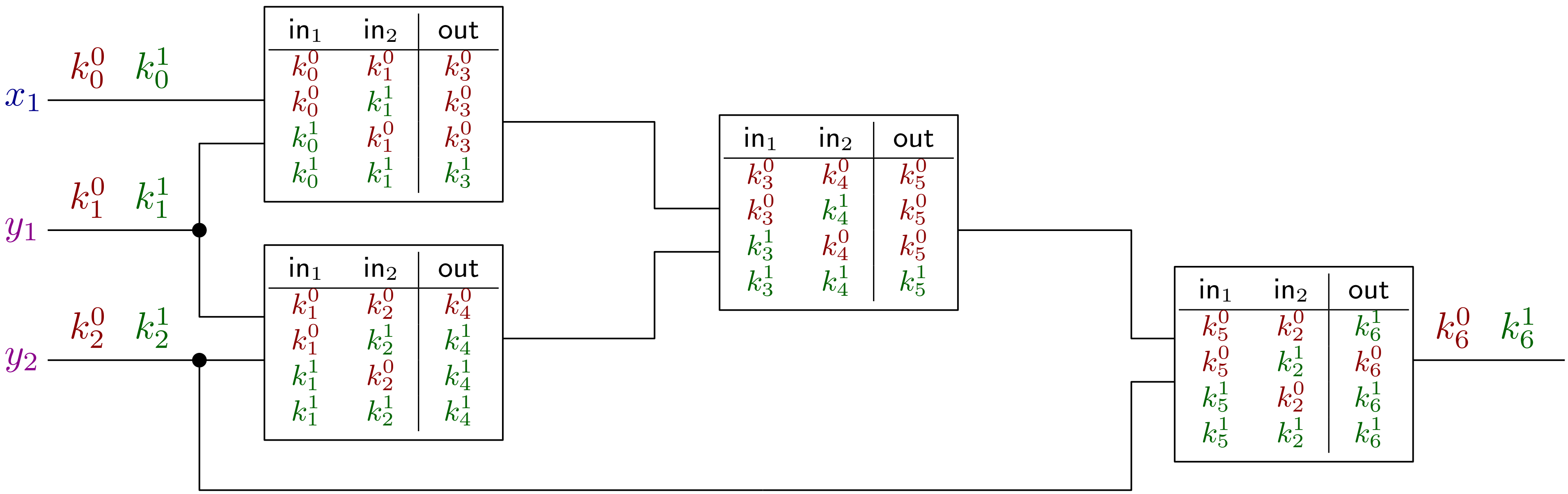
Alice now encrypts the outputs in each truth table using a **secure authenticated encryption scheme**



The key used to encrypt an output consists of the two corresponding input wire labels

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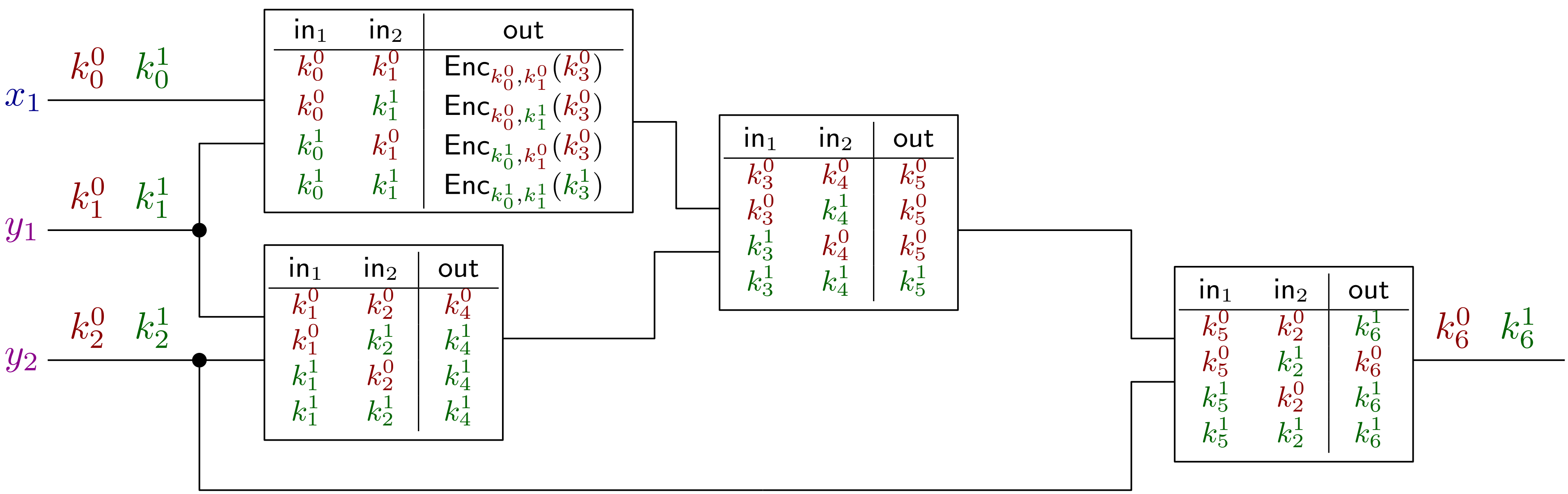

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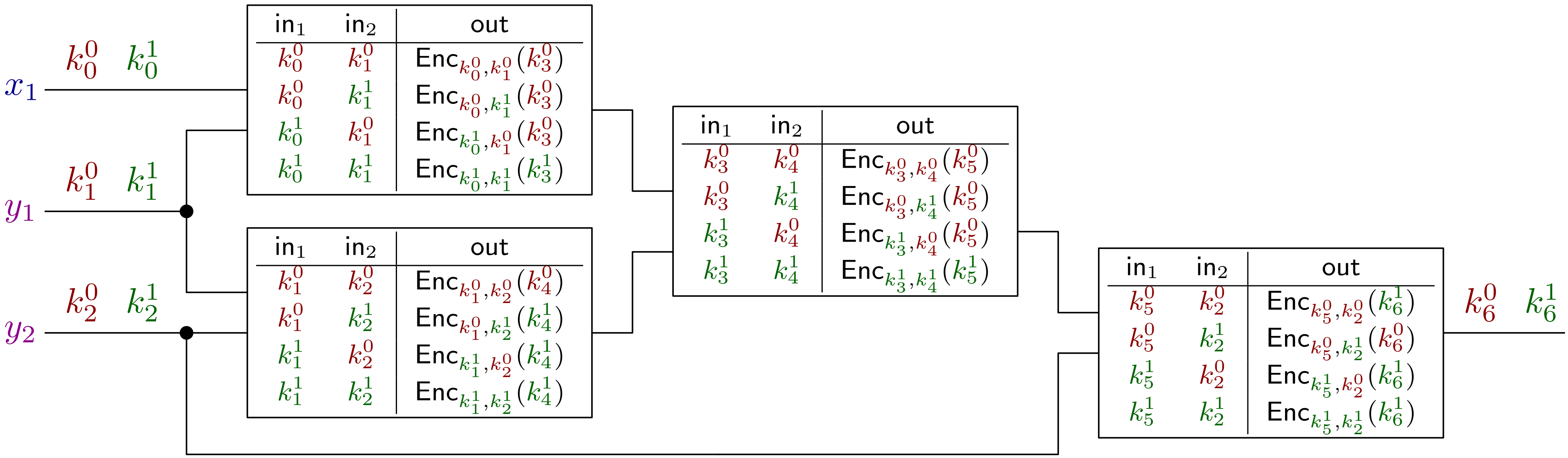

