Perfect Zero-Knowledge Proof Systems

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Definition: An interactive proof system (P, V) for a language L is **perfect zero-knowledge** if for every probabilistic polynomial-time verifier V^* there exists a probabilistic polynomial-time algorithm M^* such that the following two random variables are identically distributed for every $x \in L$:

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The algorithm M^* is called a **simulator** for the interaction of V^* with P.

Note: the condition must hold **for every** verifier V^*
(even one that deviates from the protocol and tries to trick the prover into leaking information)

The Simulation Paradigm

If anything that the verifier can compute after the interaction can also be computed without the interaction, then the verifier gained no information from the interaction

If M^* is a simulator for V^*, then the execution of an algorithm $A(x, \text{view}_P^{V^*})$ can always be simulated by an algorithm that returns $A(x, M^*(x))$

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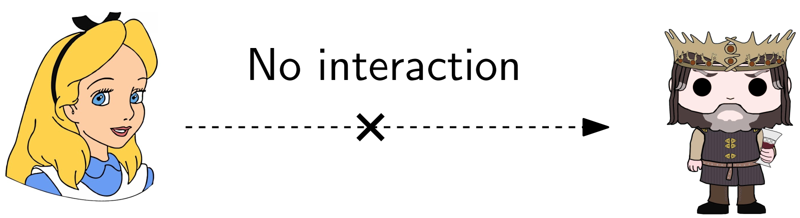
It is trivial to design a perfect zero-knowledge proof system for any language in BPP.



$V(x)$:
  return $\mathcal{A}(x)$

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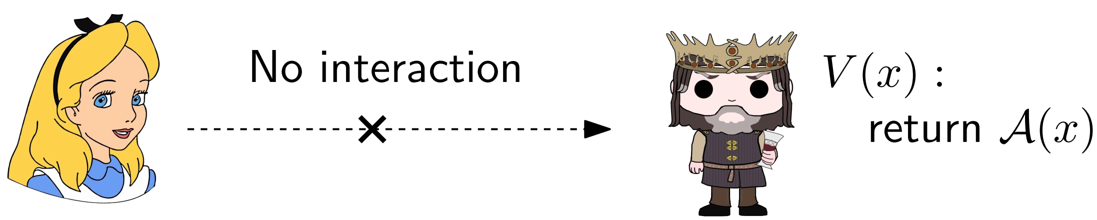

Simulator: Take an interactive protocol (P, V) in which P does not interact with V and, given V^*, define M^* as follows:

- Simulate an execution of $V^*(x)$ and record the value of all random bits used during the execution
- Return the recorded view

Relaxing the Simulator Requirements

We can relax the requirements on the simulator

For every V^*, it suffices to have a probabilistic polynomial-time algorithm $\widetilde{M}^P_{V^*}$ such that:

- $\widetilde{M}^P_{V^*}(x)$ succeeds with probability at least $\frac{1}{p(n)}$ and returns a (simulated) view, or it fails
- Let $m(x)$ be the random variable describing the output of $\widetilde{M}^P_{V^*}(x)$ conditioned on the event that the execution succeeds. The variables $m(x)$ and $\text{view}^P_{V^*}(x)$ are identically distributed

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The existence of $\widetilde{M}^P_{V^*}$ implies the existence of a simulator $M^P_{V^*}$ that runs in expected polynomial-time:

Simulator $M^P_{V^*}(x)$:

- $v \leftarrow \perp$
- While $v \neq \perp$:
  - $v \leftarrow \widetilde{M}^P_{V^*}(x)$
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$$(1 - \frac{1}{p(n)})^{k \cdot p(n)} \leq e^{-k}$$

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Let T be the number of "groups" of $p(n)$ iterations that are (completely or partially) executed

$$\mathbb{E}[T] = \textstyle\sum_{k=1}^{\infty} \Pr[T \geq k] \ \leq \sum_{k=0}^{\infty} e^{-k}$$

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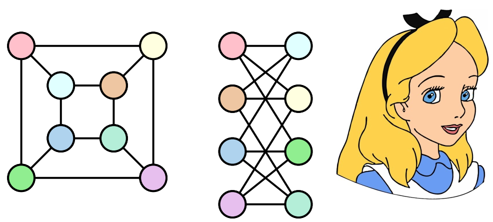
A PZK Proof System for Graph Isomorphism

