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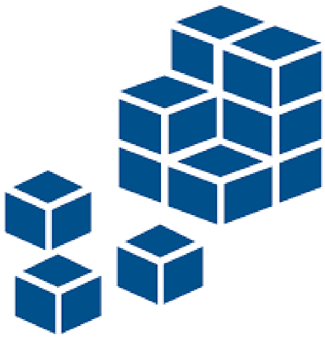
The historic ciphers from the previous lectures are intuitively "insecure". Can we prove that formally?

When is an encryption scheme secure?

Another benefit of formal definitions is *modularity*:

- A designer can replace an encryption scheme with another (that satisfies the same security definition)
- The security of the overall application is unaffected



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Threat models

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- A threat model only specifies **what** the abilities of the adversary are

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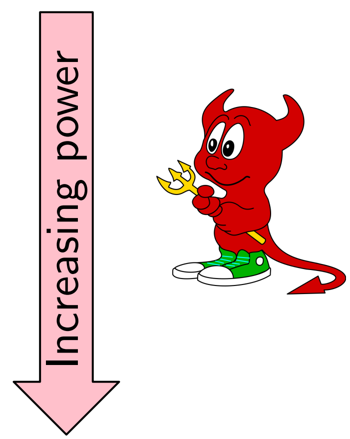

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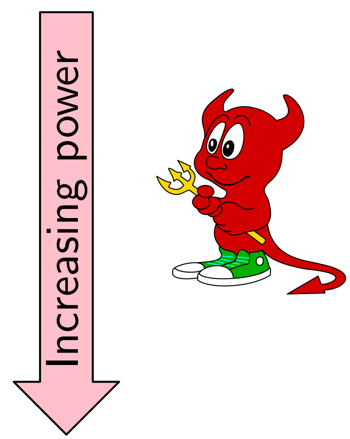

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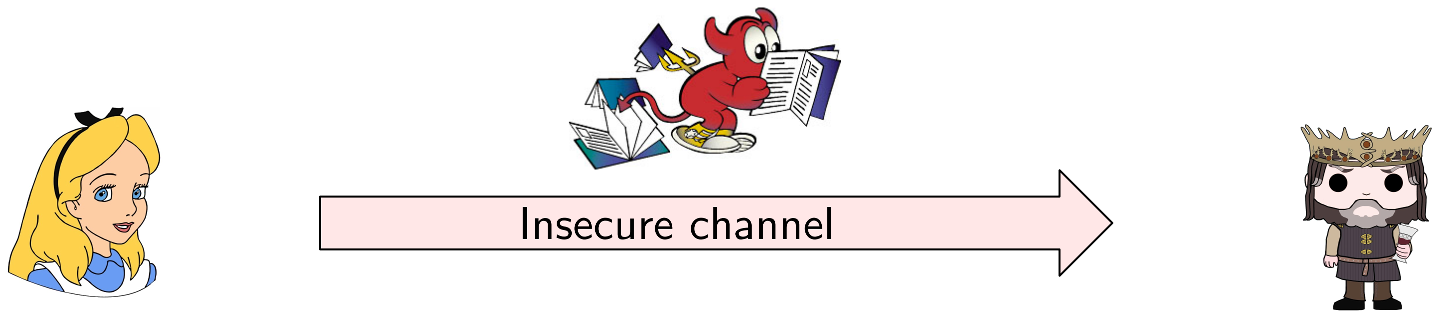

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- It observes a ciphertext (or multiple ciphertexts) and attempts to determine information about the underlying plaintext (or plaintexts).

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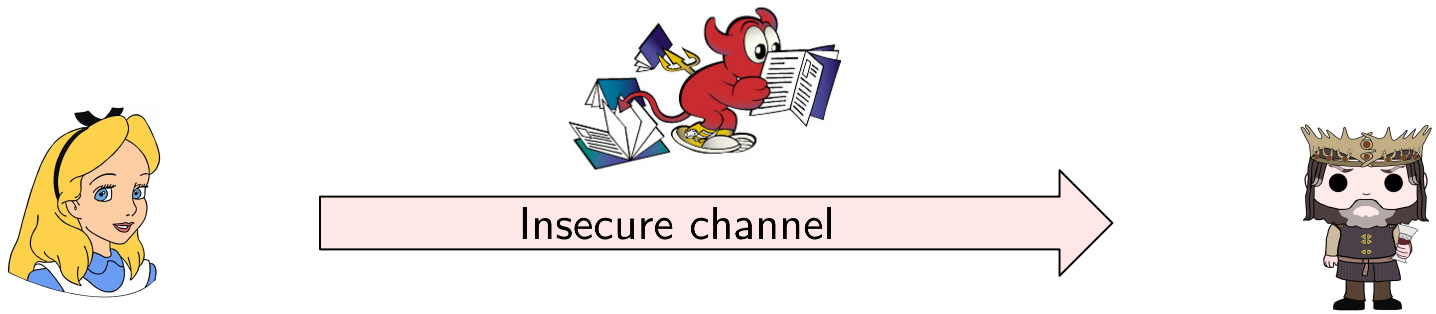

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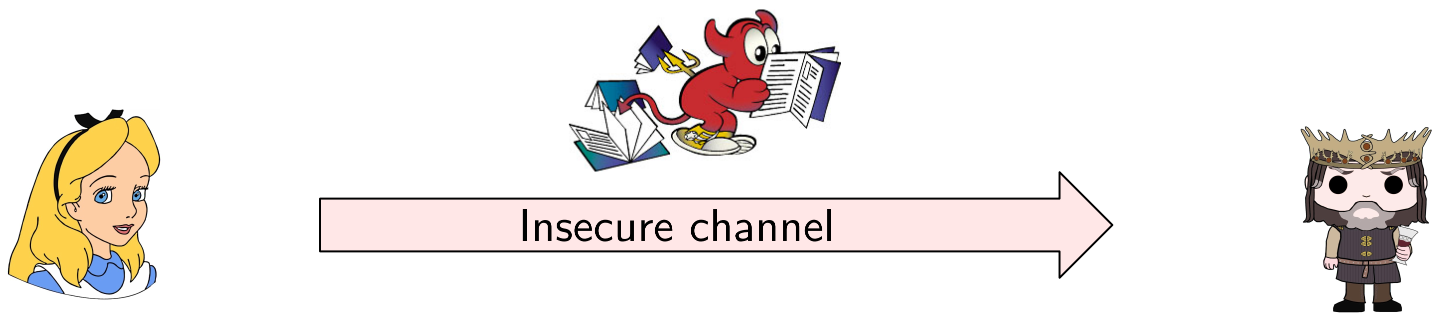

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Most basic type of attack (weakest threat model)

It is the attack type that we have been implicitly considering in our discussion about historic ciphers

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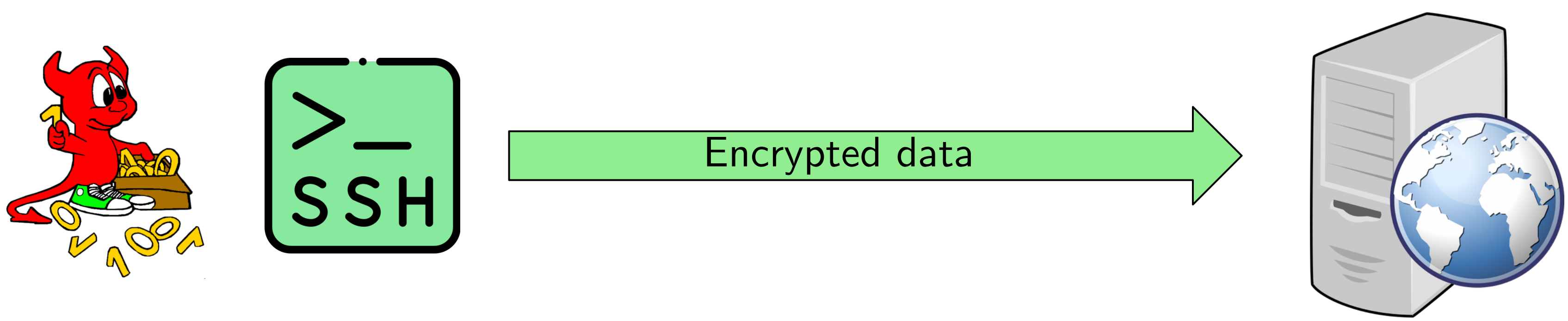

