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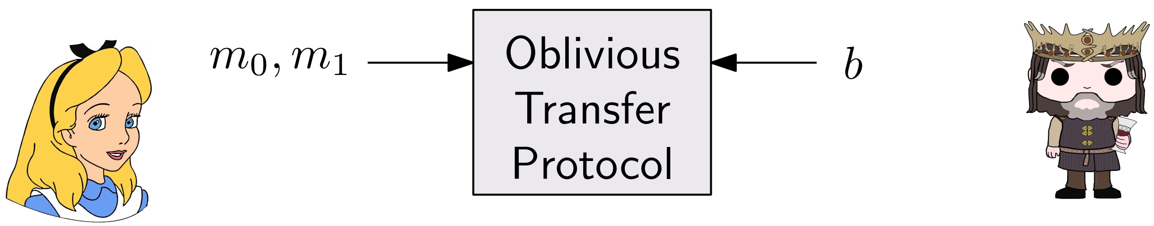

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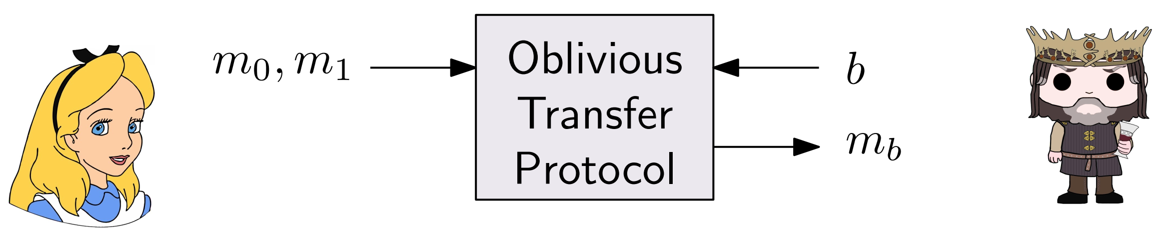

# Reminder: DDH-Based Key Encapsulation Mechanism

Gen($1^n$):

- Run $\mathcal{G}(1^n)$, where $\mathcal{G}$ is a group generation algorithm, to obtain $(G, q, g)$ where $G$ is a group of order $q$ and $g \in G$ is a generator
- Pick some key derivation function $H : G \to \{0,1\}^{\ell(n)}$
- Choose a uniform $x$ u.a.r. from $\{0, \ldots, q-1\}$
- Compute $h = g^x$
- Output $(pk, sk)$ where $pk = (G, q, g, h, H)$ and $sk = (G, q, g, x, H)$.

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Encaps$_{pk}$($1^n$):

- Here $pk = (G, q, g, h, H)$
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We can build a CPA-secure PKE scheme by combining a CPA-secure KEM with an EAV-secure DEM

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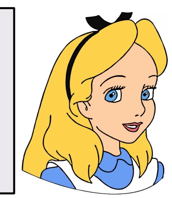
The resulting scheme is as follows:

Gen($1^n$):

- Pick a group $G$, its order $q$, a generator $g \in G$, a key-derivation function $H : G \to \{0,1\}^{\ell(n)}$
  We think of these as fixed public values agreed upon in advance between Alice and Bob.
- Pick a random $x \in G$, the **public-key** is $h = g^x$ and the **secret-key** is $x$

Enc$_h(m)$:

- Choose a uniform $y \in \{0, \ldots, q-1\}$. Return the pair $c = (g^y, H(h^y) \oplus m)$



Dec$_x((c, c'))$:

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**Idea:**

- We pick **two** public keys $h_0, h_1$ for Bob
- We ensure that Bob knows the secret key $x_b$ corresponding to **exactly** one of the public keys (of his choice)
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**How can Bob "prove" to Alice that he knows exactly one private key?**

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Secure under the Random Oracle model and the CDH assumption

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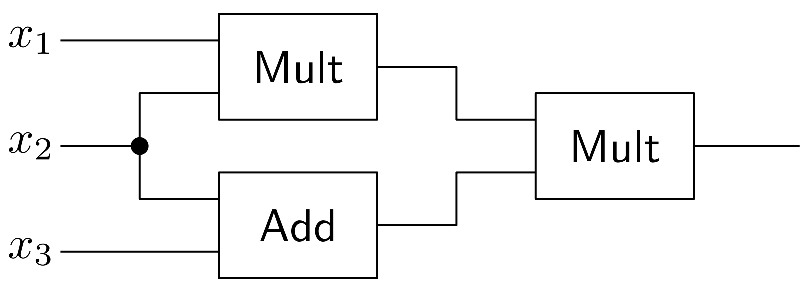