The language L contains all pairs of graphs (G_1, G_2) such that $G_2 = \phi(G_1)$ for some permutation $\phi : V \to V$

Common input: $x = (G_1, G_2)$ where $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and $V = \{1, \ldots, n\}$



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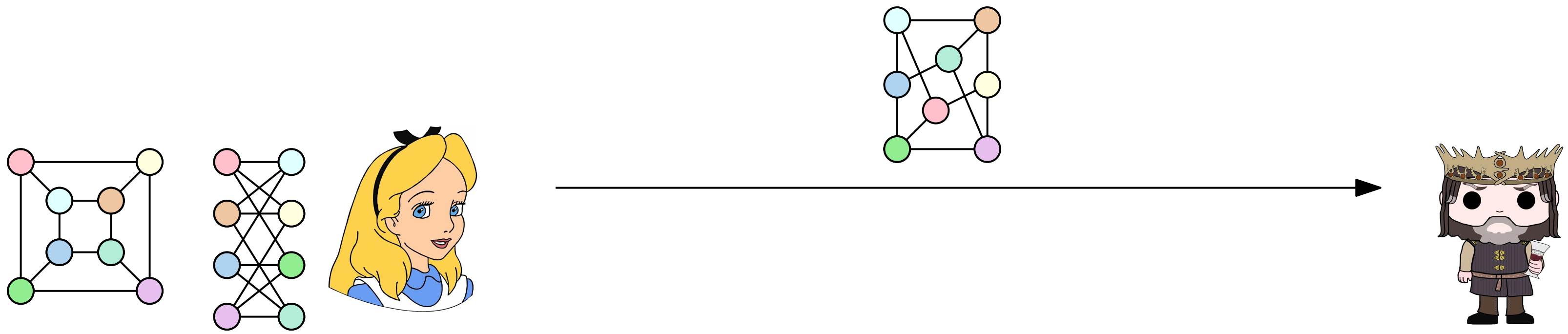

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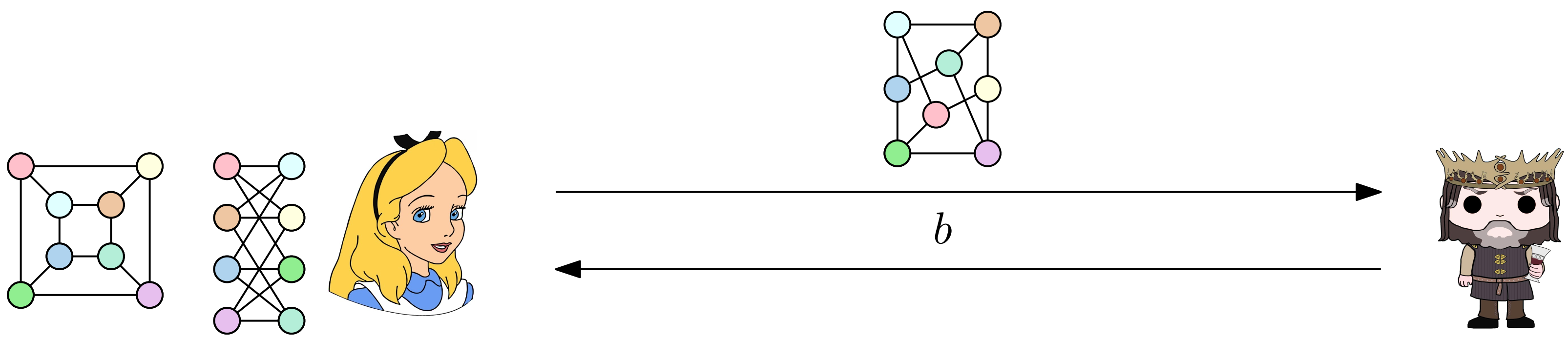