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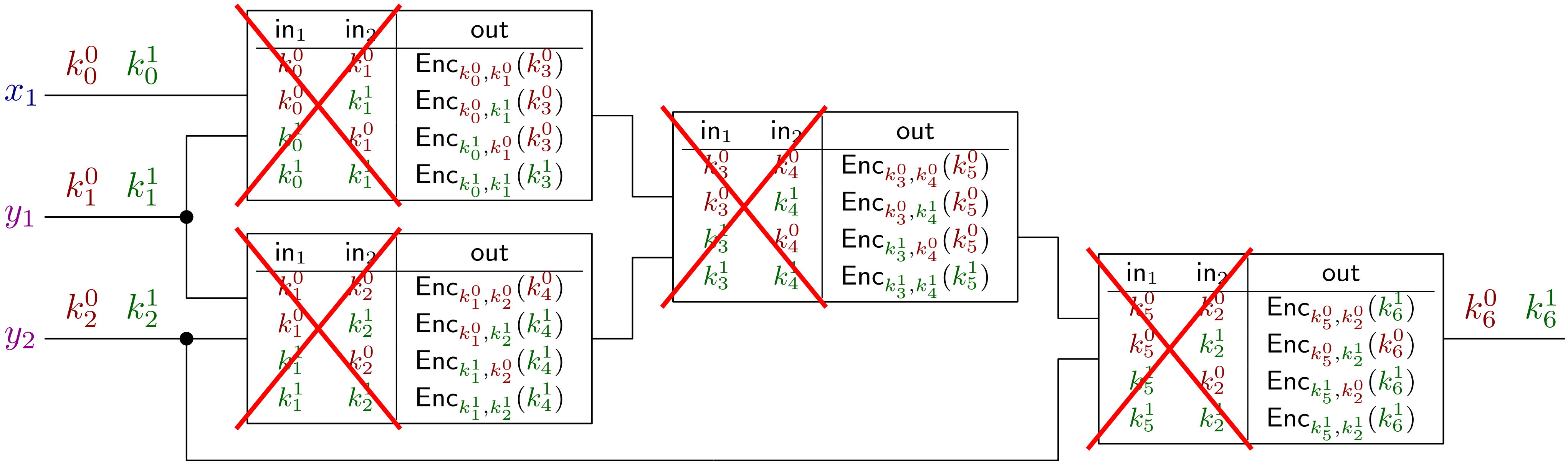

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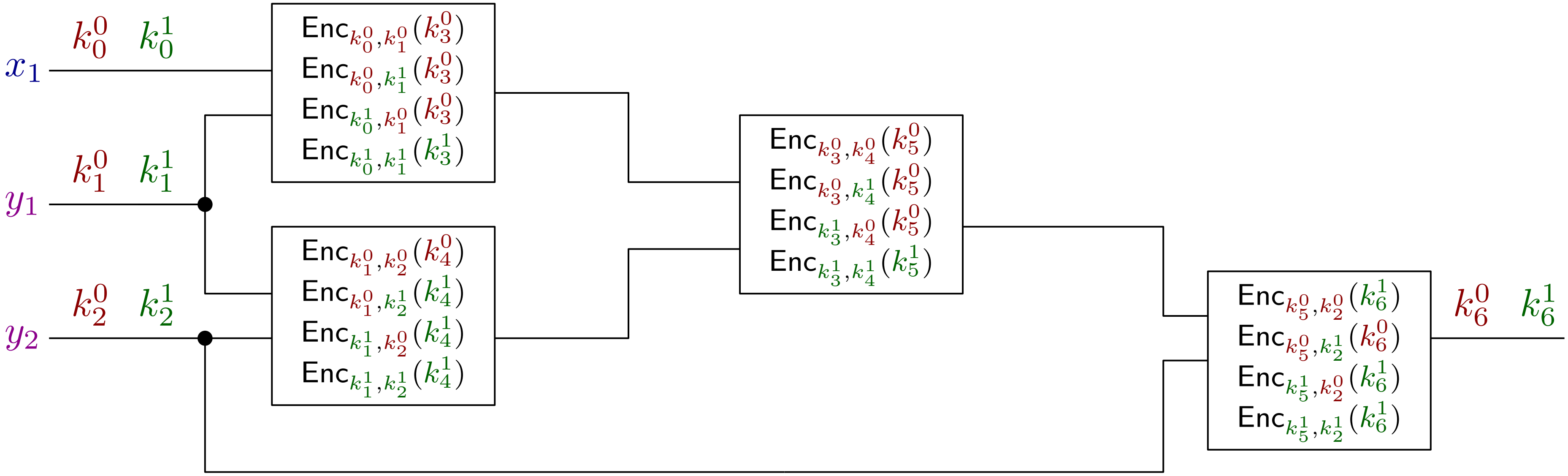

