Reminder: Complexity Classes

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Zero Knowledge Proofs

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3			7				
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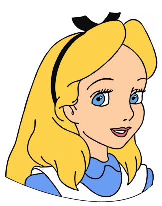

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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3			7				
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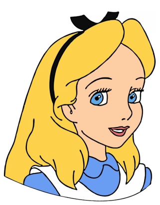
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

I know a solution to this Sudoku instance



Zero Knowledge Proofs

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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3	4	5	2	8	6	1	7	9

5	3			7				
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8				6				3
4			8		3			1
7				2				6
	6					2	8	
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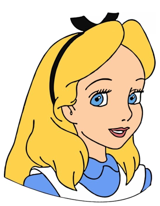
				8			7	9

Really? Show it to me!



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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
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5	3			7				
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4			8		3			1
7				2				6
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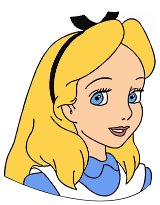
				8			7	9

I don't want to reveal it to you



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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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5	3			7				
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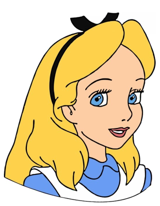
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Then I don't believe you really have a solution



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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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3	4	5	2	8	6	1	7	9

5	3			7				
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7				2				6
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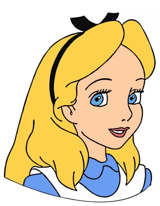
				8			7	9

I can prove to you I have a solution without revealing anything about it



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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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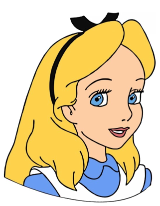

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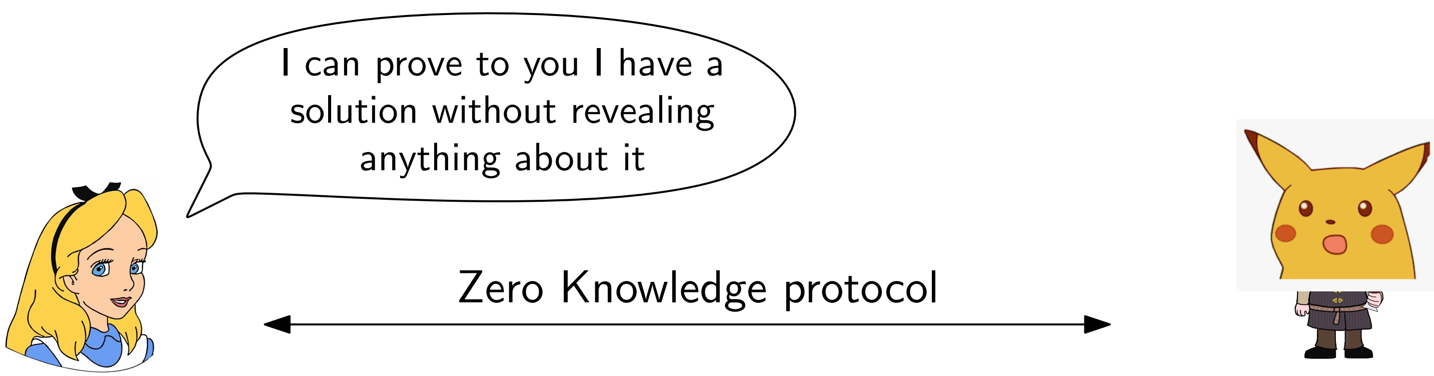

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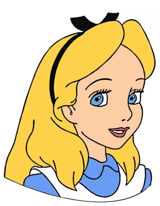
I am now convinced
you have the solution



Zero Knowledge protocol

Zero Knowledge Proofs

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6			1	9	5			
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Hey, Charlie!
Alice has a solution to
this Sudoku instance

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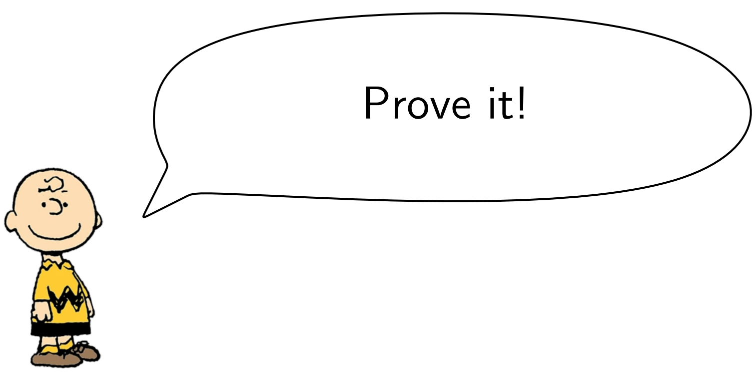

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Even if I definitely know she has a solution, somehow I have no way of proving that

What are proofs anyway?

There is some **claim** x known to both Alice and Bob

A **prover** (Alice) wants to convince a **verifier** (Bob) that the claim is true

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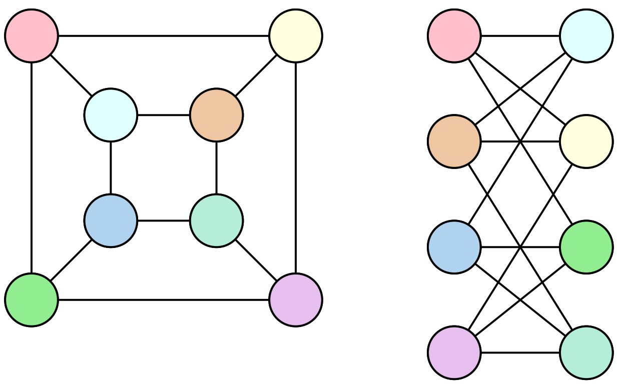
We can model the verifier as a polynomial-time algorithm $\mathcal{V}(x, w)$ that takes as input the claim x and the proof w, and outputs 1 if and only if it accepts w as a proof of x

Example: Graph Isomorphism

Graph isomorphism problem

G_1 is isomorphic to G_2 iff $\exists$ bijection $\pi : V_1 \to V_2$ s.t. $(u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2$.