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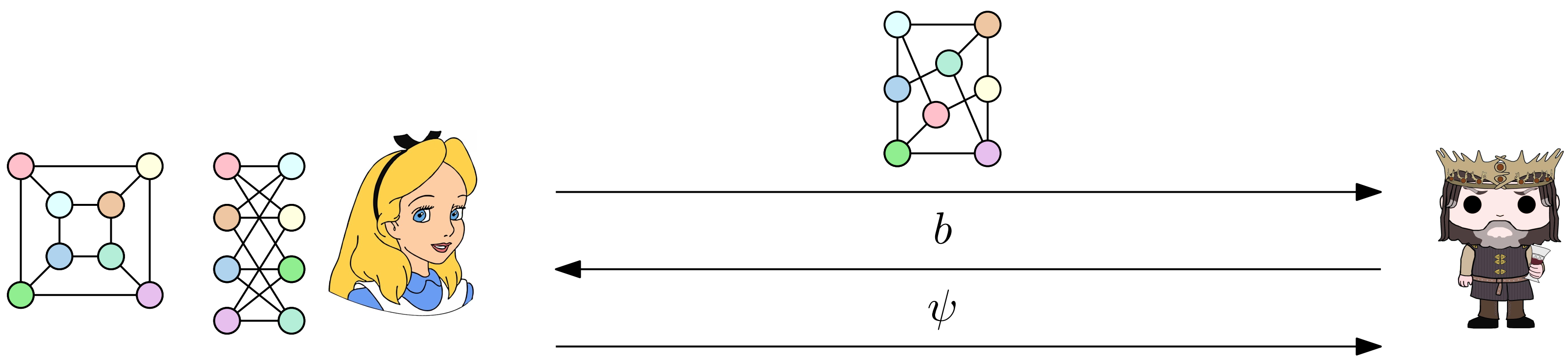
- If $b = 2$, the prover replies with $\psi = \pi$.

  Otherwise the prover replies with $\psi = \pi \circ \phi$ (i.e., $\psi(v) = \pi(\phi(v))$)



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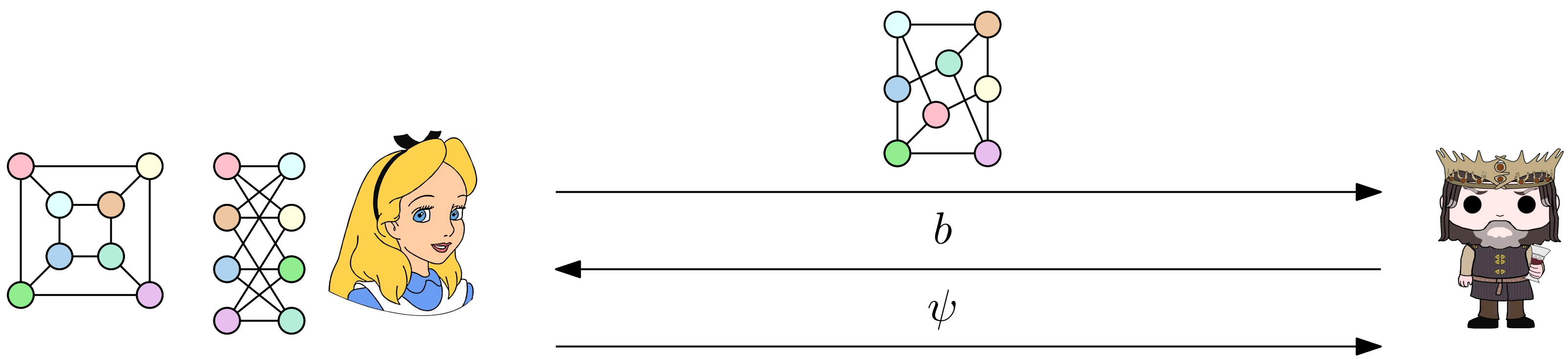
- The verifier accepts iff $\psi(G_b) = G'$



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Completeness: $\Pr[V \text{ accepts}] = 1$

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Soundness:

Consider $(G_1, G_2) \notin L$, i.e., there is no isomorphism between G_1 and G_2

The graph G' sent by the prover is isomorphic to at most one of G_1 and G_2

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Soundness: $\Pr[V \text{ accepts}] \leq \frac{1}{2}$

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Since the verifier chooses b u.a.r. in $\{1, 2\}$, there is a probability of at least $\frac{1}{2}$ that G_b is not isomorphic to G'