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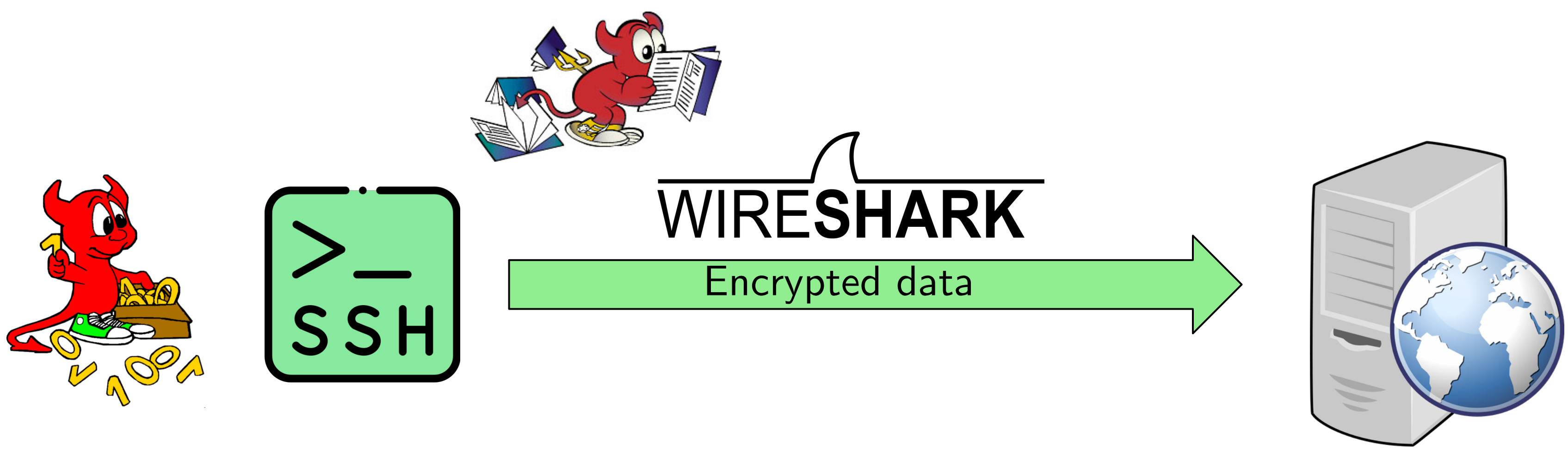

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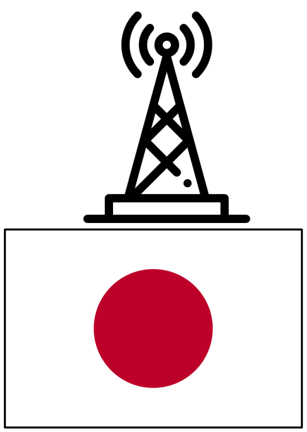

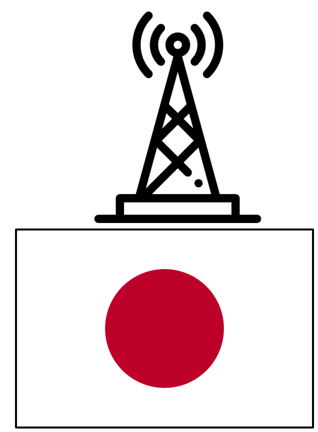

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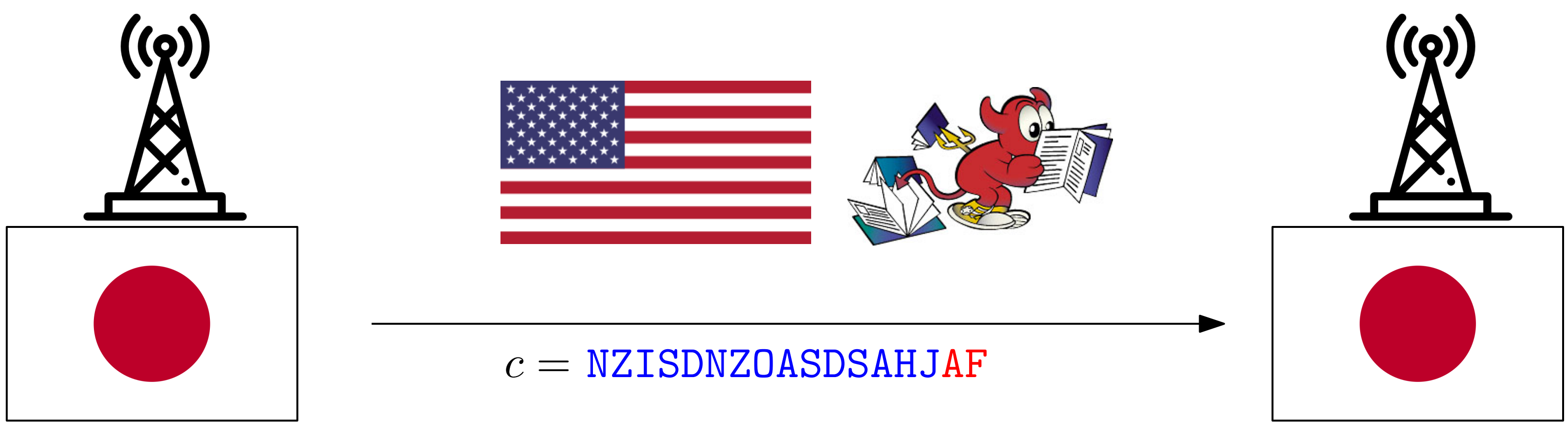

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The U.S. cryptanalysts believed that AF meant Midway Island, but they were not 100% sure

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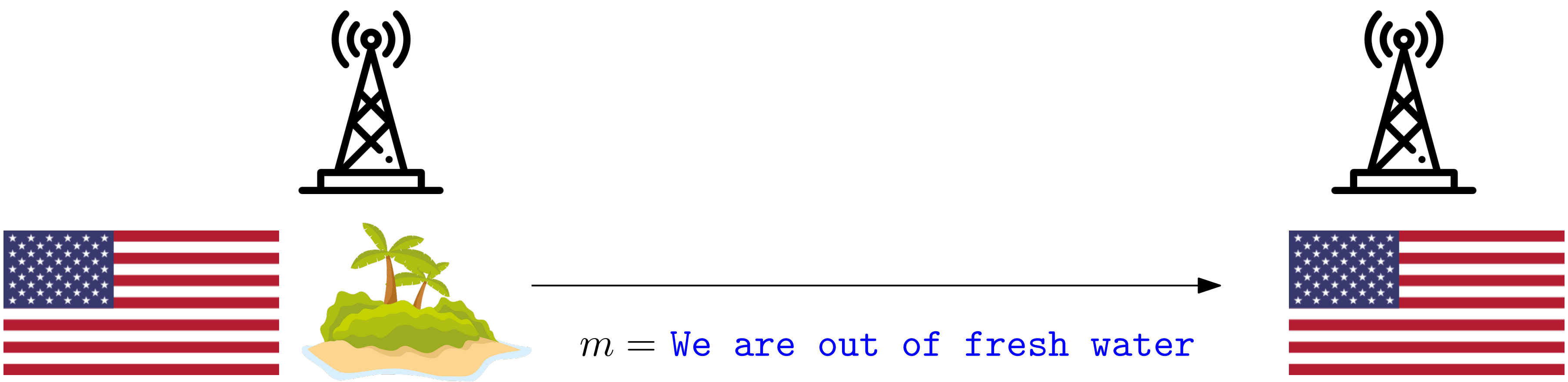