$G_1 = (V_1, E_1)$ $\quad$ $G_2 = (V_2, E_2)$



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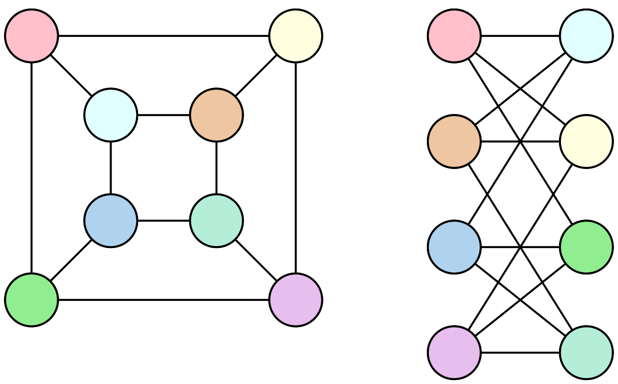

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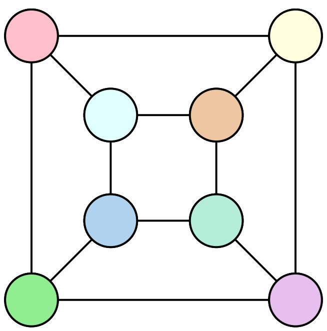
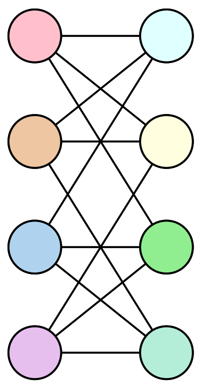


$G_2 = (V_2, E_2)$



Claim: G_1 and G_2 are isomorphic.

$$L = \{(G_1, G_2) \mid \exists \text{ bijection } \pi : V_1 \to V_2 \text{ s.t. } (u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2\}$$

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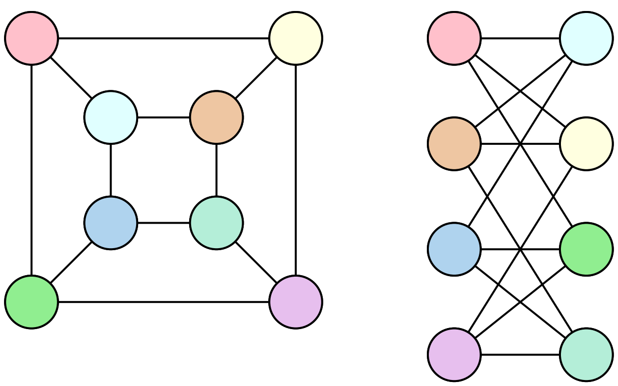

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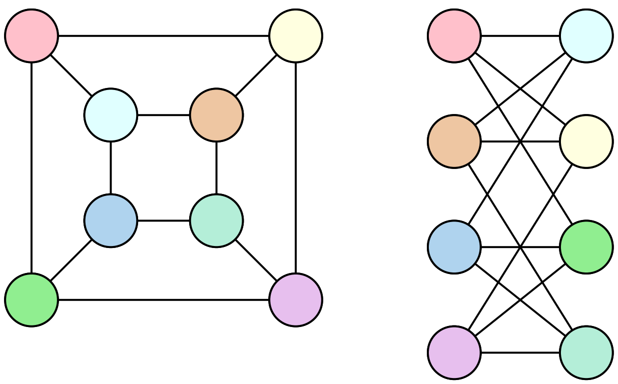

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$x = (G_1, G_2)$

$w = \pi$

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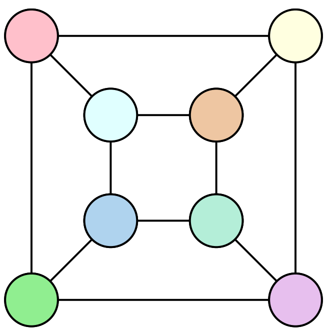
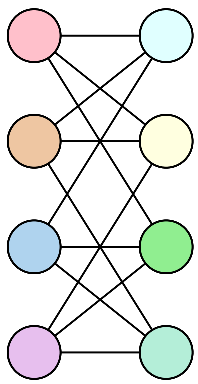


$G_2 = (V_2, E_2)$



Claim: G_1 and G_2 are isomorphic.

$$L = \{(G_1, G_2) \mid \exists \text{ bijection } \pi : V_1 \to V_2 \text{ s.t. } (u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2\}$$

$x = (G_1, G_2)$

$w = \pi$

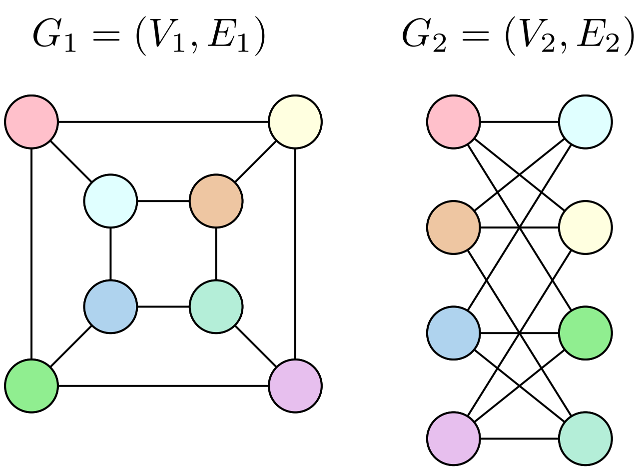
Algorithm $\mathcal{V}((G_1, G_2), \pi)$:

- For all $u, v \in V_1^2$:
  - If $((u, v) \in E_1 \wedge (\pi(u), \pi(v)) \notin E_2) \vee ((u, v) \notin E_1 \wedge (\pi(u), \pi(v)) \in E_2)$: Return 0
- Return 1

Example: Graph Isomorphism

Graph isomorphism problem

G_1 is isomorphic to G_2 iff $\exists$ bijection $\pi : V_1 \to V_2$ s.t. $(u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2$.