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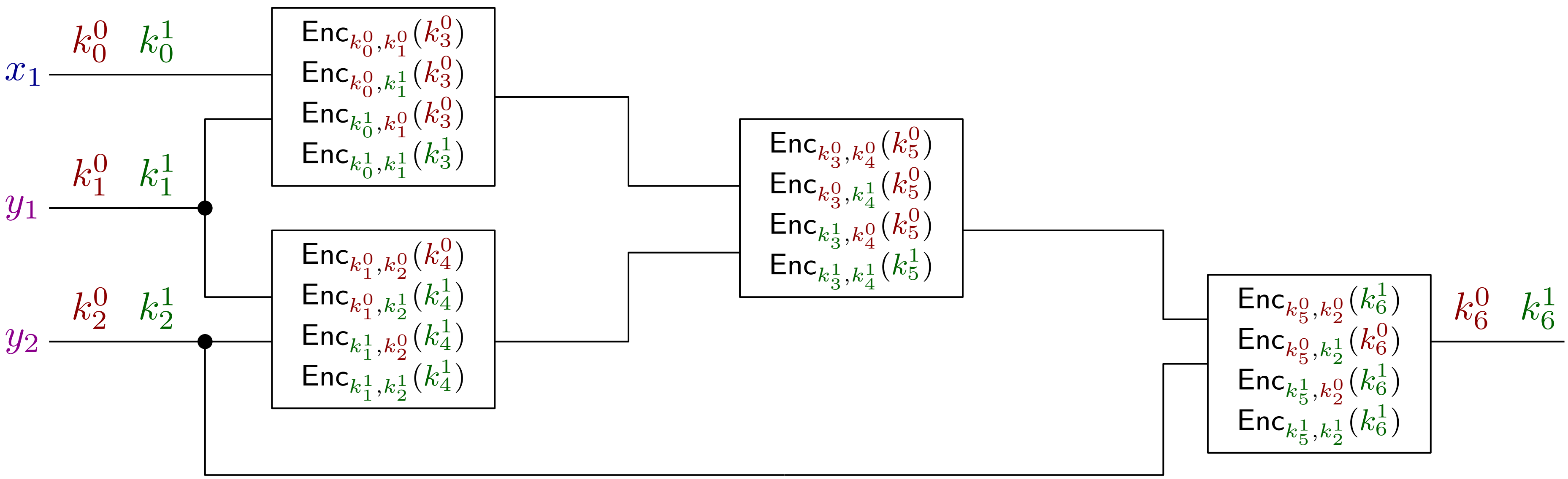

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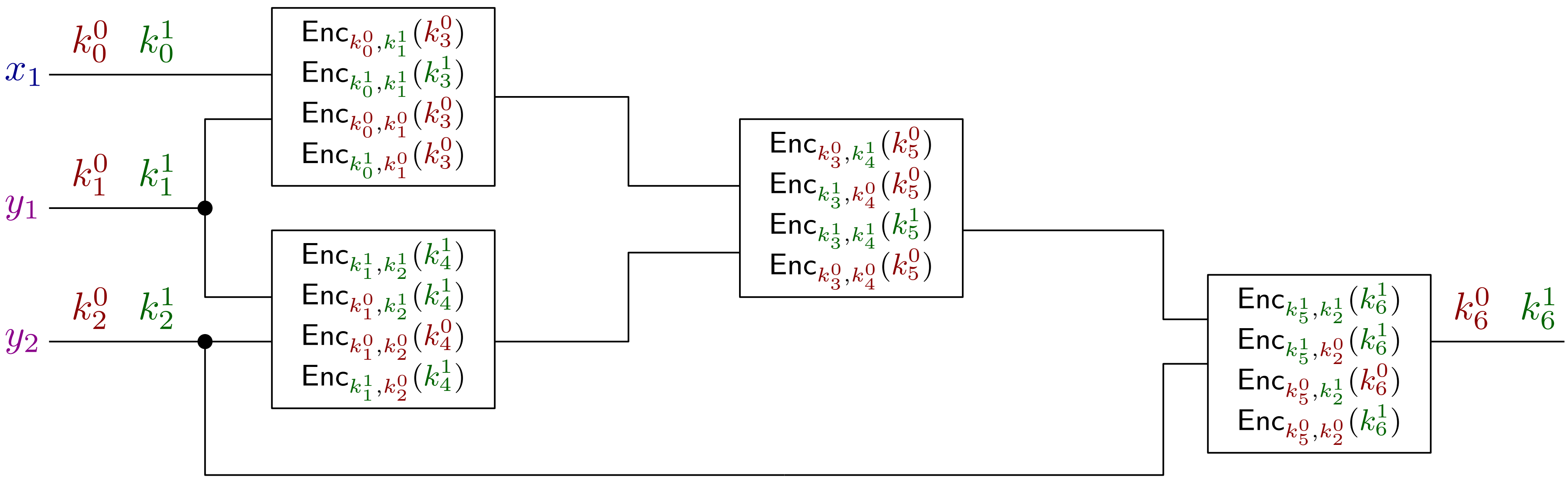

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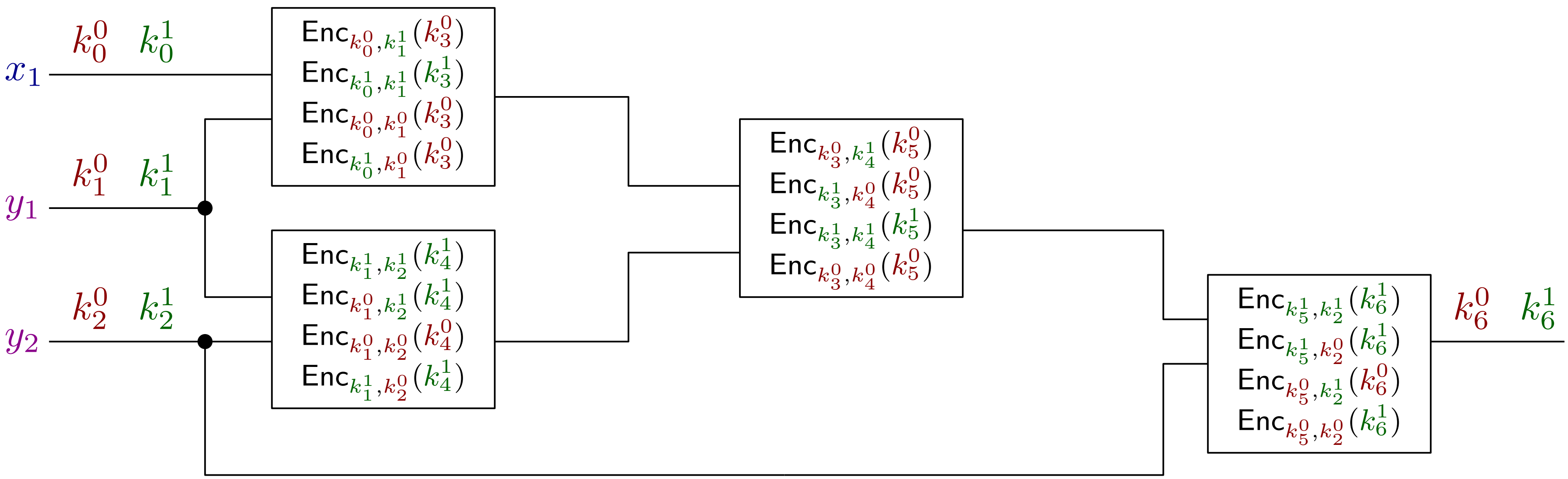