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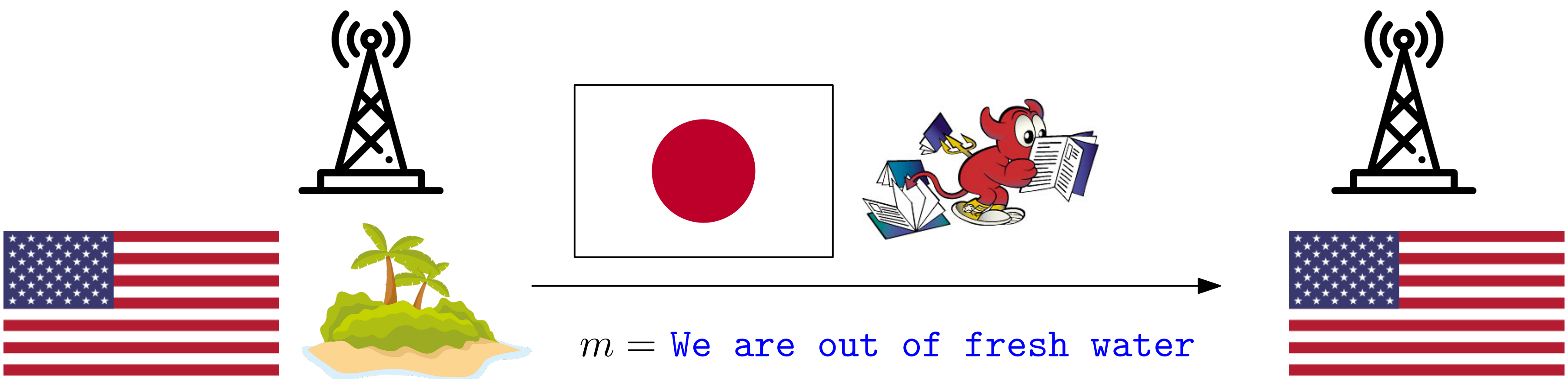

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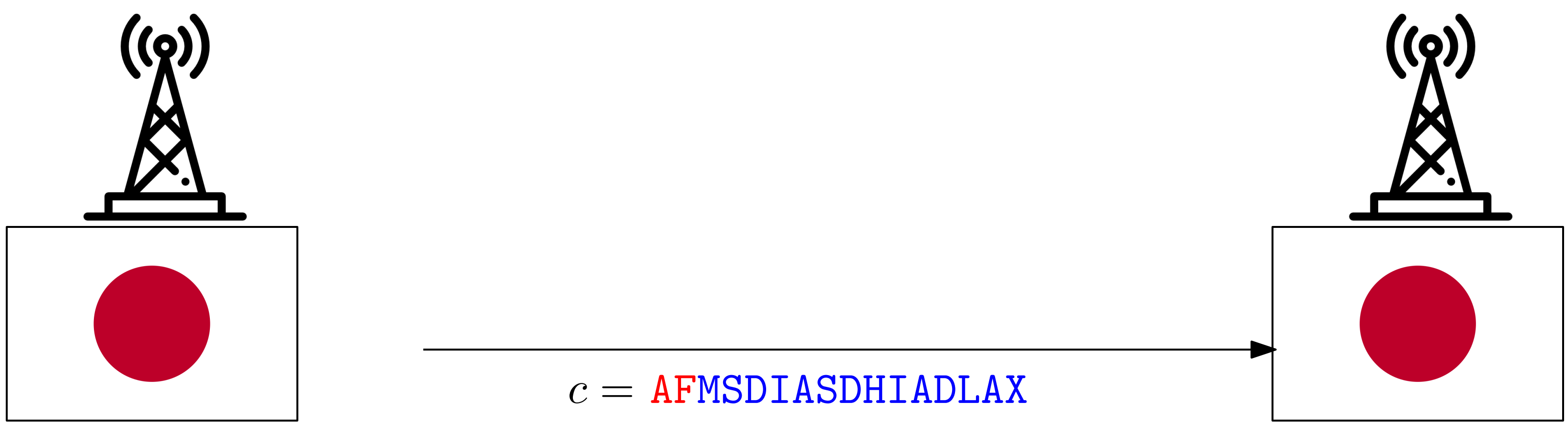

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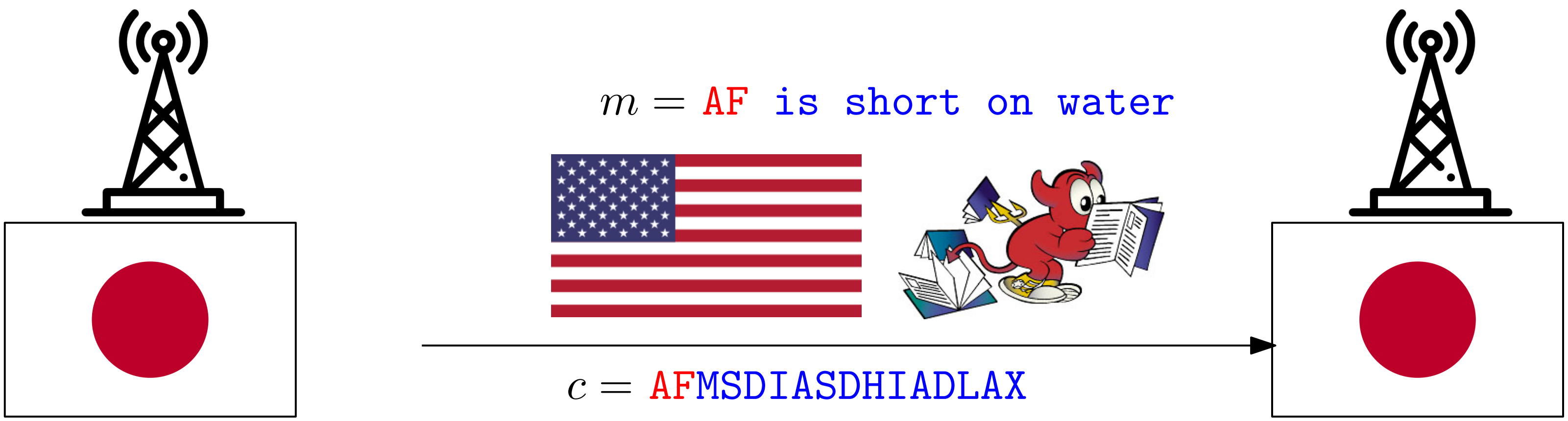

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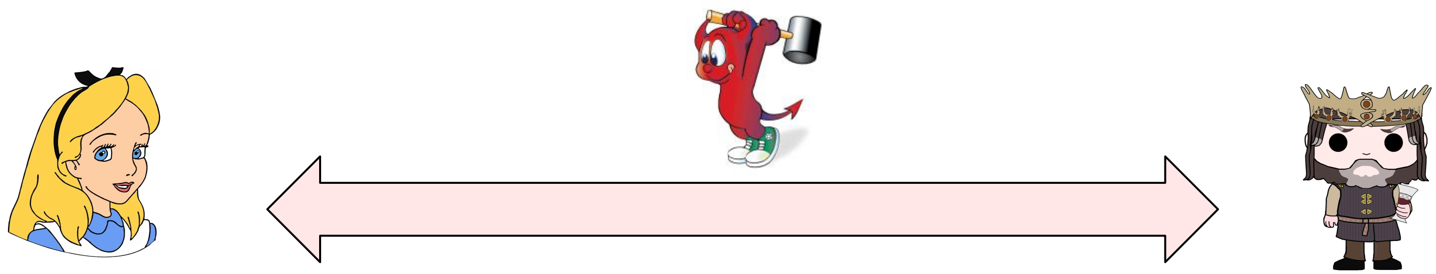

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Many protocols close a connection or request a retransmission when a bad message is received

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Being able to know whether a ciphertext is valid enables "Padding oracle" attacks:

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What about the following private-key encryption scheme?