In this case, the verifier cannot reply with an isomorphism ψ that satisfies $\psi(G_b) = G'$

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Use probability amplification

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For any $(G_1, G_2) \in L$ and any (possibly cheating) verifier V^*, we need to come up with a simulator $M_{V^*}^P$ for the interaction between P and V^*

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For any $(G_1, G_2) \in L$ and any (possibly cheating) verifier V^*, we need to come up with a simulator $M^P_{V^*}$ for the interaction between P and V^*

It is easy to come up with a simulator for the interaction between P and the honest verifier V

Simulator $M^P_V(x)$:

- Choose $\widetilde{b}$ u.a.r. in $\{1, 2\}$
- Pick a random permutation $\psi : V \to V$
- Pretend to "send" $\psi(G_{\widetilde{b}})$ from the prover to the verifier
- (The verifier "chooses" $b = \widetilde{b}$)
- Pretend to "send" ψ from the prover to the verifier
- The view consists of all the simulated messages sent from the prover to the verifier and of the value b

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How do we handle cheating verifiers?

A PZK Proof System for Graph Isomorphism

Idea: the only thing that a cheating verifier can do is to choose the value of b (as a function of G')

If the chosen value b happens to be $\widetilde{b}$, then the previous strategy works

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- Simulate the verifier $V^*(x)$ and send $G' = \psi(G_{\widetilde{b}})$ to it
- "Receive" the bit b that the simulated verifier sends to the prover

Assume that $b \in \{1, 2\}$
(can be handled)

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Assume that $b \in \{1, 2\}$
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What's the failure probability?

A PZK Proof System for Graph Isomorphism

Since G_1 and G_2 are isomorphic: $\Pr[\psi(G_1) = H] = \Pr[\psi(G_2) = H] = p_H$

Let $R(G')$ be the (possibily randomized) process used by the verifier to compute b from G' (and x)

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$\Pr[M_{V^*}^{P} \text{ fails}] = \Pr[\widetilde{b} = 1 \wedge b = 2] + \Pr[\widetilde{b} = 2 \wedge b = 1]$

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$$\Pr[\widetilde{b} = 1 \wedge b = 2] = \Pr[b = 2 \mid \widetilde{b} = 1] \cdot \Pr[\widetilde{b} = 1] = \Pr[R(\psi(G_1)) = 2] \cdot \frac{1}{2}$$

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$$= \tfrac{1}{2} \sum_H \Pr[R(H) = 2 \wedge \psi(G_1) = H]$$

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If G_1 and G_2 are isomorphic then, after interacting with the prover, the verifier will be convinced that:

- There **exists** an isomorphism ψ between G_1 and G_2 (i.e., there exists a witness for $x \in L$)
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In general, a Zero-Knowledge proof system for a language $L \in$ NP only needs to convince the verifier of the **existence** of a witness

If the proof additionally convinces the verifier that the prover must **know** a witness, then it is called a **zero-knowledge proof-of-knowledge**

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- To obtain a CZK proof system for L, the prover and the verifier can first reduce the instance x of L to an instance $x' = f(x)$ of G3C and then use the CZK proof for G3C

Computational Zero-Knowledge

Idea: If we consider a computation to be feasible if it can be performed in polynomial time, then we do not need to perfectly simulate the view of V^*

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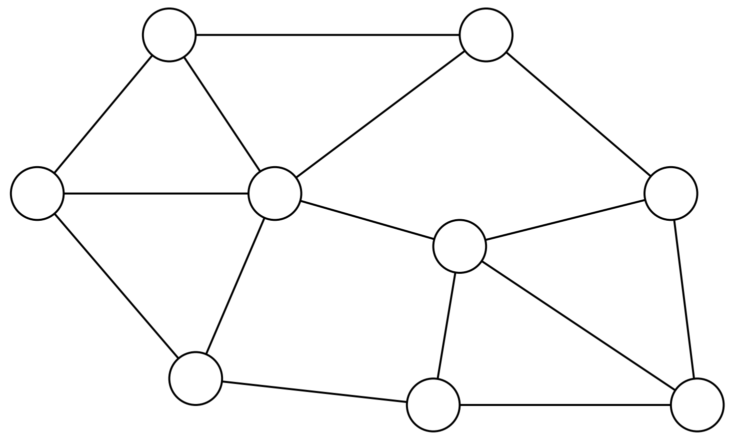
We define CZK (resp. PZK) as the class of all languages that admit a computational (resp. perfect) zero-knowledge proof system