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The verifier learns the isomorphism π

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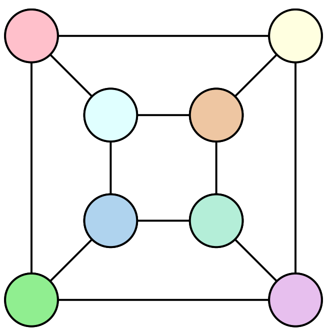

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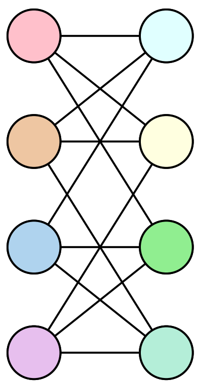

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The verifier learns the isomorphism π

Is it necessary to reveal π?

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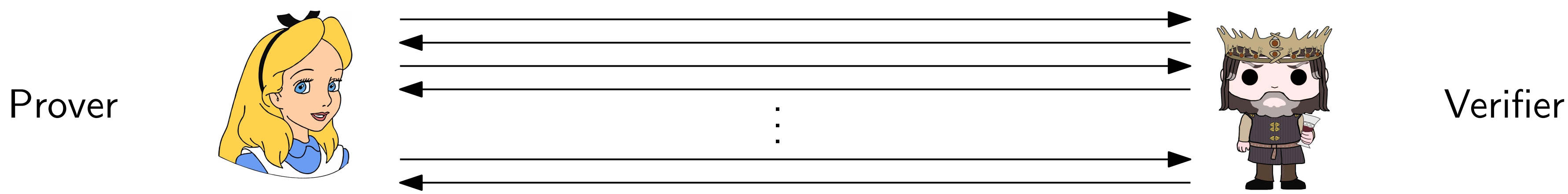
Interactive Proofs Systems

In an **interactive proof system**, the prover and the verifier exchange multiple messages following some **protocol**



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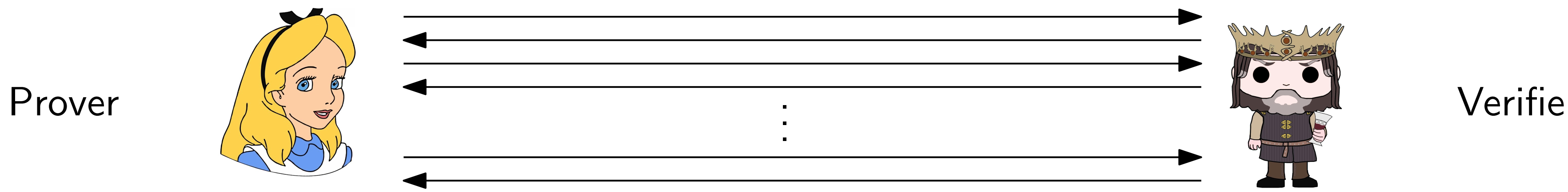

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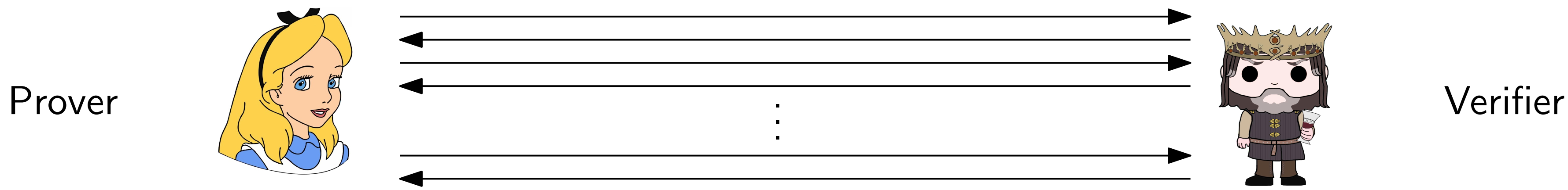

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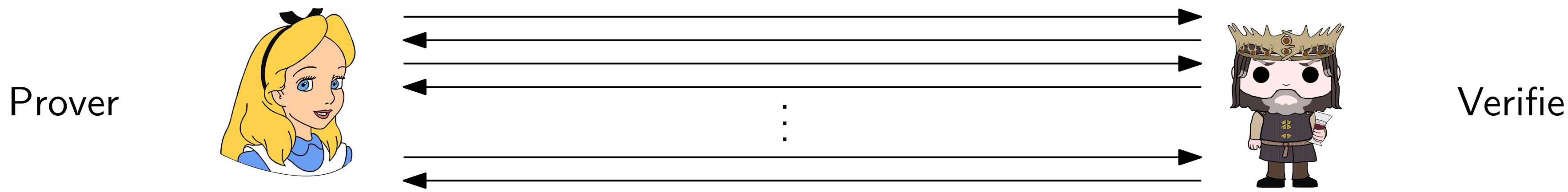

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Ideally, at the end of the interaction:

- If $x \in L$, the prover knows a proof, and follows the protocol, then the verifier will be convinced that $x \in L$ (**completeness**)