- Gen returns a random key
- $\mathsf{Enc}_k(m) = m$
- $\mathsf{Dec}_k(c) = c$

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- $\mathsf{Enc}_k(m) = \begin{cases} \texttt{A} \| f_k(m) & \text{if } m \geq 100 \\ \texttt{B} \| f_k(m) & \text{if } m < 100 \end{cases}$, for some $f_k(\cdot)$?

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What about $f(m) = 42$?

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Is it a "good" definition? Maybe...

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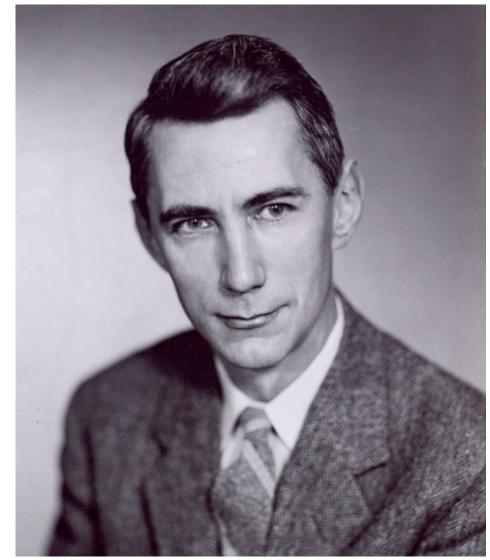
Shannon's Treatment

Messages come from a probability distribution over the message space $\mathcal{M}$



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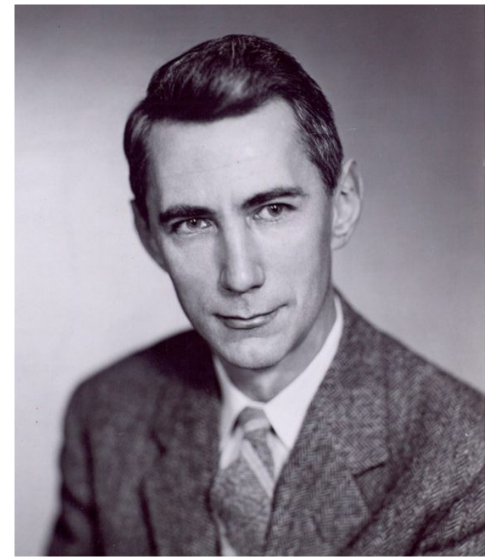

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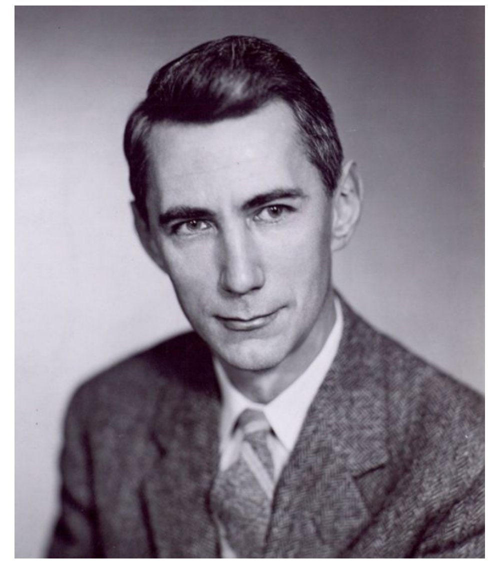

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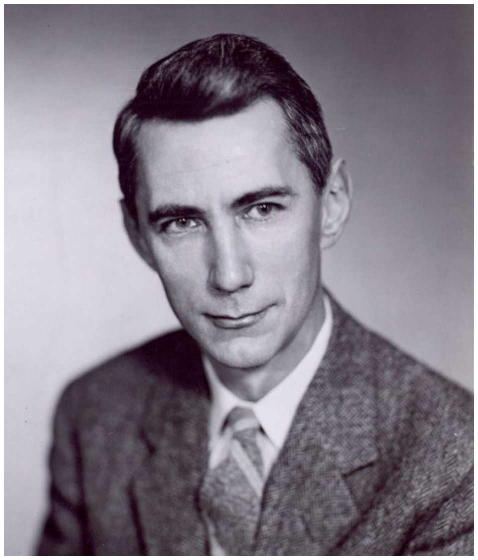

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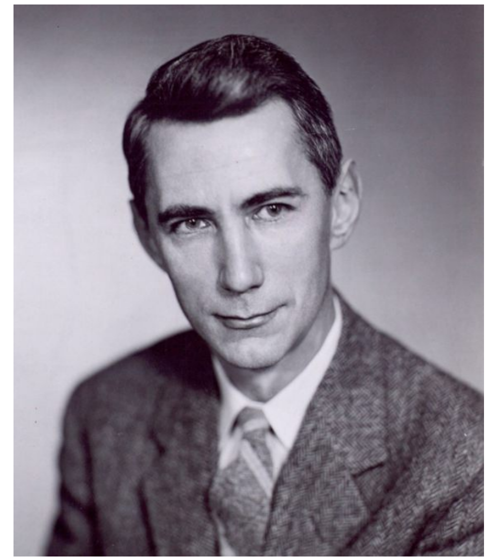

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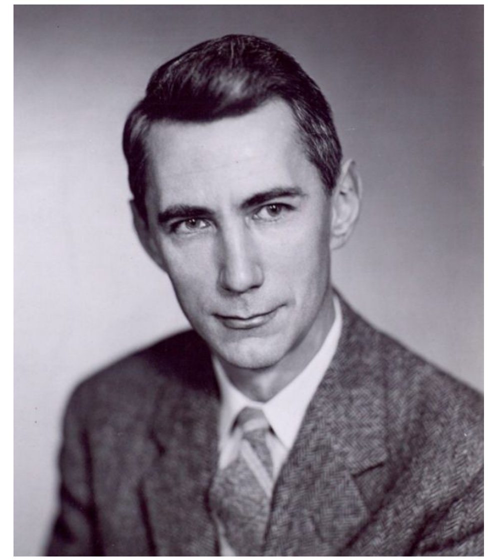

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C is a random variable (over $\mathcal{C}$) denoting the resulting ciphertext.