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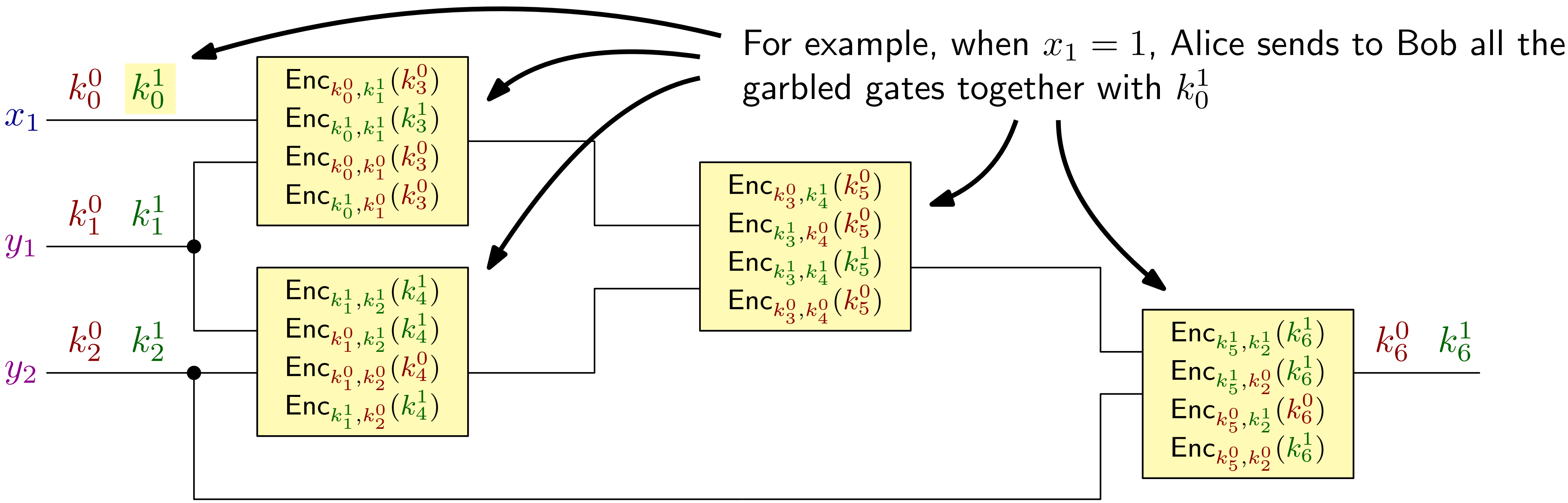

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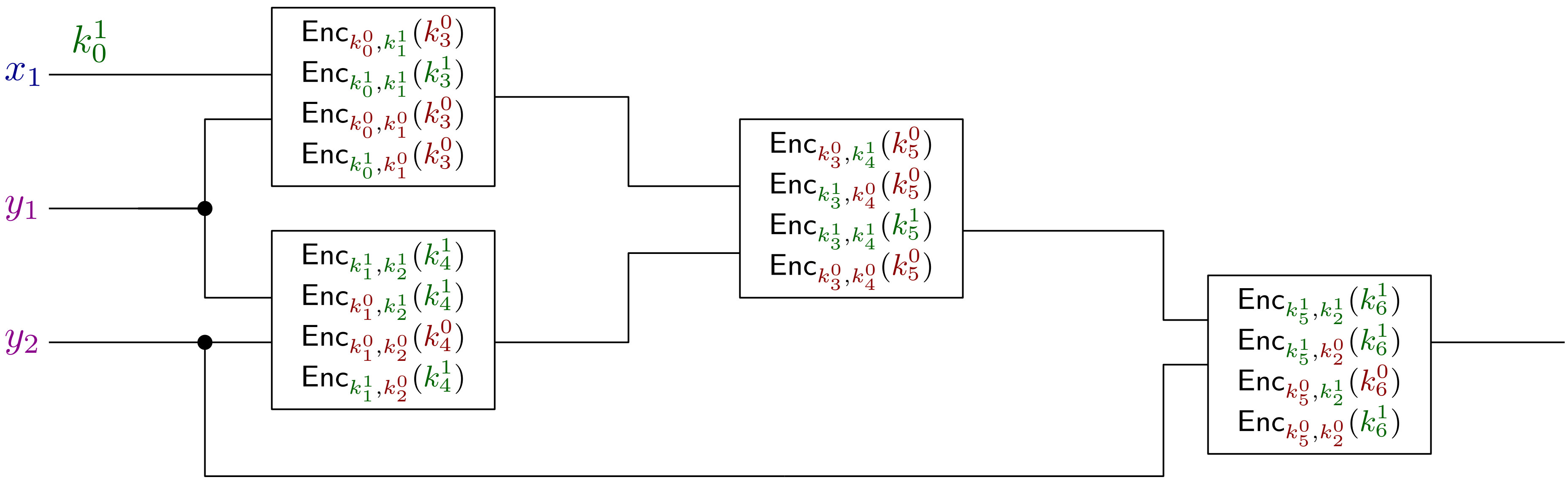

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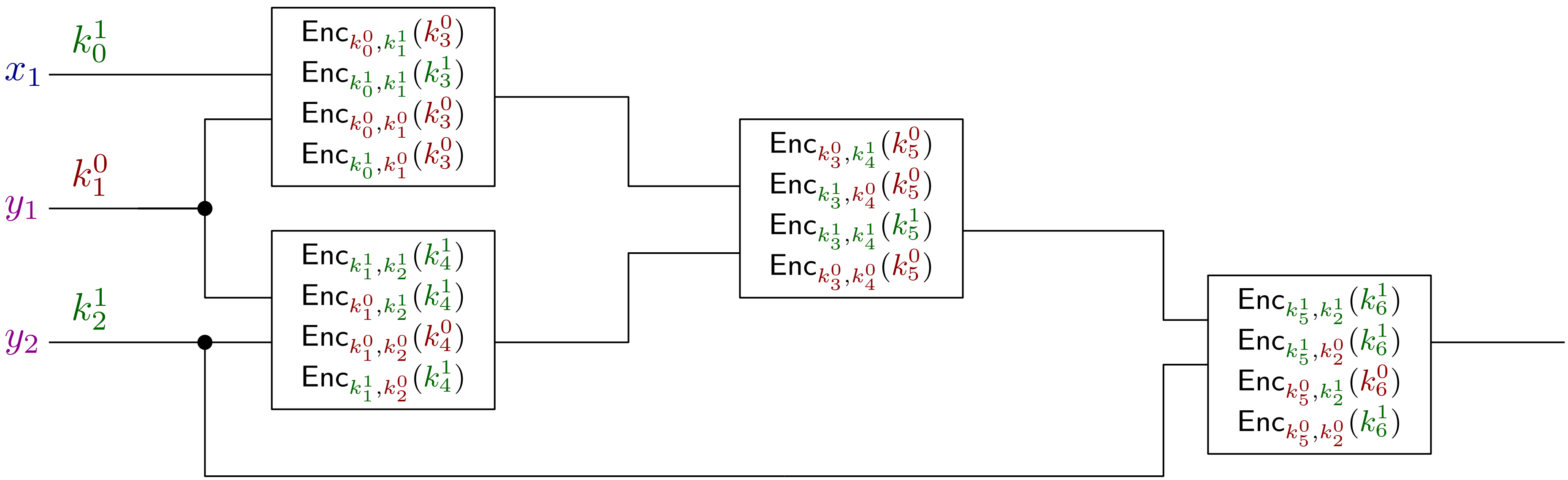