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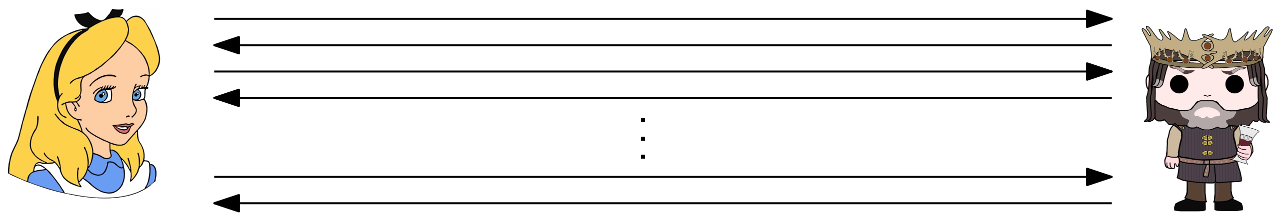
Prover



Verifier

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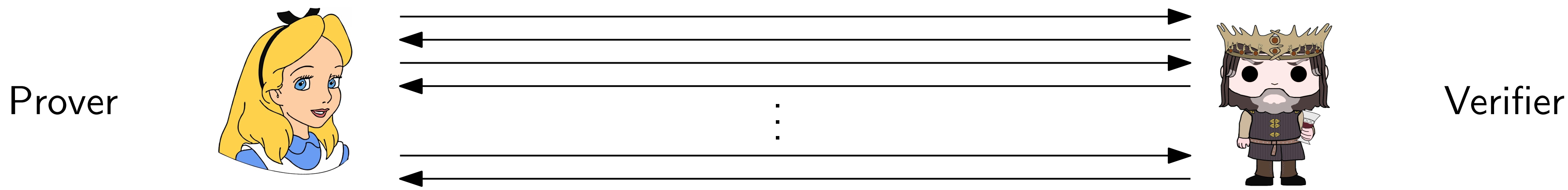

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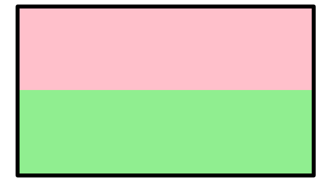
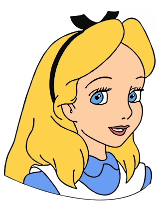
We relax the above requirements by allowing the verifier to commit errors (with a small probability).

A Physical Interactive Proof System

Suppose that the prover needs to convince a colorbrind verifier that this sheet of paper contains multiple colors:





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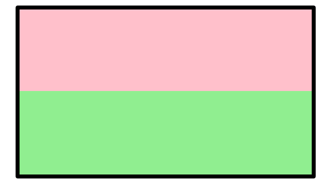

The verifier secretly chooses a random bit b

- If $b = 0$ the verifier keeps the sheet in the original orientation 
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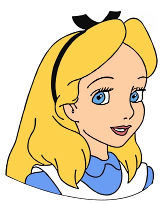

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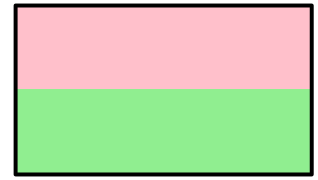

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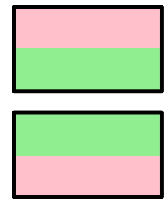

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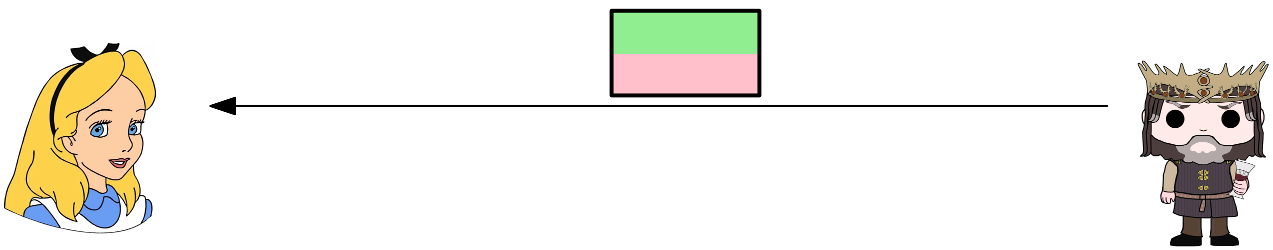

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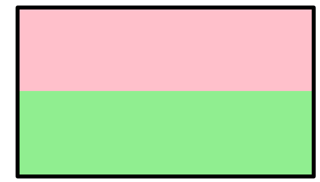

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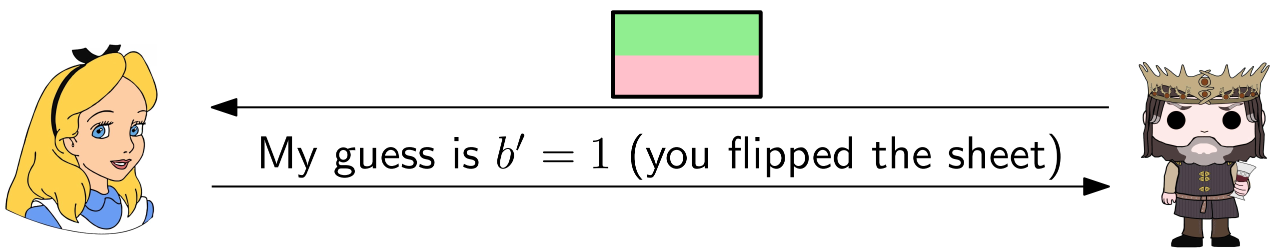

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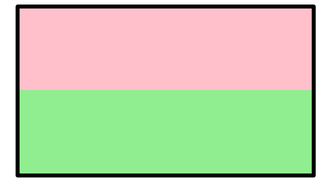