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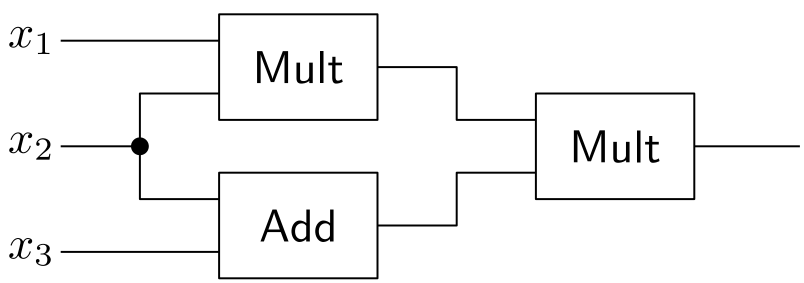


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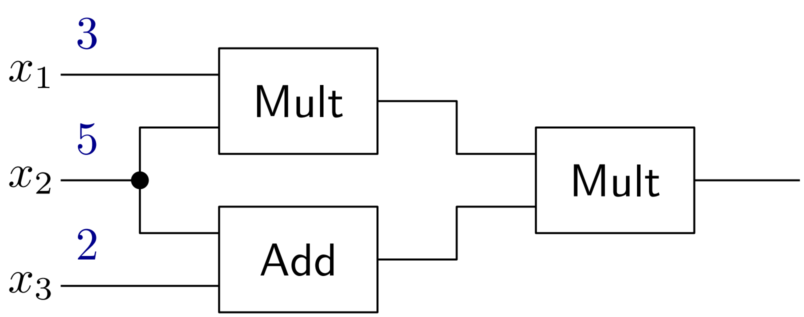


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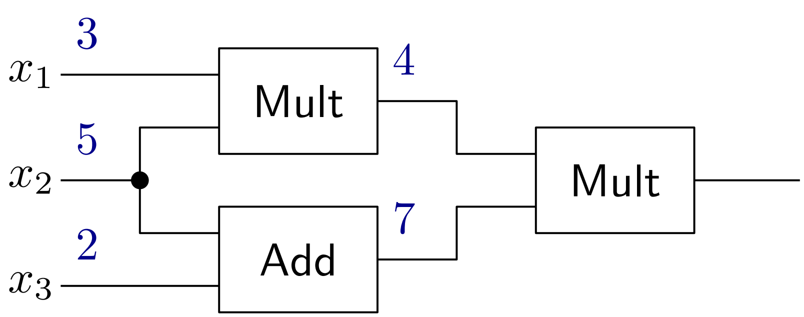


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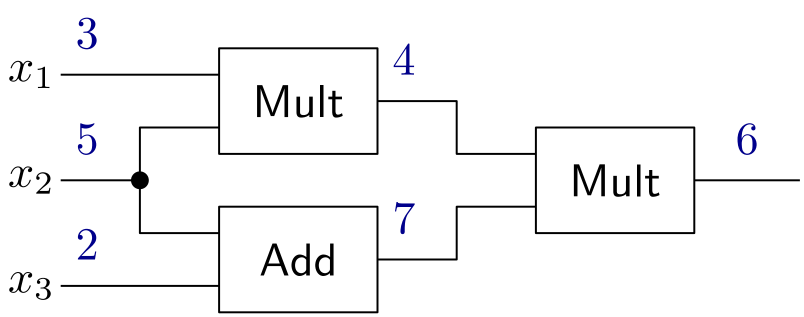


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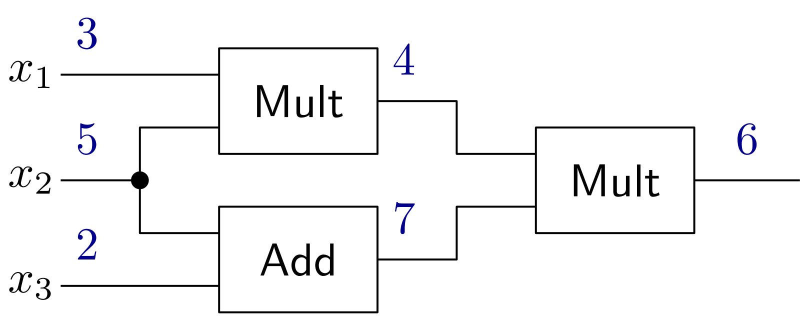


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- Replace each wire of the arithmetic circuit with $\lceil \log p \rceil$ Boolean wires
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# Multiparty Computation

How do we "garble" an arithmetic circuit for multiple parties?

How do we evaluate it?

We can perform computation on shares, without having to recover their secrets first!