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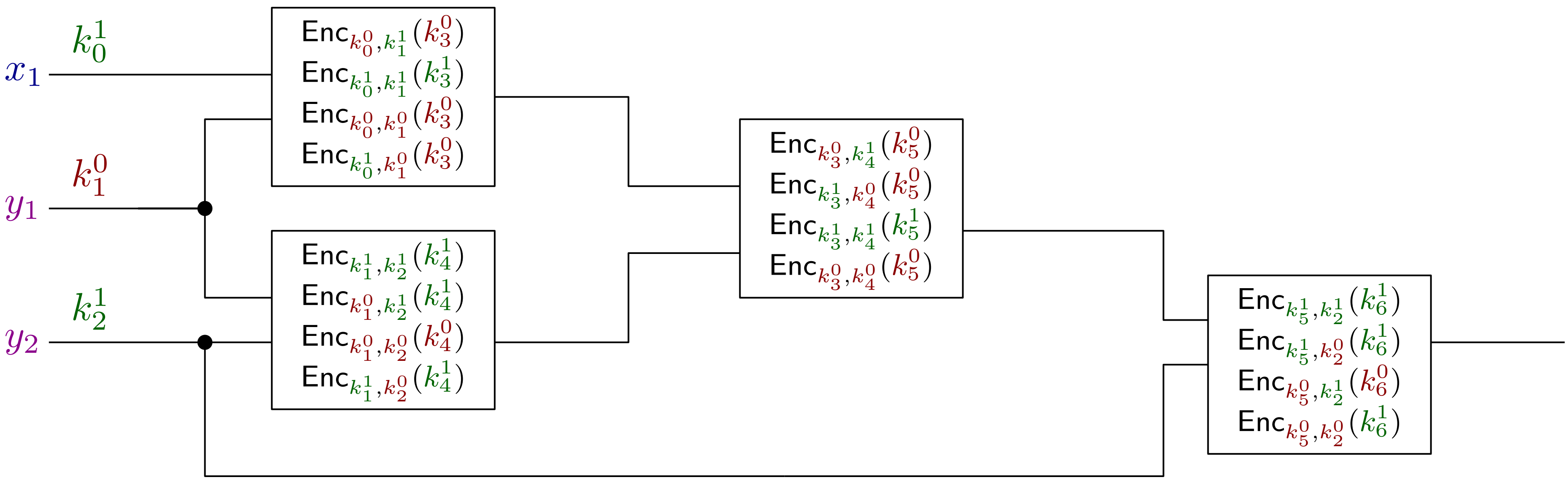

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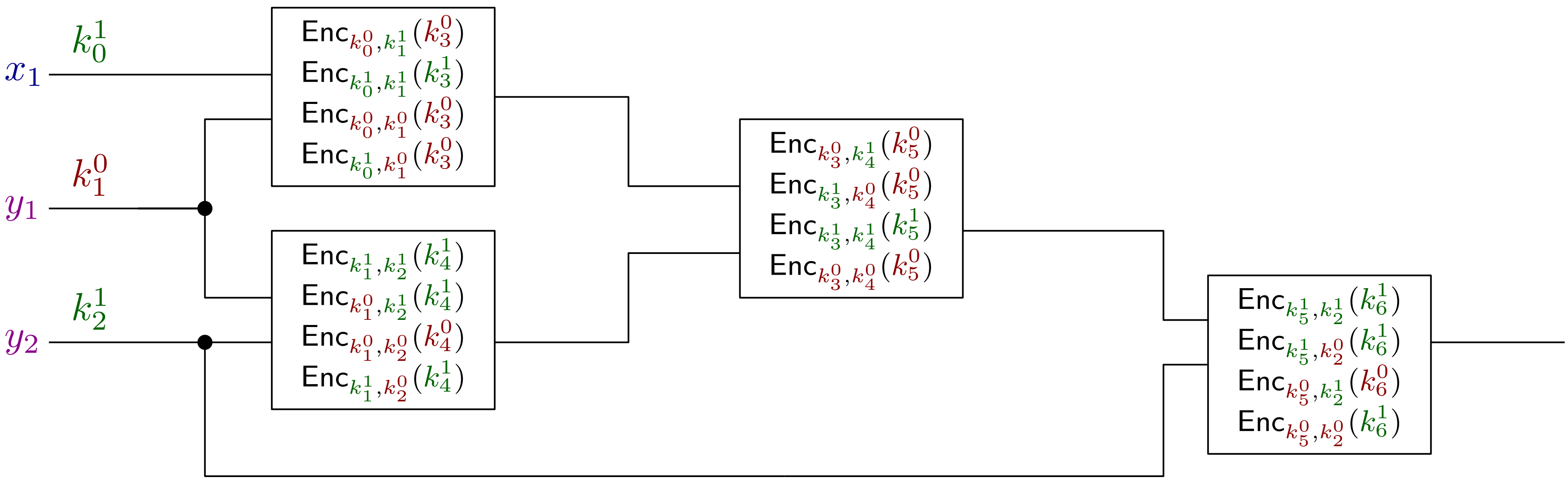