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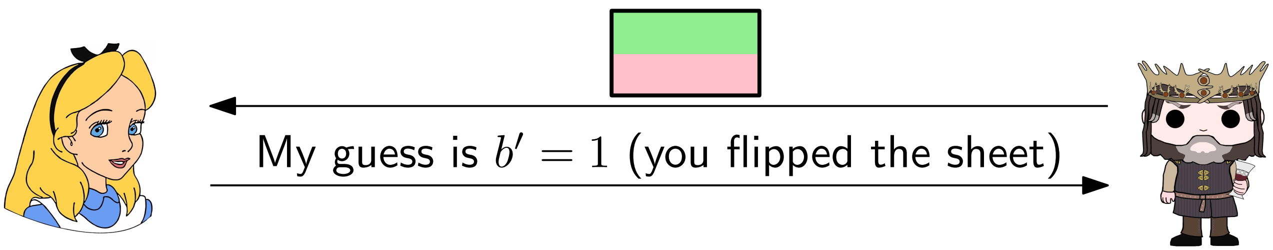

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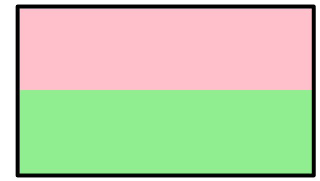

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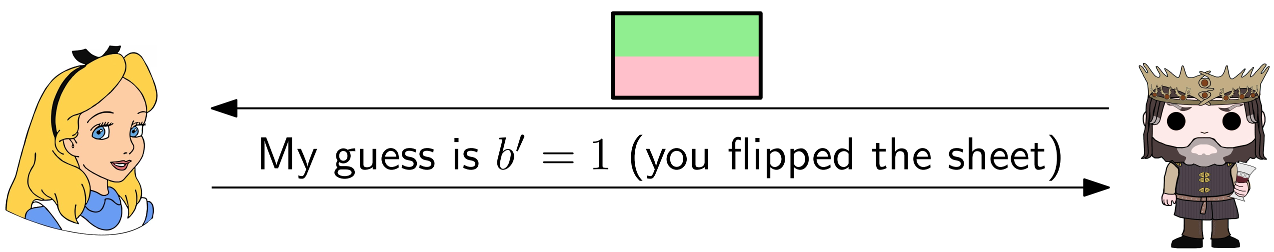

Honest prover (completeness):

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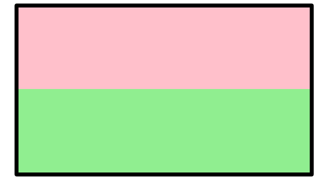

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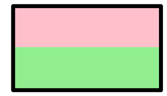
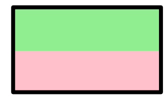
- If $b = 0$ the verifier keeps the sheet in the original orientation 
- If $b = 1$ the verifier flips the sheet upside down 

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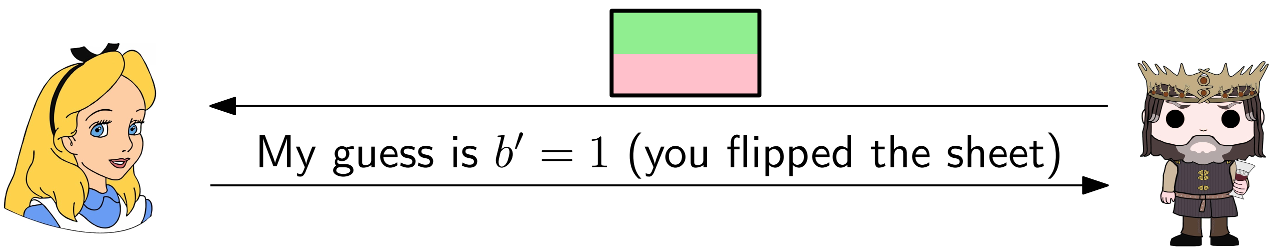

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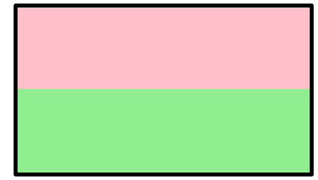

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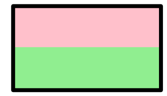
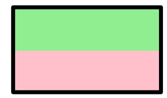
- If $b = 0$ the verifier keeps the sheet in the original orientation 
- If $b = 1$ the verifier flips the sheet upside down 

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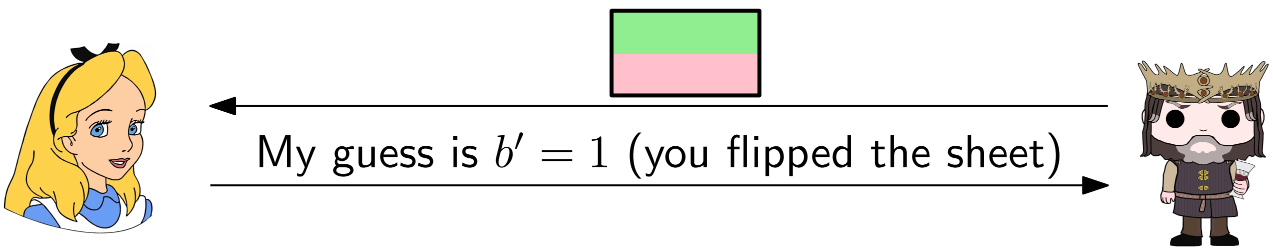

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What if the verifier does not like those odds?