**Idea:**

- Do not garble the circuit, use the **homomorphic properties** of Shamir secret sharing instead
- Each party shares its input with all other parties using Shamir's $k$-out-of-$n$ threshold secret sharing scheme
- Each party evaluates the arithmetic circuit: a gate takes a share for each of the two inputs and produces a share of the output
- The output of the circuit is a share of $f(x_1, \ldots, x_n)$
- The parties combine their output shares and recover the value of $f(x_1, \ldots, x_n)$

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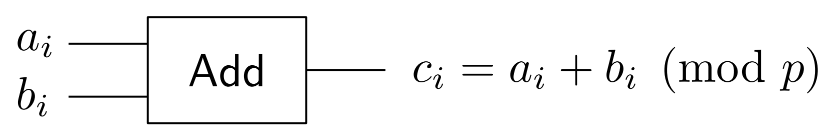


**Addition gates do not require any special care!**

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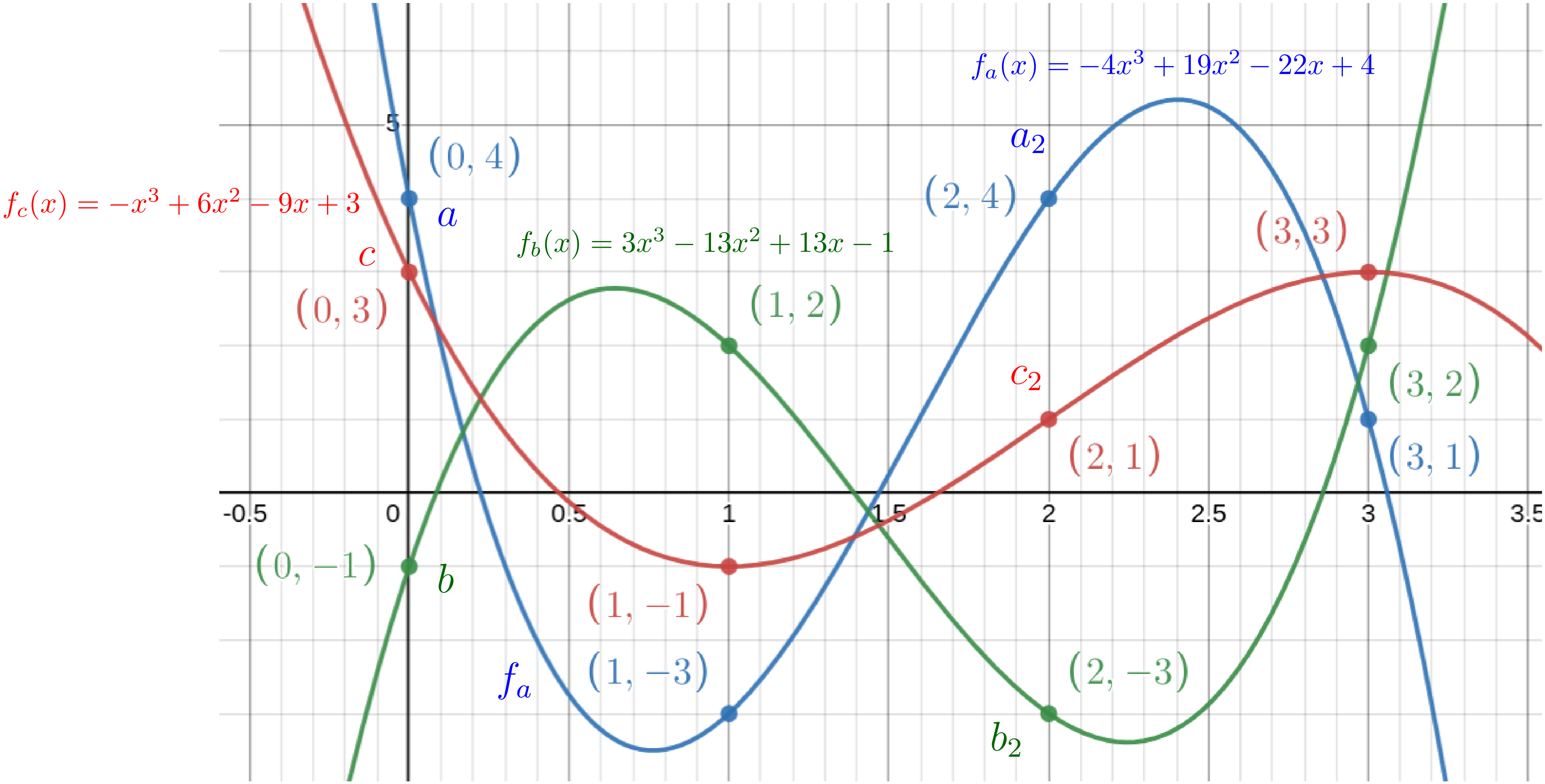

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We need to use another property of interpolating polynomials...

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**Lemma:** Given distinct $x_1, \ldots, x_n$, define $r_j = \prod_{\substack{i=1,\ldots,n \\ i \neq j}} x_i \cdot (x_i - x_j)^{-1}$.

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- $h$ is a polynomial of degree at most $k-1$ s.t. $h(0) = c$ and $c_i$ is exactly the $i$-th share of $h$