Example 0

The adversary knows that the message is going to be either `ATTACK` or `RETREAT`

Moreover, he believes that the probability of `ATTACK` is 70%

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a priori probability

Gen outputs a binary string of length 3 chosen uniformly at random (u.a.r.):

$\Pr[K = \texttt{011}] = \frac{1}{8}$

Example 1

Consider a shift cipher:

$\mathcal{M} = \{\texttt{a}, \ldots, \texttt{z}\}^*$ $\mathcal{C} = \{\texttt{A}, \ldots, \texttt{Z}\}^*$ $\mathcal{K} = \{0, \ldots, 25\}$

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Lower-case for plaintexts $\qquad$ Upper-case for ciphertexts

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$\Pr[C = \texttt{B}]$

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$$\Pr[C = \mathtt{B}] \quad = \sum_{m \in \mathcal{M}} \Pr[C = \mathtt{B} \wedge M = m]$$

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&= \Pr[C = \mathtt{B} \mid M = \mathtt{a}] \cdot \Pr[M = \mathtt{a}] \ + \ \Pr[C = \mathtt{B} \mid M = \mathtt{b}] \cdot \Pr[M = \mathtt{b}] \\
&= \Pr[K = 1] \cdot \Pr[M = \mathtt{a}] \ + \ \Pr[K = 0] \cdot \Pr[M = \mathtt{b}]
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&= \Pr[K = 1] \cdot \Pr[M = \mathtt{a}] \ + \ \Pr[K = 0] \cdot \Pr[M = \mathtt{b}] \ = \tfrac{1}{26} \cdot \tfrac{7}{10} \ + \ \tfrac{1}{26} \cdot \tfrac{3}{10}
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$\Pr[M = \mathtt{a} \mid C = \mathtt{B}]$

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$$\Pr[M = \texttt{a} \mid C = \texttt{B}] = \Pr[C = \texttt{B} \mid M = \texttt{a}] \cdot \frac{\Pr[M=\texttt{a}]}{\Pr[C=\texttt{B}]}$$

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a posteriori probability

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The adversary has the following a priori distribution over $\mathcal{M}$:

$\Pr[M = \mathtt{kim}] = 0.5$ $\quad$ $\Pr[M = \mathtt{ann}] = 0.2$ $\quad$ $\Pr[M = \mathtt{boo}] = 0.3$

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$$\begin{aligned}\Pr[C = \texttt{DQQ}] = \quad & \Pr[C = \texttt{DQQ} \mid M = \texttt{kim}] \Pr[M = \texttt{kim}] \\ & + \Pr[C = \texttt{DQQ} \mid M = \texttt{ann}] \Pr[M = \texttt{ann}] \\ & + \Pr[C = \texttt{DQQ} \mid M = \texttt{boo}] \Pr[M = \texttt{boo}]\end{aligned}$$

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\Pr[C = \mathtt{DQQ}] = & \Pr[C = \mathtt{DQQ} \mid M = \mathtt{ann}] \Pr[M = \mathtt{ann}] + \Pr[C = \mathtt{DQQ} \mid M = \mathtt{boo}] \Pr[M = \mathtt{boo}] \\
= & \Pr[K = 3] \cdot 0.2 + \Pr[K = 2] \cdot 0.3
\end{aligned}$$

Example 2

Consider a shift cipher:

$\mathcal{M} = \{\texttt{a}, \ldots, \texttt{z}\}^*$ $\qquad$ $\mathcal{C} = \{\texttt{A}, \ldots, \texttt{Z}\}^*$ $\qquad$ $\mathcal{K} = \{0, \ldots, 25\}$

K is distributed uniformly over $\mathcal{K}$

The adversary has the following a priori distribution over $\mathcal{M}$:

$$\Pr[M = \texttt{kim}] = 0.5 \qquad \Pr[M = \texttt{ann}] = 0.2 \qquad \Pr[M = \texttt{boo}] = 0.3$$

What is the probability that the ciphertext is DQQ?

$$\begin{aligned}
\Pr[C = \texttt{DQQ}] &= \Pr[C = \texttt{DQQ} \mid M = \texttt{ann}] \Pr[M = \texttt{ann}] + \Pr[C = \texttt{DQQ} \mid M = \texttt{boo}] \Pr[M = \texttt{boo}] \\
&= \Pr[K = 3] \cdot 0.2 + \Pr[K = 2] \cdot 0.3 \\
&= \tfrac{1}{26} \cdot 0.2 + \tfrac{1}{26} \cdot 0.3 = \tfrac{1}{52}
\end{aligned}$$

Perfect secrecy

Candidate definition 5 (inf.): *Regardless of any information an attacker already has, a ciphertext should leak <u>no additional information</u> about the underlying plaintext.*

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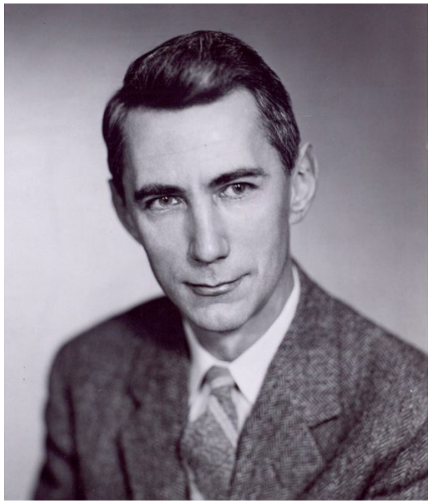

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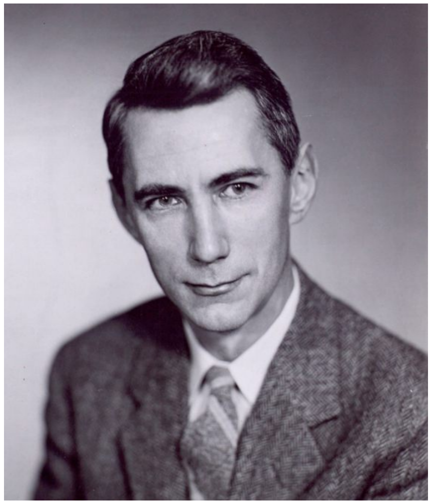

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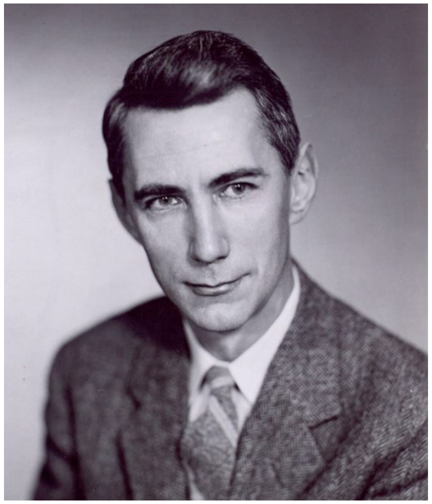

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A posteriori probability

The knowledge the adversary has about m after observing c

All the a priori information known by the adversary about the plaintexts

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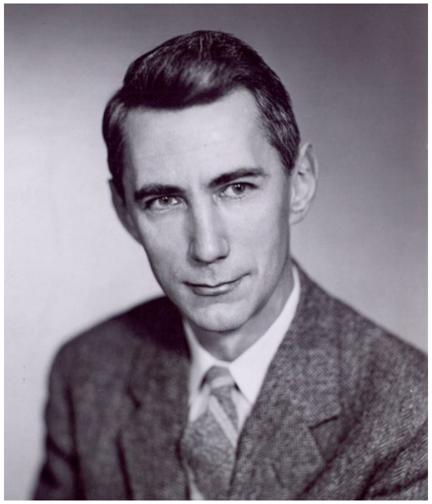

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$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

The adversary learns nothing **new**

Example

Are shift ciphers perfectly secret?

Our intuition says "no"... can we prove that formally?

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Idea: Two occurrences of the same characters in the plaintext must produce the same characters in the ciphertext

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This is a valid choice since:

$$\begin{aligned}\Pr[C = \mathtt{XX}] &\geq \Pr[C = \mathtt{XX} \wedge M = \mathtt{aa}] \\ &= \Pr[C = \mathtt{XX} \mid M = \mathtt{aa}] \Pr[M = \mathtt{aa}] \\ &= \Pr[K = 23] \Pr[M = \mathtt{aa}] > 0\end{aligned}$$

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$$\Pr[M = \texttt{ab} \mid C = \texttt{XX}] \qquad\qquad \Pr[M = \texttt{ab}]$$

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$\square$

Another definition

What about the following definition of *perfect secrecy*?