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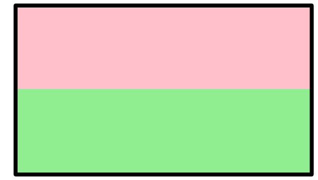

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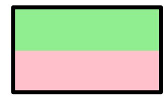
- If $b = 1$ the verifier flips the sheet upside down 

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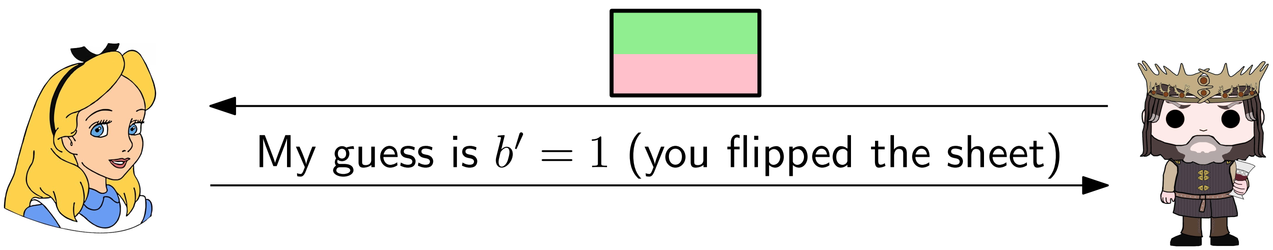

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What if the verifier does not like those odds?

Repeat the experiment k times. Accept iff all trials succeed.

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- **Completeness:** For every $x \in L$, $\Pr[(P, V)(x) = 1] \geq \frac{2}{3}$
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IP is the class of all languages that admit an interactive proof system

Probability Amplification

The constants $\frac{2}{3}$ and $\frac{1}{3}$ are arbitrary. Any pair of error probabilities that differ by at least $\frac{1}{p(n)}$, for some polynomial $p(n) > 0$, is enough.

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- $\Pr[(P, V)(x) = 1] \geq c(n)$ for $x \in L$
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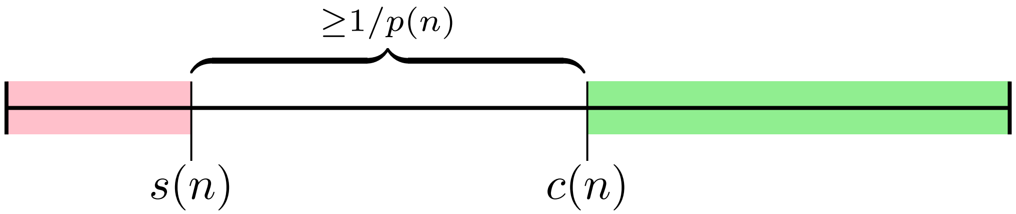
- $s(n) - c(n) \geq \frac{1}{p(n)}$



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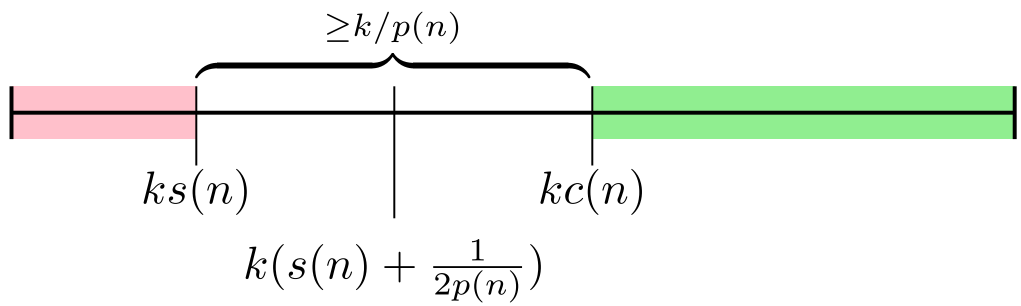

Strategy: Repeat the protocol k times and accept if the $(P, V)(x) = 1$ in at least $k(s(n) + \frac{1}{2p(n)})$ executions

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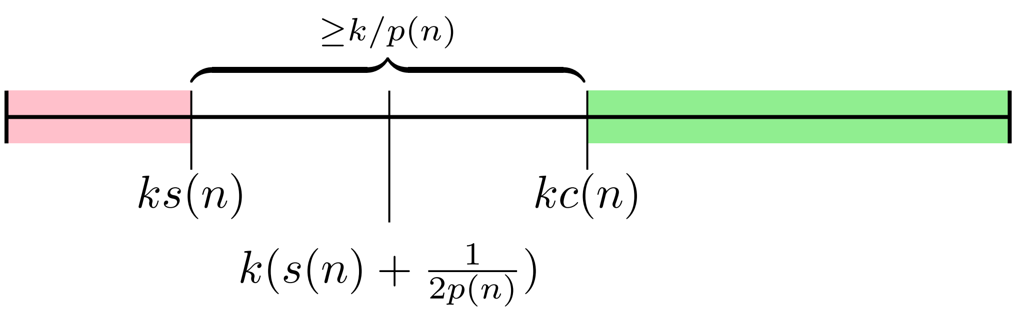

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Chernoff bound: Let $X = X_1 + X_2 + \ldots$ where the X_is are independent binary random variables and let $\mu = \mathbb{E}[X]$. Then, for any $\delta > 0$:

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\delta^2 \mu/2}$$

$O(p(n)^3)$ repetitions suffice

Efficiently verifiable proofs

The class NP can be thought of as the set of all claims that admit a short, efficiently verifiable proof

A language $L \subseteq \{0,1\}^*$ is in NP iff there exists a non-deterministic polynomial-time Turing machine T such that $x \in L$ iff (at least one computation path of) $T(x)$ accepts

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Proof sketch:

$\implies$: A (poynomial-length) accepting computation path of $T(x)$ is a witness w
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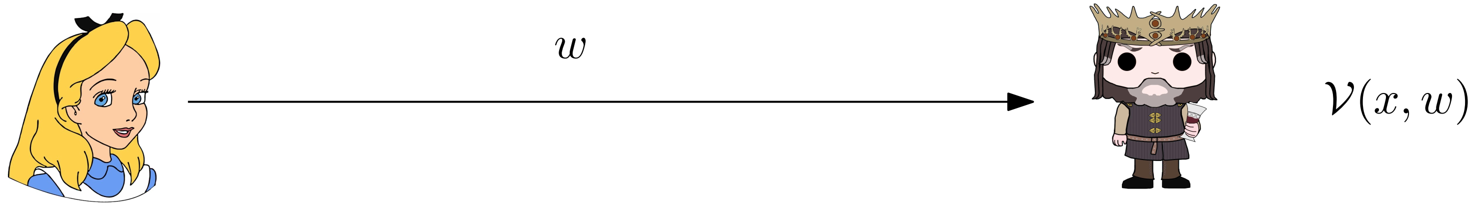
$\Longleftarrow$: A non-determistic Turing machine T can "guess" the witness w, and then check if $\mathcal{V}(x, w) = 1$