$$\text{BPP} \subseteq \text{PZK} \subseteq \text{CZK} \subseteq \text{IP}$$

Graph 3-Coloring

Let $G = (V, E)$ be a graph



A k-coloring of G is a function $c : V \rightarrow \{1, 2, \ldots, k\}$ such that each edge $(u, v) \in E$ has endpoints of the different "colors" (i.e., $c(u) \neq c(v)$)

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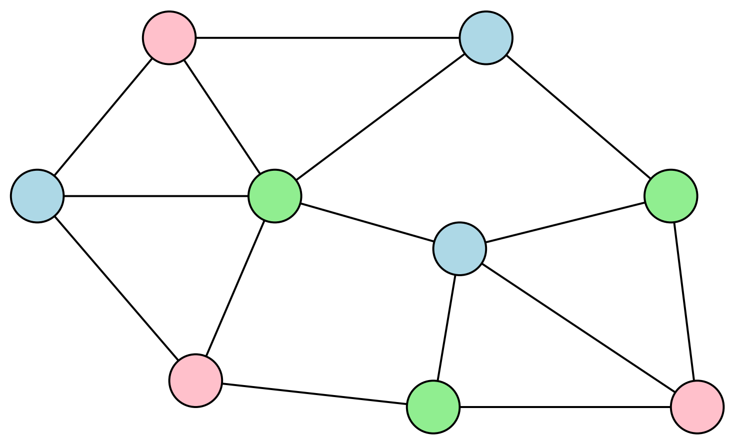

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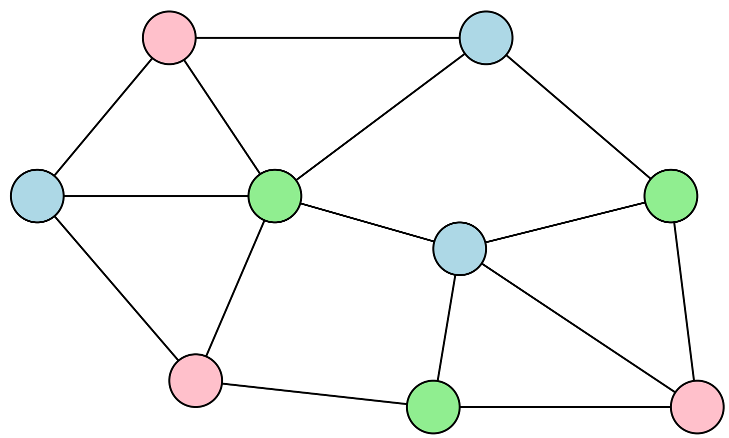

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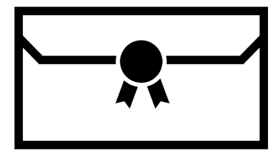
G3C is the set of all graphs G that admit a 3-coloring

Commitment schemes (Informal)

A commitment scheme is an interactive protocol that works in two phases, called **commit** and **reveal**, and allows a party, called **sender**, to:



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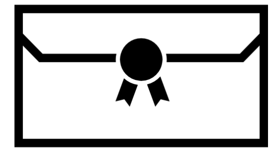

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Even if the receiver cheats by deviating from the protocol!

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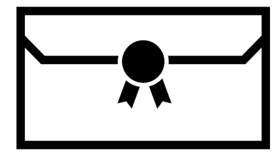

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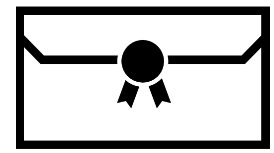

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- **Viability:** if both parties follow the protocol, then after the reveal phase, the receiver learns m

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  Let $(r, \overline{m})$ be the view of R, where r are the random coins used by R and $\overline{m}$ is a transcript of the messages received from the sender.