Definition: *An encryption scheme* (Gen, Enc, Dec) *with message space* $\mathcal{M}$ *is* **perfectly secret** *if* **for every** $m, m' \in \mathcal{M}$*, and* **every** $c \in \mathcal{C}$*:*

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Intuition: the distribution of the ciphertexts does not depend on the plaintext

- If the distribution of the ciphertexts obtained when m is encrypted is identical to the distribution obtained when m' is encrypted, then it is impossible to tell m and m' apart when observing c

Another definition: Example

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$$\Pr[\mathsf{Enc}_K(\mathtt{aa}) = \mathtt{CC}] = \Pr[K = 2] = \tfrac{1}{26}$$

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$$\Pr[\mathsf{Enc}_K(\mathtt{aa}) = \mathtt{CC}] = \Pr[K = 2] = \frac{1}{26}$$

$$\nparallel$$

$$\Pr[Enc_K(\mathtt{ab}) = \mathtt{CC}] = 0$$

$\square$

Relating the two definitions

Definition: *An encryption scheme* $(\mathit{Gen}, \mathit{Enc}, \mathit{Dec})$ *with message space* $\mathcal{M}$ *is* **perfectly secret** *if* **for every** *probability distribution over* $\mathcal{M}$, **every** *message* $m \in \mathcal{M}$, *and* **every** *ciphertext* $c \in \mathcal{C}$ *with* $\Pr[C = c] \neq 0$*:*

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How do the two definitions compare?

Which one is "better"?

They are equivalent!

Proof of equivalence

$\forall$ *probability distribution over* $\mathcal{M}$, $\forall m \in \mathcal{M}, c \in \mathcal{C}$ *with* $\Pr[C = c] \neq 0$:

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Pick the uniform distribution over $\mathcal{M}$ and any c s.t. $\Pr[C = c] \neq 0$. For an arbitrary m:

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This does not depend on the choice of m!

Proof of equivalence

$\forall$ *probability distribution over* $\mathcal{M}$*,* $\forall m \in \mathcal{M}, c \in \mathcal{C}$ *with* $\Pr[C = c] \neq 0$*:*

$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

$$\Downarrow$$

$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}$*:*

$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

Pick the uniform distribution over $\mathcal{M}$ and any c s.t. $\Pr[C = c] \neq 0$. For an arbitrary m:

$$\cancel{\Pr[M = m]} = \Pr[M = m \mid C = c] = \Pr[C = c \mid M = m] \cdot \frac{\Pr[M=m]}{\Pr[C=c]} = \Pr[\mathsf{Enc}_K(m) = c] \cdot \frac{\cancel{\Pr[M=m]}}{\Pr[C=c]}$$

$$\Pr[\mathsf{Enc}_K(m) = c] = \Pr[C = c] = \Pr[\mathsf{Enc}_K(m') = c] \qquad \text{(repeating the same argument for } m'\text{)}$$

This does not depend on the choice of m!

Proof of equivalence

$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}$:

$$\Pr[\textit{Enc}_K(m) = c] = \Pr[\textit{Enc}_K(m') = c]$$

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Consider an *arbitrary* distribution over $\mathcal{M}$, any $m \in \mathcal{M}$, and any c s.t. $\Pr[C = c] \neq 0$.

We only need to consider $\Pr[M = m] > 0$ (otherwise the thesis is trivially true)

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Proof of equivalence

$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}$:

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$\square$

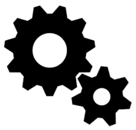
Perfect indistinguishability

Adversary $\mathcal{A}$ (deterministic, computationally unbounded algorithm)

Verifier

Perfect indistinguishability

Adversary $\mathcal{A}$ (deterministic, computationally unbounded algorithm)



$m_0, m_1 \in \mathcal{M}$

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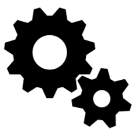

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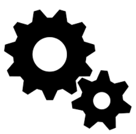

$m_0, m_1 \in \mathcal{M}$ $\longrightarrow$

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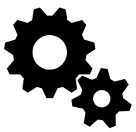