Interactive Proofs for NP and BPP

We immediately have an interactive proof for all languages $L \in \mathsf{NP}$



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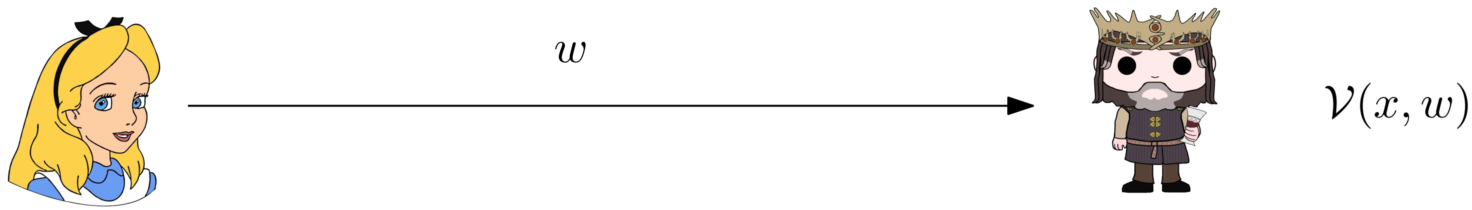

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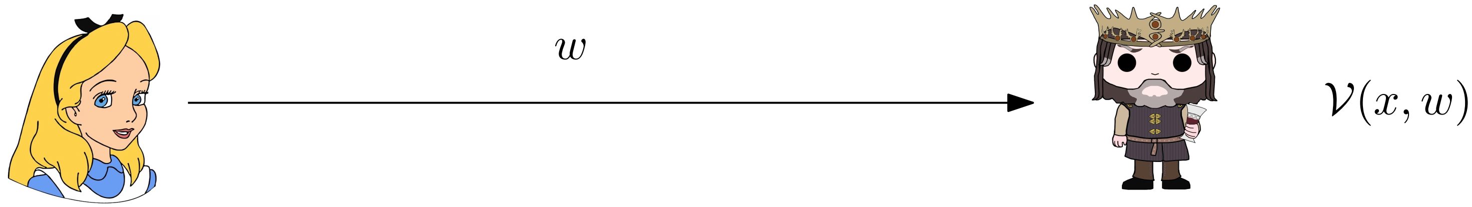

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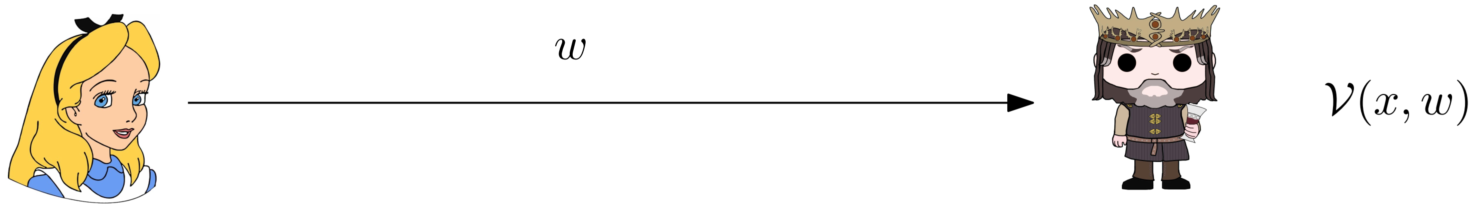

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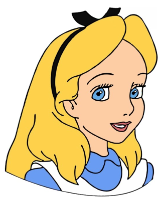

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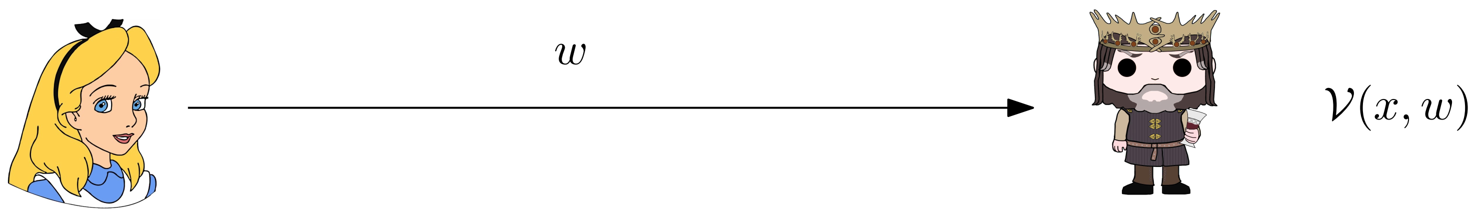

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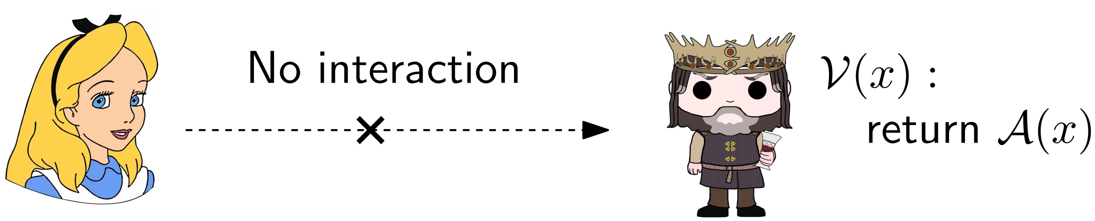