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  $(r, \overline{m})$ is **ambiguous** if its **both** a possible 0-commitment and a possible 1-commitment

  We require that for all but a negligible fraction of the strings $r \in \{0, 1\}^{\mathsf{poly}(n)}$, **there exists no $\overline{m}$** such that $(r, \overline{m})$ is ambiguous

Commitment schemes

The viability requirement is implicitly enforced by the formalization of the **binding** condition

There is a canonical way to **open** the commitment:

- The sender sends to the receiver the secret value b and sequence s of the random bits it used in the commitment phase
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- **Perfectly binding:** except that for a negligible probability, the sender is bound to the committed value (regardless of its computational power)
- **Computationally hiding:** the committed value is hidden from a computationally bounded receiver. (A computationally unbounded receiver can bruteforce s)

(the commitment scheme based on hash functions does not suit our needs since it is not perfectly binding)

Commitment schemes from PRGs

We can build a commitment scheme from a PRG $G : \{0,1\}^n \to \{0,1\}^{3n}$

Commit phase:

- The receiver chooses r u.a.r. from $\{0,1\}^{3n}$ and sends it to the sender
- The sender commits to b by selecting s u.a.r. from $\{0,1\}^n$ and sending $\alpha = G(s)$ if $b = 0$ and $\alpha = G(s) \oplus r$ otherwise

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(Canonical) Reveal Phase:

- The sender reveals b and s
- The receiver accepts iff $b = 0$ and $G(s) = \alpha$, or $b = 1$ and $G(s) \oplus r = \alpha$

Commitment schemes from PRGs

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Hiding:

Let U be a uniform random variable over $\{0,1\}^{3n}$ and let $\approx$ denote computational indistinguishability

For any $r \in \{0,1\}^{3n}$:

$$\underbrace{G(s) \approx U}_{G \text{ is a PRG}} \equiv \overbrace{U \oplus r}^{U \text{ is uniform}} \approx \underbrace{U \oplus r \approx G(s) \oplus r}_{G \text{ is a PRG}}$$

Therefore, $G(s)$ and $G(s) \oplus r$ are computationally indistinguishable

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$\implies$ For any polynomial-time algorithm R^*, the output distributions of $R^*(1^n, G(s))$ and $R^*(1^n, G(s) \oplus r)$ are computationally indistinguishable

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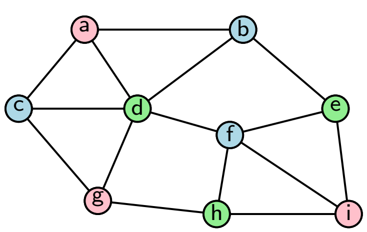
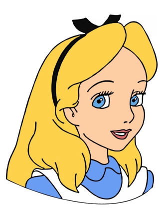
- The fraction of strings r that can result in ambiguous (r, α) is at most $\frac{2^{2n}}{2^{3n}} = \frac{1}{2^n}$, which is negligible

A CZK Proof System for Graph 3-Coloring

Common input: $G = (V, E)$

The prover knows a 3-coloring c of G



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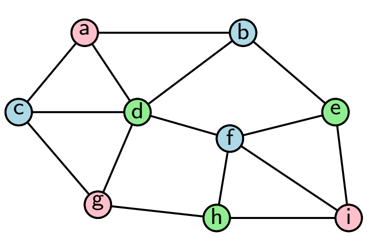

A commitment to a color for $c'(z)$ every vertex z →

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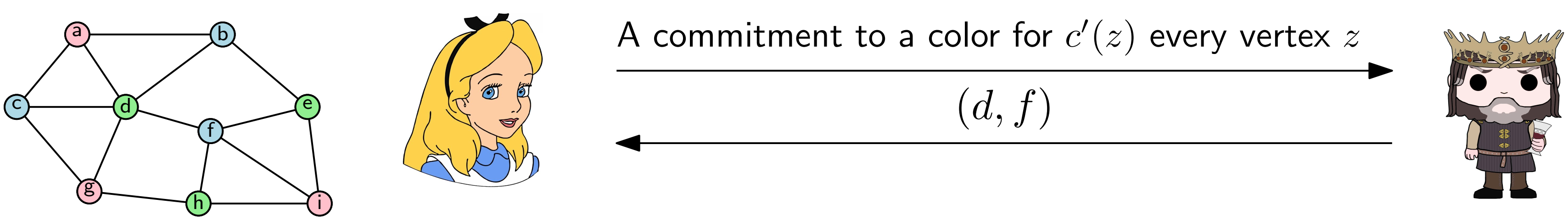