$m_0, m_1 \in \mathcal{M}$

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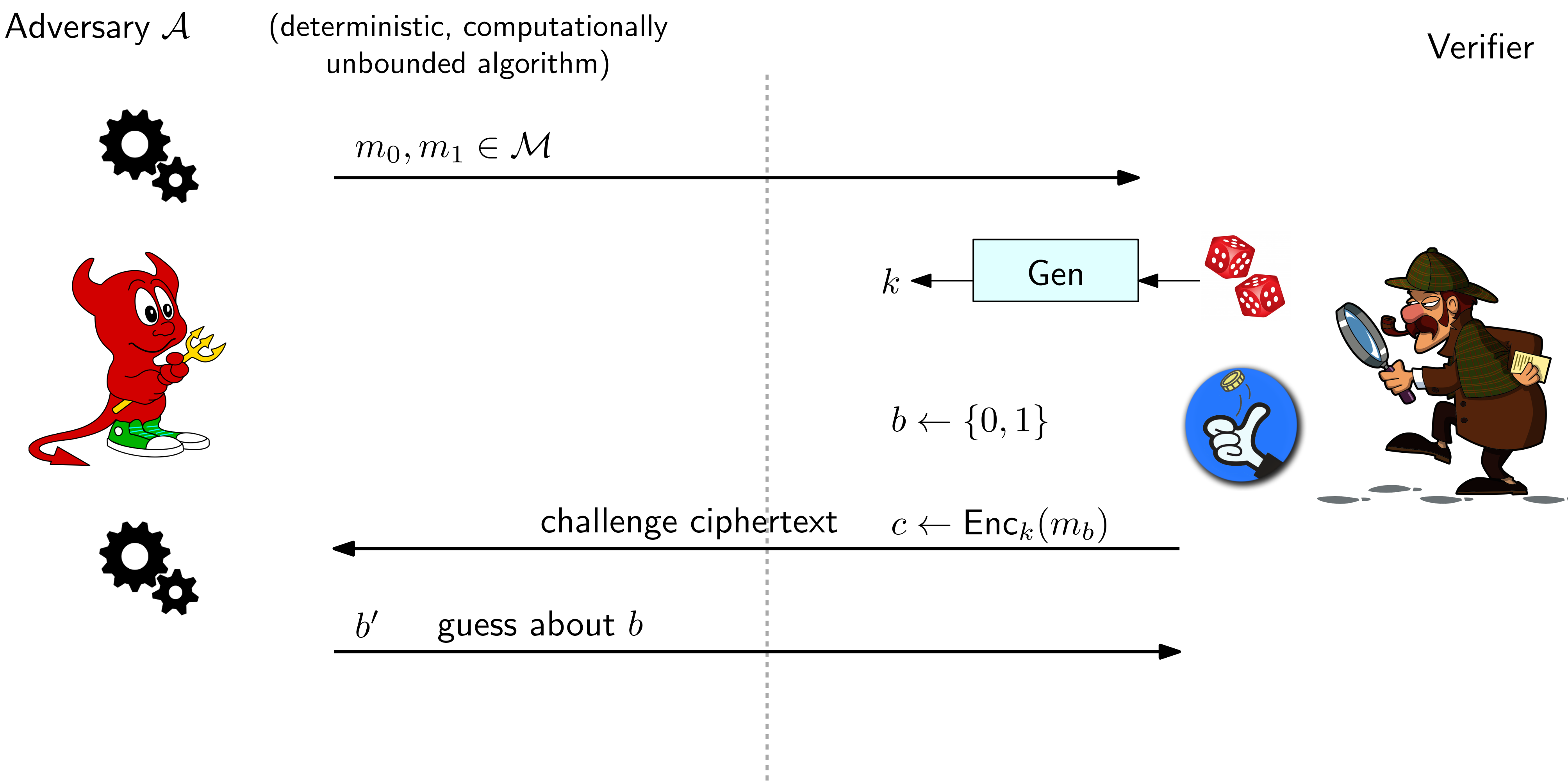
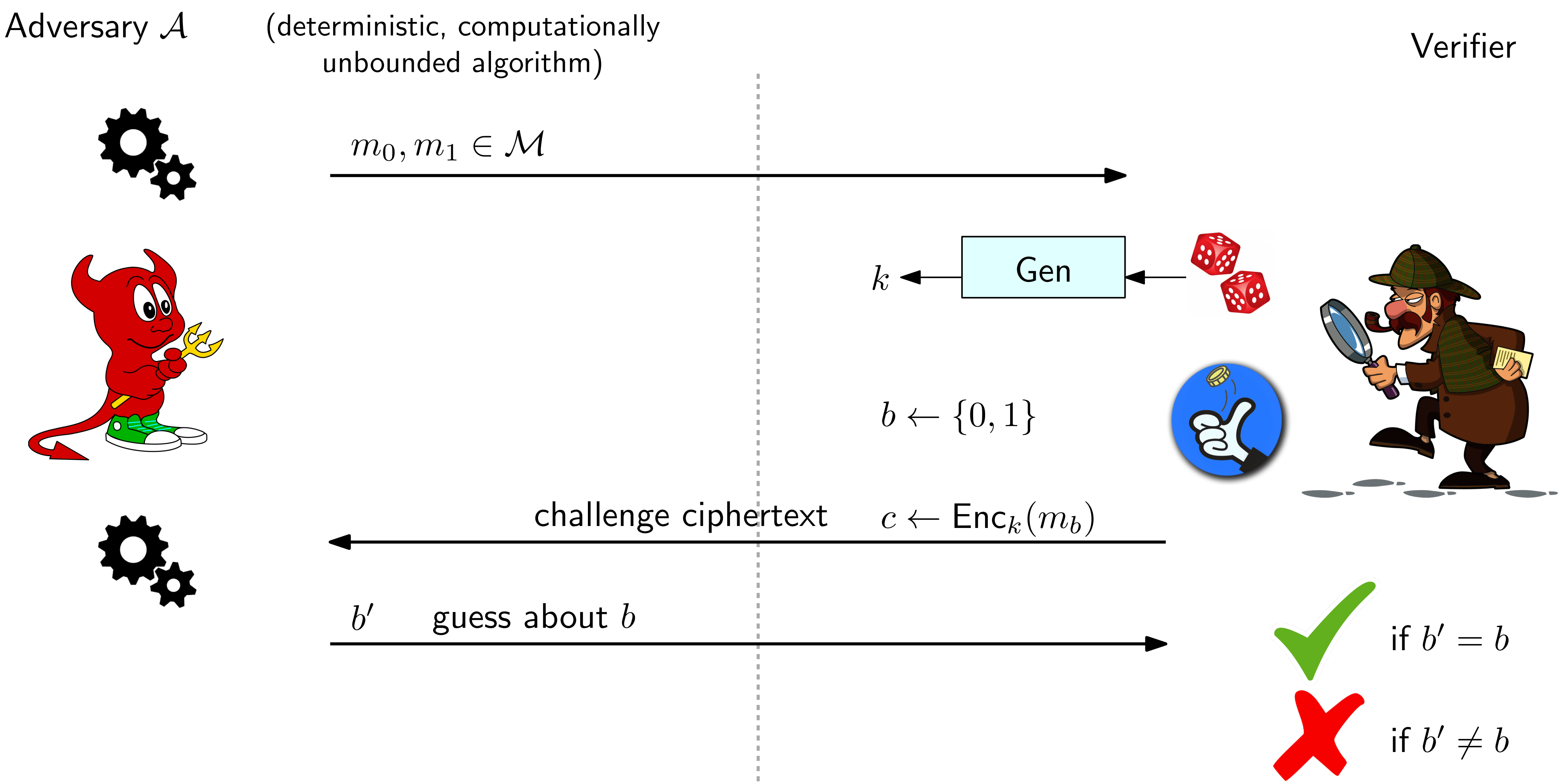
$b \leftarrow \{0,1\}$

challenge ciphertext $c \leftarrow \mathsf{Enc}_k(m_b)$

Perfect indistinguishability



Perfect indistinguishability



Perfect indistinguishability

Formally, if $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is a private key encryption scheme with message space $\mathcal{M}$, we denote the previous experiment by $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$

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The algorithm $\mathcal{A}$ used by the adversary

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The considered encryption scheme

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- $\mathcal{A}$ chooses two messages $m_0, m_1 \in \mathcal{M}$
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- A uniform random bit $b \in \{0, 1\}$ is generated
- The *challenge ciphertext* c is computed by running $\mathsf{Enc}_k(m_b)$, and it is given to $\mathcal{A}$
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- The *output of the experiment* is defined to be 1 if $b' = b$, and 0 otherwise

We write $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1$ (resp. $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 0$) to denote that the output of the experiment is 1 (resp. 0)

Perfect indistinguishability

Definition: *A private key encryption scheme* $\Pi = (\mathit{Gen}, \mathit{Enc}, \mathit{Dec})$ *with message space* $\mathcal{M}$ *is* **perfectly indistinguishable** *if for every* $\mathcal{A}$ *it holds:*

$$\Pr[\mathit{PrivK}^{\mathit{eav}}_{\mathcal{A},\Pi} = 1] = \frac{1}{2}$$

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Advantage of $\mathcal{A}$

Perfect indistinguishability: Example

Consider the Vigenère cipher Π with:

$\mathcal{M} = \{a, b, \ldots, z\}^2$ $\quad$ $\mathcal{K} = \{A, \ldots, Z\} \cup \{A, \ldots, Z\}^2$ $\quad$ $\mathcal{C} = \{A, B, \ldots, Z\}^2$

Where the key is selected as follows:

- Pick a key length ℓ uniformly at random in $\{1, 2\}$
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Is Π perfectly indistinguishable?

We need to devise a "distinguisher", i.e., an algorithm $\mathcal{A}$ that wins the $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$ experiment with probability greater than $\frac{1}{2}$

Perfect indistinguishability: Example

Algorithm $\mathcal{A}$:

- Output $m_0 = \texttt{aa}$, $m_1 = \texttt{ab}$
- Upon receiving the challenge ciphertext $c = c^{(1)}c^{(2)}$:
  - If $c^{(1)} = c^{(2)}$ output $b' = 0$
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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] \; = \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1]$$

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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] = \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{26}$$

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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] \;=\tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1]$$

- When $b = 0$, $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \iff \ell = 1$ or $\ell = 2$ and the two characters of the key are equal

  $$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] = \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{26}$$

- When $b = 1$, $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \iff$

Perfect indistinguishability: Example

Algorithm $\mathcal{A}$:

- Output $m_0 = \texttt{aa}$, $m_1 = \texttt{ab}$
- Upon receiving the challenge ciphertext $c = c^{(1)}c^{(2)}$:
  - If $c^{(1)} = c^{(2)}$ output $b' = 0$
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  $$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] = \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{26}$$

- When $b = 1$, $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \iff \ell = 1$ or $\ell = 2$ and the key $k = k_1k_2$ satisfies $k_1 \neq k_2 + 1 \pmod{26}$

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  $$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1] = \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{25}{26}$$

Perfect indistinguishability: Example

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$$\begin{aligned}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] &= \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1] \\ &= \tfrac{1}{2}\cdot(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{26}) + \tfrac{1}{2}\cdot(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{25}{26})\end{aligned}$$

Perfect indistinguishability: Example

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- Output $m_0 = \texttt{aa}$, $m_1 = \texttt{ab}$
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$$\begin{aligned}
\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] &= \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] + \tfrac{1}{2}\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1] \\
&= \tfrac{1}{2}\cdot(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{26}) + \tfrac{1}{2}\cdot(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{25}{26}) \quad = \tfrac{3}{4} = \tfrac{1}{2} + \tfrac{1}{4} > \tfrac{1}{2}
\end{aligned}$$

Perfect indistinguishability: Example

Algorithm $\mathcal{A}$:

- Output $m_0 = \texttt{aa}$, $m_1 = \texttt{ab}$
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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] = \frac{1}{2} \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 0] + \frac{1}{2} \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1 \mid b = 1]$$

$$= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{26}\right) + \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{25}{26}\right) = \frac{3}{4} = \frac{1}{2} + \frac{1}{4} > \frac{1}{2}$$

Advantage of $\mathcal{A}$

Perfect secrecy & perfect indistinguishability

A private key encryption scheme is **perfectly secret** if and only if it is **perfectly indistinguishable**.