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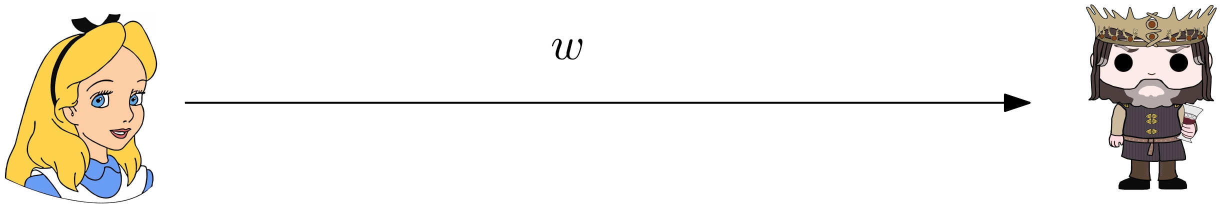

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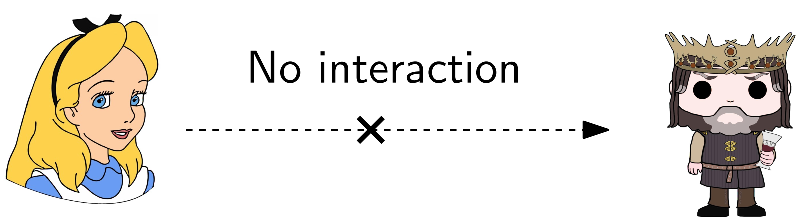

$\mathcal{V}(x)$:
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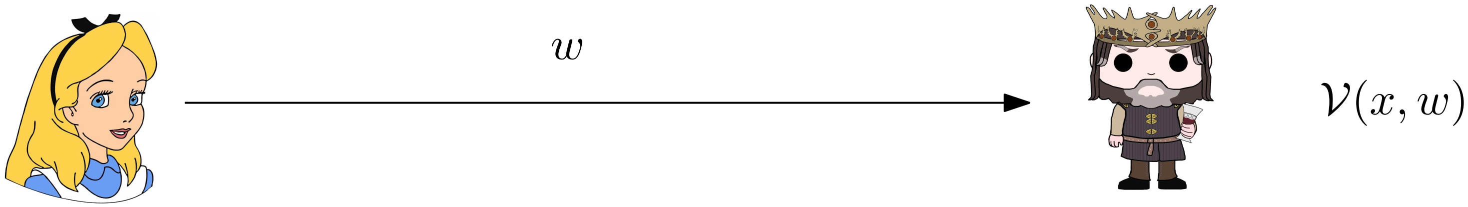

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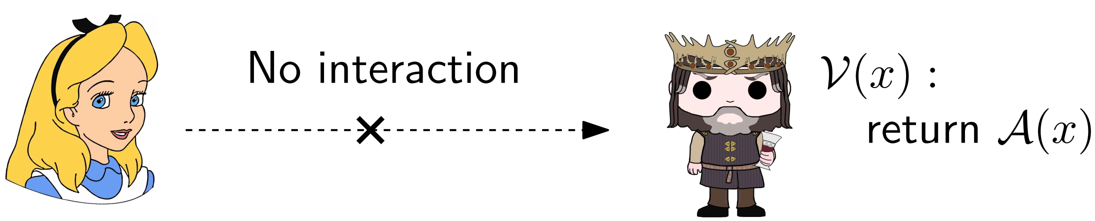

An interactive proof in which the verifier never talks to the prover is **degenerate**

An Interactive Proof for Graph Non-Isomorphism

Let's look at a non-degenerate interactive proof for a problem that is not known to be in $\mathsf{NP} \cup \mathsf{BPP}$

The language L contains all pairs of graphs (G_1, G_2) such that G_1 and G_2 are **not** isomorphic

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Common input: $x = (G_1, G_2)$ where $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and $V = \{1, \ldots, n\}$

- The verifier chooses b u.a.r. in $\{1, 2\}$
- The verifier picks a random permutation $\pi : V \to V$ and sends the graph $G' = \pi(G_b)$ to the prover
- The prover checks whether G' is isomorphic to G_1. If so it replies with $b' = 1$, otherwise it replies with $b' = 2$.
- If $b' = b$, the verifier accepts. Otherwise it rejects

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The verifier accepts with probability:

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An Interactive Proof for Graph Non-Isomorphism

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If $(G_1, G_2) \in L$ then G_b will be isomorphic to exactly one of G_1 and G_2. The (computationally unbounded) prover always guesses correctly

Soundness

Idea: If the input graphs are isomorphic (the prover is cheating), then a random isomorphic copy of one graph will be distributed identically to a random isomorphic copy of the other graph.

> If G_1 and G_2, are isomorphic and π is a random permutation then $\Pr[\pi(G_1)=H]=\Pr[\pi(G_2)=H]$

For any graph G' isomorphic to G_1 (and G_2):

$\Pr[b = 1 \mid \pi(G_b) = G'] \;= \Pr[b = 2 \mid \pi(G_b) = G'] \;= \frac{1}{2}$

Let R be the (possibily randomized) process used by the prover to compute the reply to the verifier

The verifier accepts with probability:

$\Pr[R(G_b) = b] \;= \sum_{G'} \Pr[R(G') = b \mid \pi(G_b) = G'] \cdot \Pr[\pi(G_b) = G'] \;\leq \sum_{G'} \frac{1}{2} \cdot \Pr[\pi(G_b) = G']$

$\Pr[R(G') = b \mid \pi(G_b) = G'] \leq \frac{1}{2}$

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$\Pr[R(G') = b \mid \pi(G_b) = G'] \leq \frac{1}{2}$ **Use probability amplification** □