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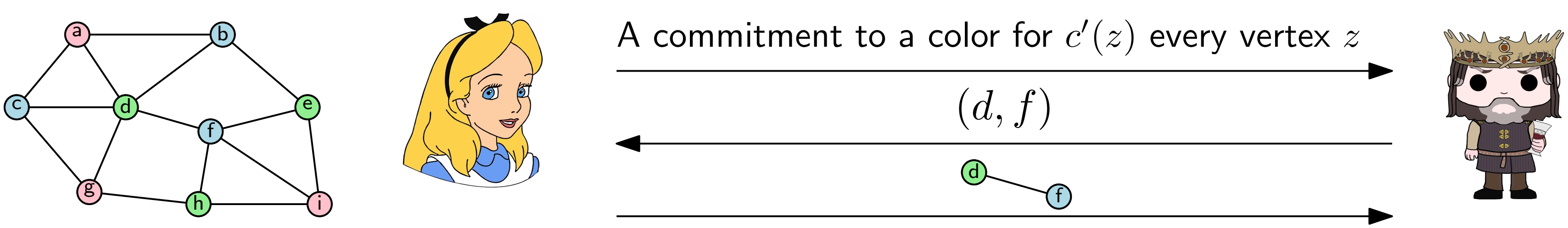

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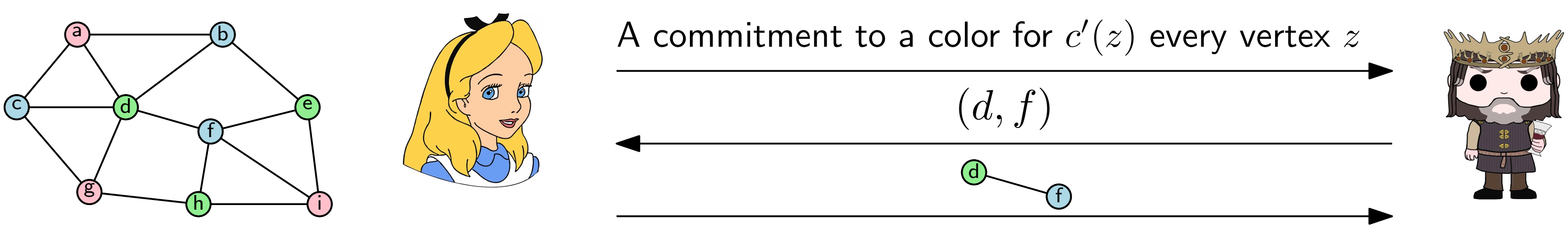

A CZK Proof System for Graph 3-Coloring

Completeness:

- When c is a 3 coloring of G, $c(u^*) \neq c(v^*) \implies c'(u^*) \neq c'(v^*)$ and the verifier accepts

$$\Pr[\text{the verifier accepts}] = 1$$

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- Consider a (possibly cheating) prover B and let c'' be the coloring resulting from its commitments
- Since G is not 3-colorable, there is at least one edge $(u, v) \in E$ such that $c''(u) = c''(v)$
- With probability at least $\frac{1}{|E|}$, the verifier picks $(u^*, v^*) = (u, v)$

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Use probability amplification

A CZK Proof System for Graph 3-Coloring

Zero-Knowledge:

It is easy to come up with a simulator for the interaction between P and the honest verifier V

Simulator $M_V^P(x)$**:**

- Choose (u^*, v^*) u.a.r. in E
- Choose two random distinct colors $c'(u^*), c'(v^*) \in \{1, 2, 3\}$ and a color $c'(z) \in \{1, 2, 3\}$ for each $z \in V \setminus \{u^*, v^*\}$ independently and u.a.r.

Note: c' may not be a valid coloring.

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A CZK Proof System for Graph 3-Coloring

Zero-Knowledge: **How do we handle a cheating verifier V^*?**

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Probability of failure? At most $1 - \frac{1}{|E|}$ Run the simulator until it succeeds. Expected polynomial time

Zero-Knowledge Proof Systems for Languages not in NP?