$\forall$ *probability distribution over* $\mathcal{M}$, $\forall m \in \mathcal{M}, c \in \mathcal{C}$ *with* $\Pr[C = c] \neq 0$*:*

$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

$$\Updownarrow$$

$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}$*:*

$$\Pr[Enc_K(m) = c] = \Pr[Enc_K(m') = c]$$

$$\Updownarrow$$

$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] = \frac{1}{2} \quad \forall \mathcal{A}$$

Proof of equivalence

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$

$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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Fix any algorithm $\mathcal{A}$, and let m_0, m_1 be the messages output by $\mathcal{A}$

Partition $\mathcal{C}$ into $\mathcal{C}_0, \mathcal{C}_1$, where C_i is the set of ciphertexts for which $\mathcal{A}$ guesses $b' = i$

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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] \ = \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1]$$

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$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$

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$$\begin{aligned} \Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] &= \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1] \\ &= \tfrac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_0) \in \mathcal{C}_0] + \tfrac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_1) \in \mathcal{C}_1] \end{aligned}$$

Proof of equivalence

$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}$:

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\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] &= \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1] \\
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&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_0} \Pr[\mathsf{Enc}_K(m_0) = c] + \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_1} \Pr[\mathsf{Enc}_K(m_1) = c]
\end{aligned}$$

Proof of equivalence

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}:$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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&= \tfrac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_0) \in \mathcal{C}_0] + \tfrac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_1) \in \mathcal{C}_1] \\
&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_0} \Pr[\mathsf{Enc}_K(m_0) = c] + \tfrac{1}{2} \cdot \sum_{c \in \mathcal{C}_1} \Pr[\mathsf{Enc}_K(\color{red}{m_0}) = c]
\end{aligned}$$

Proof of equivalence

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

$$\Downarrow$$

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&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_0} \Pr[\mathsf{Enc}_K(m_0) = c] + \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_1} \Pr[\mathsf{Enc}_K(\color{red}{m_0}) = c] \\
&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}} \Pr[\mathsf{Enc}_K(m_0) = c]
\end{aligned}$$

Proof of equivalence

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}:$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

$$\Downarrow$$

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\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] &= \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1] \\
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&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_0} \Pr[\mathsf{Enc}_K(m_0) = c] + \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}_1} \Pr[\mathsf{Enc}_K({\color{red}m_0}) = c] \\
&= \tfrac{1}{2} \cdot \textstyle\sum_{c \in \mathcal{C}} \Pr[\mathsf{Enc}_K(m_0) = c] \quad = \tfrac{1}{2}
\end{aligned}$$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}:$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

$$\Downarrow$$

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Pick $m_0, m_1 \in \mathcal{M}, c^* \in \mathcal{C}$ s.t. $\Pr[\mathsf{Enc}_K(m_0) = c^*] \neq \Pr[\mathsf{Enc}_K(m_1) = c^*]$

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$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
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Algorithm $\mathcal{A}$:

- Output m_0, m_1
- Upon receiving the challenge ciphertext c
  - If $c = c^*$ output $b' = 0$
  - Otherwise output a b' chosen u.a.r. in $\{0, 1\}$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
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$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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$\Pr[b' = 0 \mid b = 0] = \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) \neq c^*]$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}:$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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$$\begin{aligned}\Pr[b' = 0 \mid b = 0] &= \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) \neq c^*] \\ &= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \mid \mathsf{Enc}_K(m_0) \neq c^*] \cdot \Pr[\mathsf{Enc}_K(m_0) \neq c^*]\end{aligned}$$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$

$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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Pick $m_0, m_1 \in \mathcal{M}, c^* \in \mathcal{C}$ s.t. $\Pr[\mathsf{Enc}_K(m_0) = c^*] \neq \Pr[\mathsf{Enc}_K(m_1) = c^*]$

$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] = \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1]$$

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\Pr[b' = 0 \mid b = 0] &= \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) \neq c^*] \\
&= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \mid \mathsf{Enc}_K(m_0) \neq c^*] \cdot \Pr[\mathsf{Enc}_K(m_0) \neq c^*] \\
&= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \tfrac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_0) \neq c^*]
\end{aligned}$$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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$$\begin{aligned}\Pr[b' = 0 \mid b = 0] &= \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) \neq c^*] \\ &= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \mid \mathsf{Enc}_K(m_0) \neq c^*] \cdot \Pr[\mathsf{Enc}_K(m_0) \neq c^*] \\ &= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \tfrac{1}{2} \cdot (1 - \Pr[\mathsf{Enc}_K(m_0) = c^*])\end{aligned}$$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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$$\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] = \tfrac{1}{2} \cdot \Pr[b' = 0 \mid b = 0] + \tfrac{1}{2} \cdot \Pr[b' = 1 \mid b = 1]$$

$$\begin{aligned}
\Pr[b' = 0 \mid b = 0] &= \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \wedge \mathsf{Enc}_K(m_0) \neq c^*] \\
&= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \Pr[b' = 0 \mid \mathsf{Enc}_K(m_0) \neq c^*] \cdot \Pr[\mathsf{Enc}_K(m_0) \neq c^*] \\
&= \Pr[\mathsf{Enc}_K(m_0) = c^*] + \tfrac{1}{2} \cdot (1 - \Pr[\mathsf{Enc}_K(m_0) = c^*]) \\
&= \tfrac{1}{2} + \tfrac{1}{2} \Pr[\mathsf{Enc}_K(m_0) = c^*]
\end{aligned}$$

Proof of equivalence

NOT

$$\forall m, m' \in \mathcal{M}, c \in \mathcal{C}\text{:}$$
$$\Pr[\mathit{Enc}_K(m) = c] = \Pr[\mathit{Enc}_K(m') = c]$$

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$\Pr[b' = 0 \mid b = 0] = \frac{1}{2} + \frac{1}{2} \cdot \Pr[\mathsf{Enc}_K(m_0) = c^*]$

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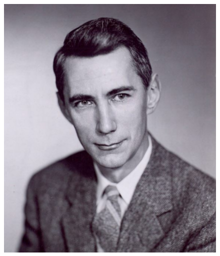
$\square$

Recap: Equivalent definitions



Definition: *An encryption scheme* (Gen, Enc, Dec) *with message space* $\mathcal{M}$ *is* **perfectly secret** *if* **for every** *probability distribution over* $\mathcal{M}$*,* **every** *message* $m \in \mathcal{M}$*, and* **every** *ciphertext* $c \in \mathcal{C}$ *with* $\Pr[C = c] \neq 0$*:*

$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

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Definition: *A private key encryption scheme* $\Pi = (Gen, Enc, Dec)$ *with message space* $\mathcal{M}$ *is* **perfectly indistinguishable** *if for every* $\mathcal{A}$ *it holds:*

$$\Pr[\mathit{PrivK}^{\mathit{eav}}_{\mathcal{A},\Pi} = 1] = \frac{1}{2}$$