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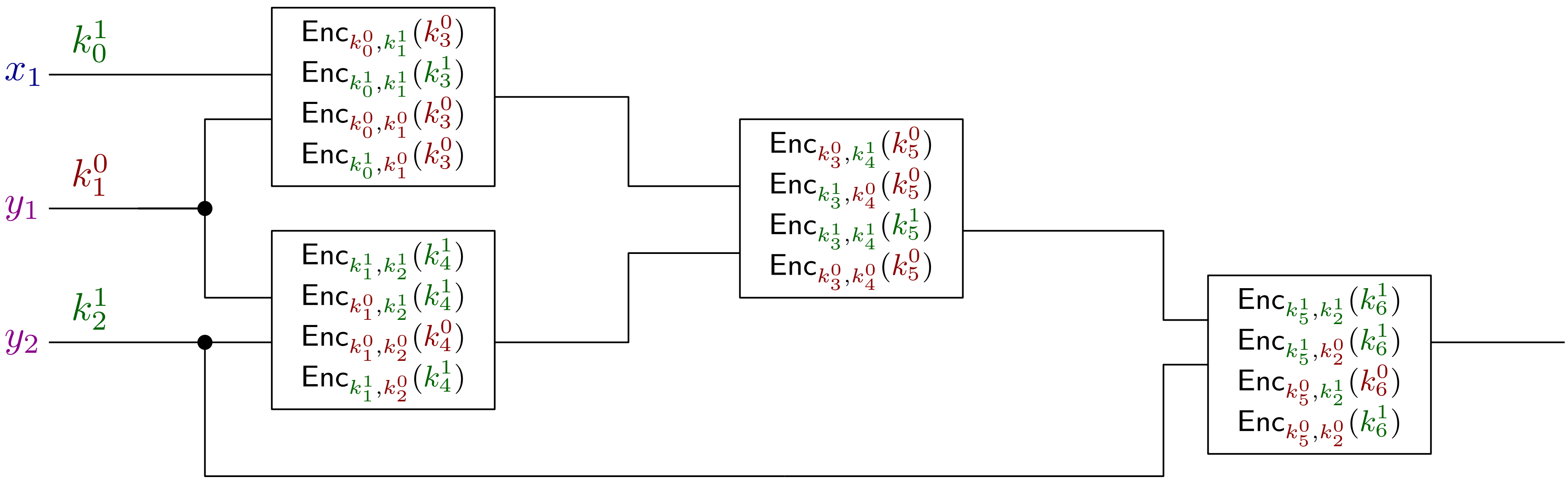

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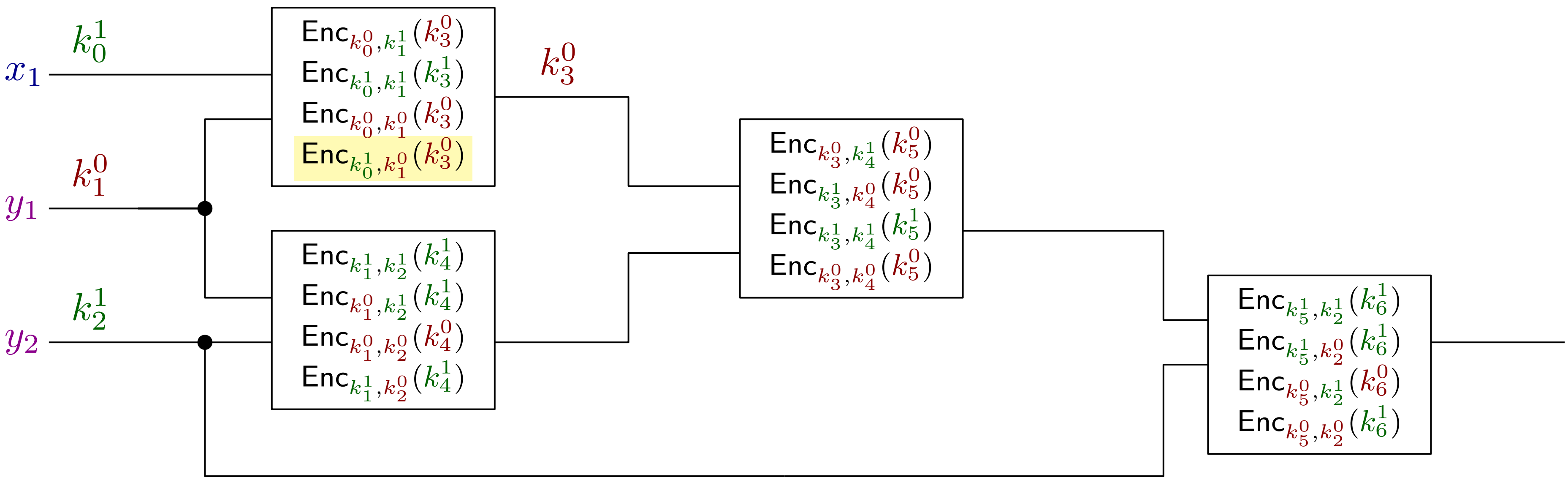

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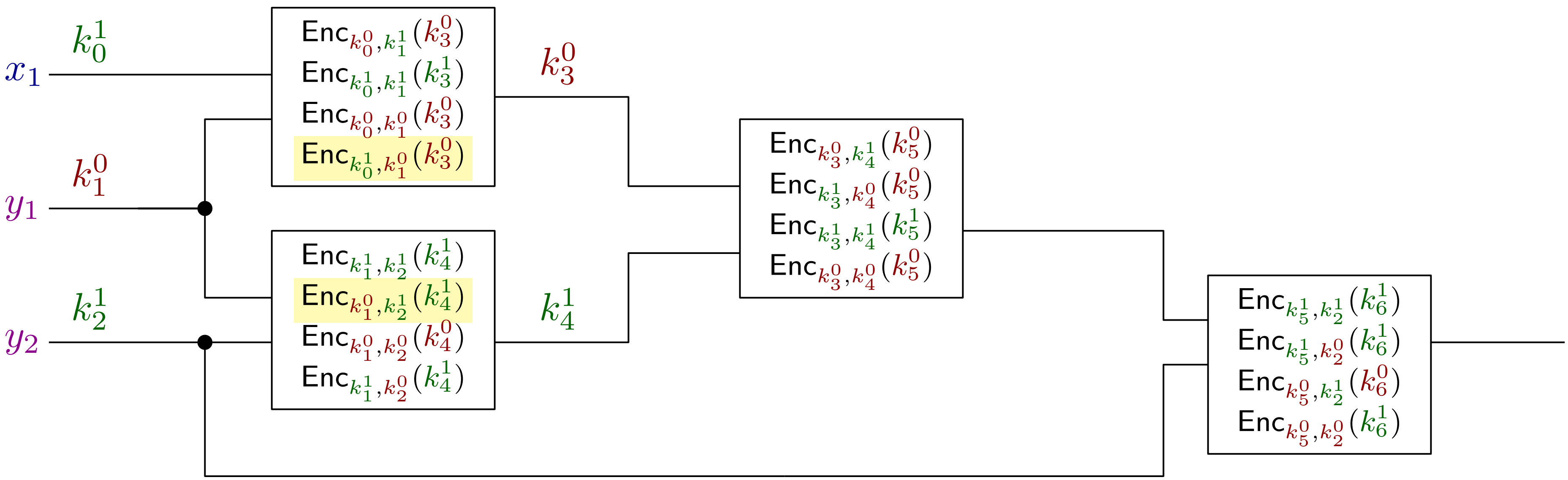