We have shown that, **if PRGs exists**, then any language in NP admits a computational zero-knowledge proof system

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This is not a zero-knowledge protocol.

Why?

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We have shown that, **if PRGs exists**, then any language in NP admits a computational zero-knowledge proof system

$$\mathsf{NP} \subseteq \mathsf{CZK}$$

What about languages outside NP**?**

An example language that is in IP but might not be in NP is graph non-isomorphism

Can we transform the previous interactive proof system into a zero-knowledge proof system?

Common input: $x = (G_1, G_2)$ where $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and $V = \{1, \ldots, n\}$

- The verifier chooses b u.a.r. in $\{1, 2\}$
- The verifier picks a random permutation $\pi : V \to V$ and sends $G' = \pi(G_b)$ to the prover
- The prover checks whether G' is isomorphic to G_1. If so it replies with $b' = 1$, otherwise it replies with $b' = 2$.
- If $b' = b$, the verifier accepts. Otherwise it rejects

This is not a zero-knowledge protocol. Why?

A cheating verifier interacting with an honest prover can learn whether an arbitrary graph G' is isomorphic to G_1

Zero-Knowledge Proof Systems for Languages not in NP?

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It is easy to come up with a simulator for a verifier that always picks a graph G' that is isomorphic to either G_1 or G_2, **and knows the correct answer**

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Idea: If only the verifier could **convince the prover** that he knows to which graph G' is isomorphic to (without leaking the answer)...

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This can be done with a perfect zero-knowledge proof-of-knowledge!

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  The verifier convices the prover that he knows a b such that G' is isomorphic to G_b
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This implies that CZK might be larger than NP How big is CZK?

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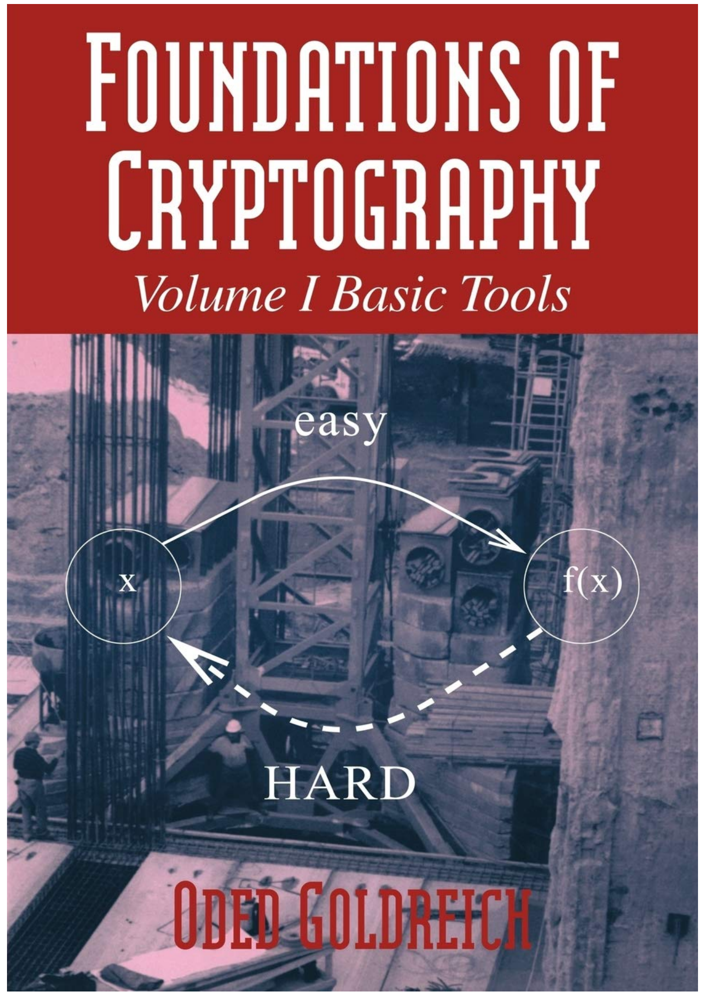
If PRGs exist:

- CZK = PSPACE
- It turns out that IP = PSPACE $\implies$ any language that admits an interactive proof system also admits a (computational) zero-knowledge proof system

References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