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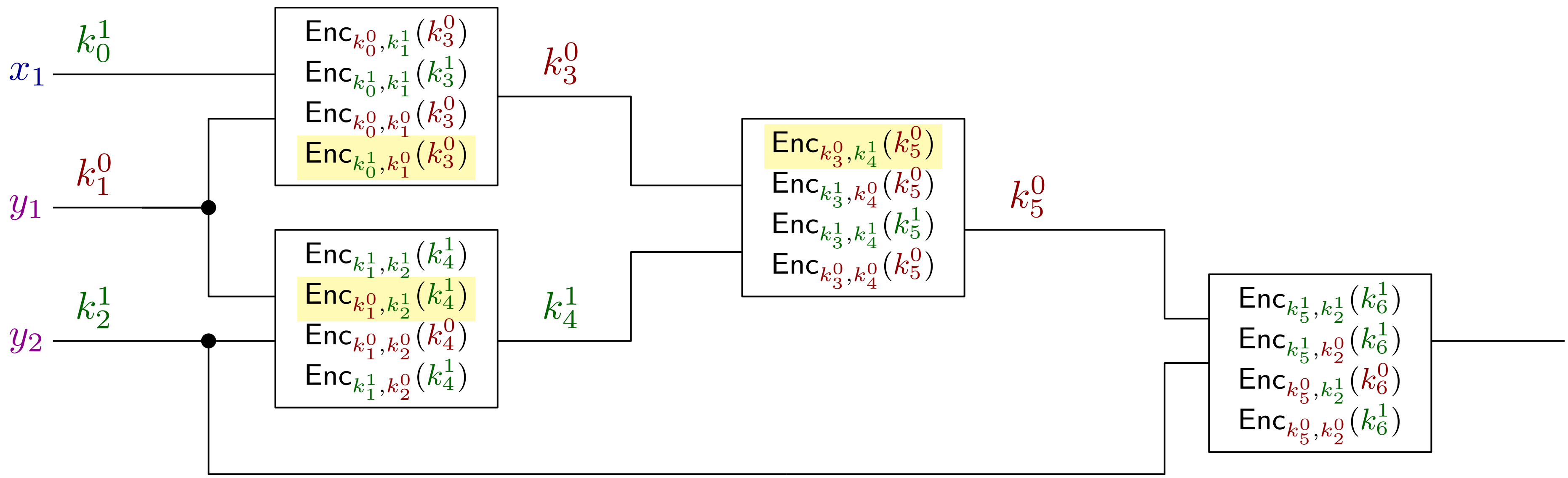

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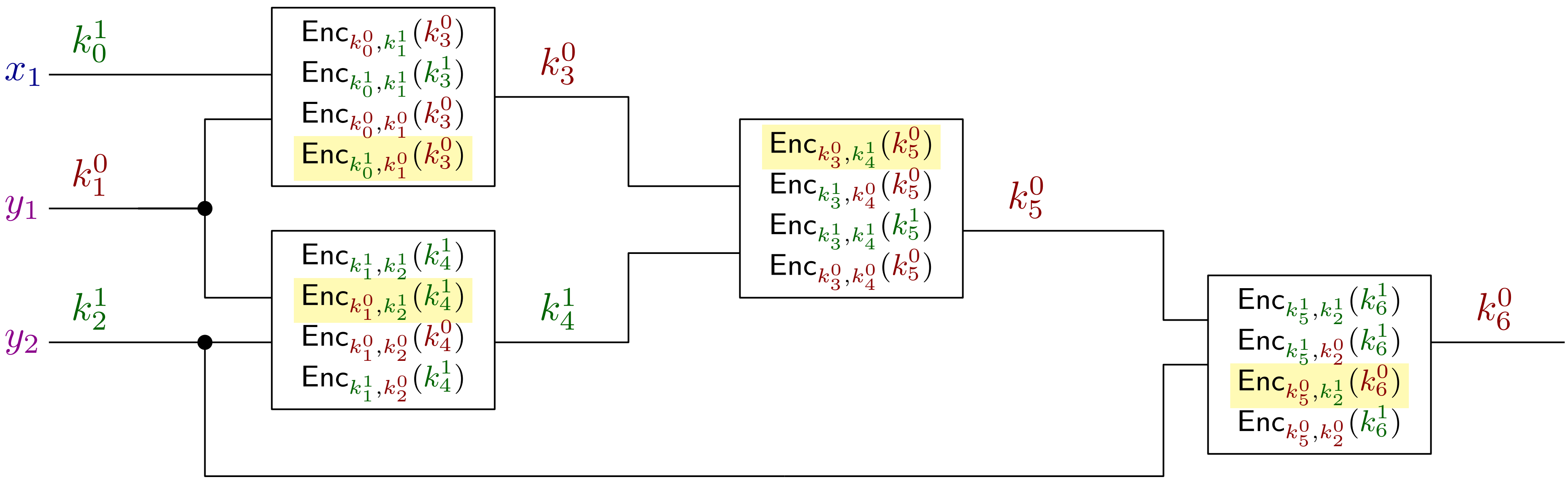

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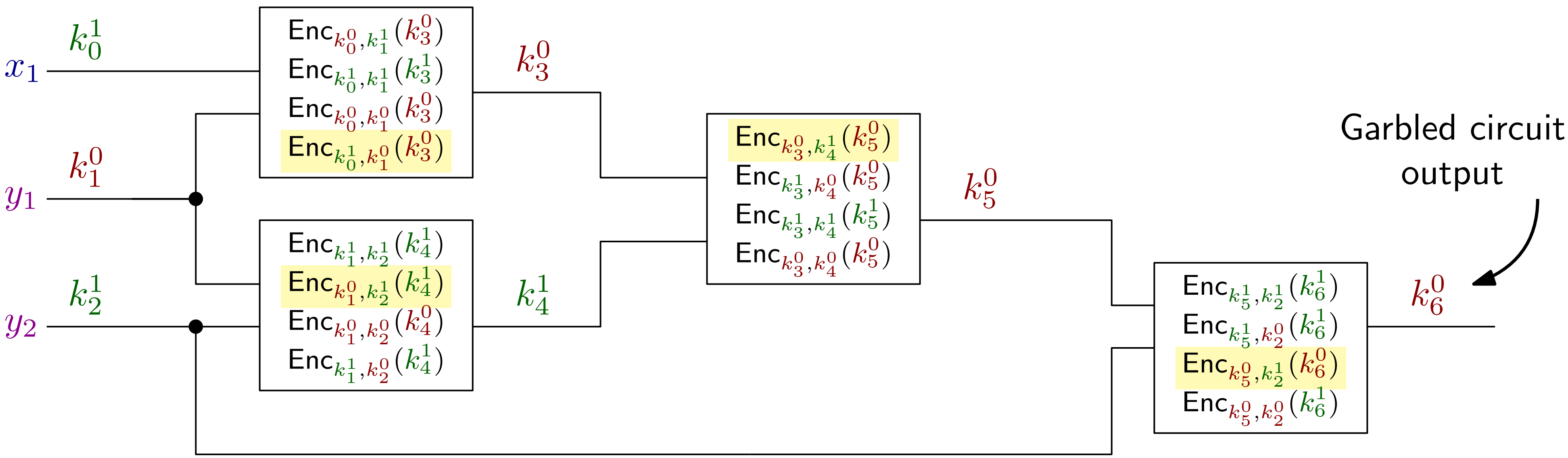

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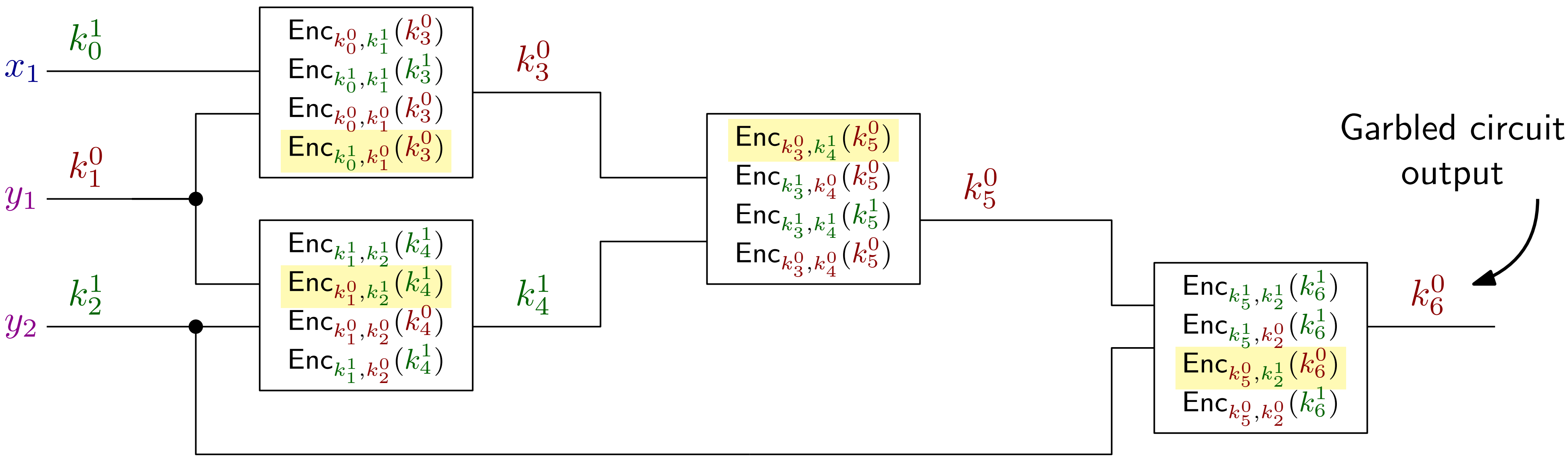

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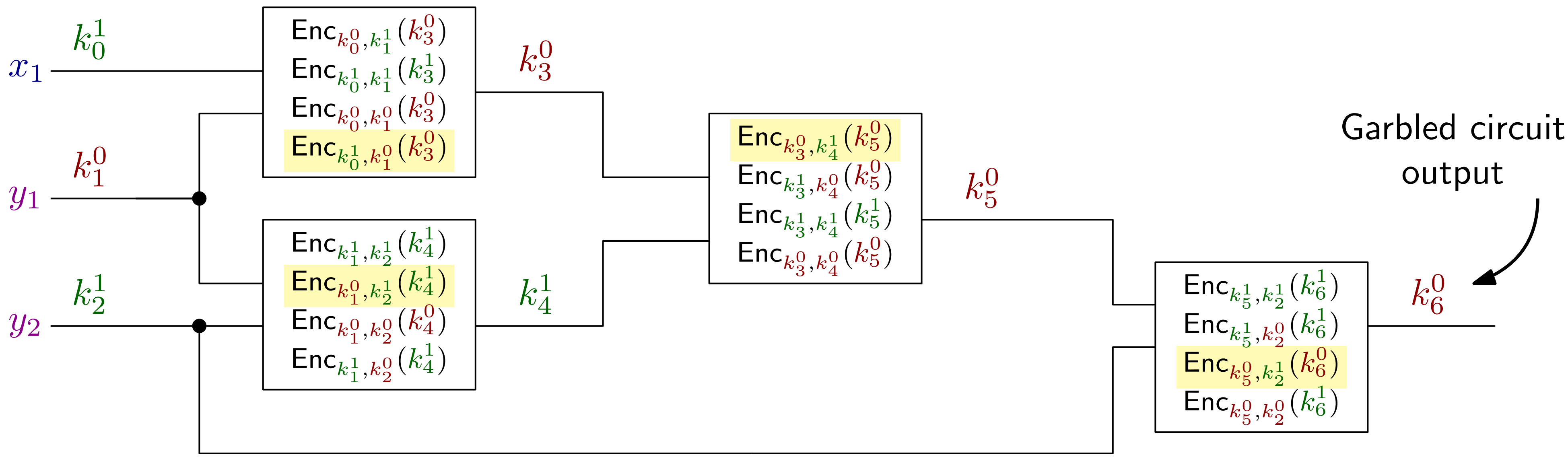

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Alice knows whether the label she received corresponds to 0 or 1.
She learns $f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$

The Oblivious Transfer Protocol

How does Bob learn the wire-labels corresponding to his input?

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Alice and Bob use a protocol known as **oblivious transfer protocol**

- In the oblivious transfer protocol Alice has two messages m_0, m_1 of length $\ell(n)$
- Bob wants to learn one of them, say m_b, without revealing which one he is interested in to Alice
- Alice wants to be sure that Bob learns exactly one of the two values