

Pseudorandom Functions

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- For EAV-security we had to rely on PRGs
- For CPA-security we need a new cryptographic primitive: **pseudorandom functions** (PRFs)

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xkcd.com

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We need to talk about probability distributions over functions instead

This is formalized using the notion of **keyed functions**

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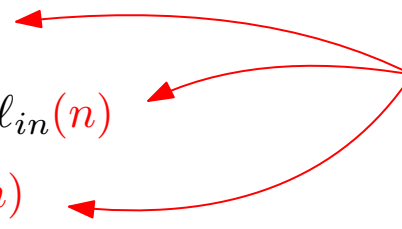
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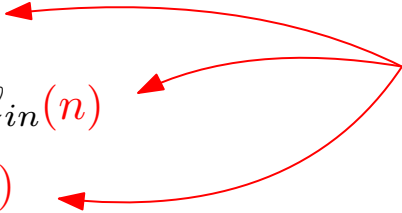
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Simplifying assumption (can be removed): F is **length-preserving**

$$\ell_{key}(n) = \ell_{in}(n) = \ell_{out}(n) = n$$

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- As a function whose outputs are decided **lazily**: whenever we need to evaluate $f(x)$:

- If $f(x)$ was never evaluated before with input x :
 - Return a binary string chosen u.a.r. from $\{0, 1\}^n$
- Otherwise, return the previously chosen string for input x

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We can only sample a **tiny** fraction of the functions in Func_n !

Defining pseudorandom functions

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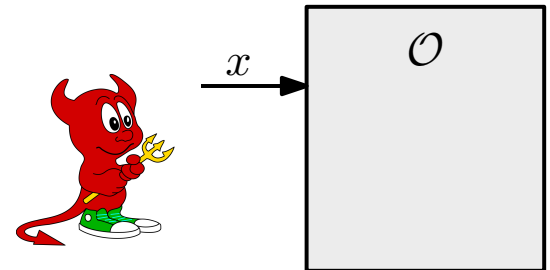
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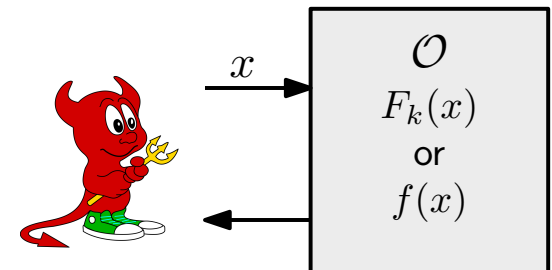
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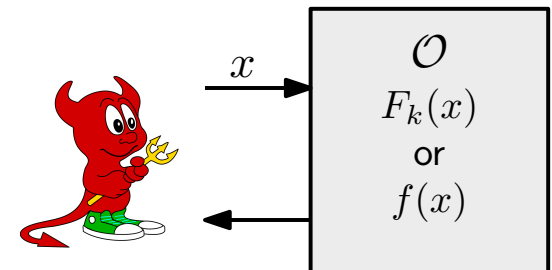
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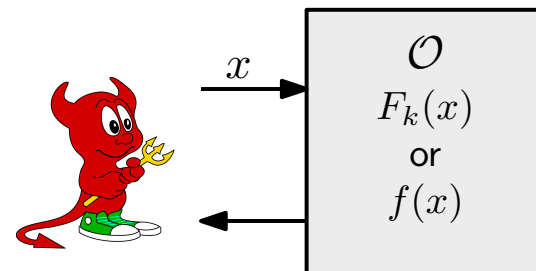
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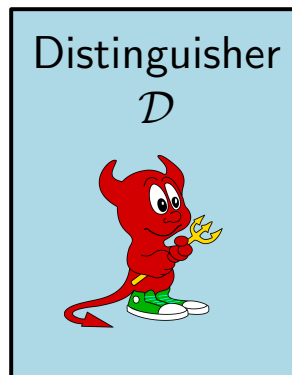
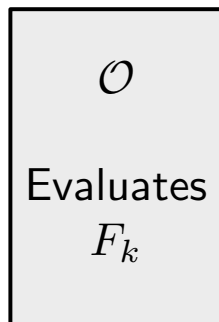
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- \mathcal{D} can query \mathcal{O} many times
- \mathcal{D} needs to guess whether \mathcal{O} is evaluating F_k or f



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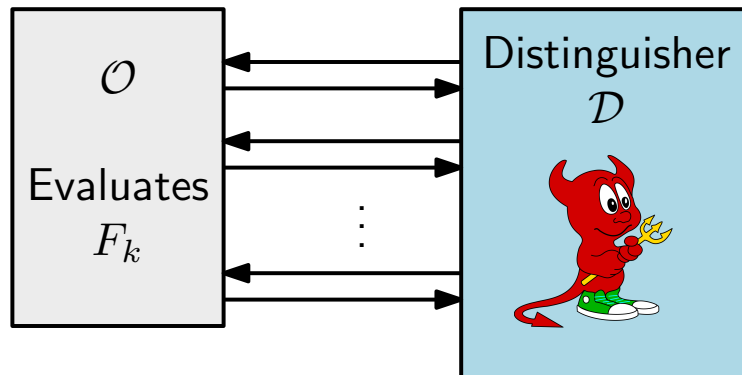
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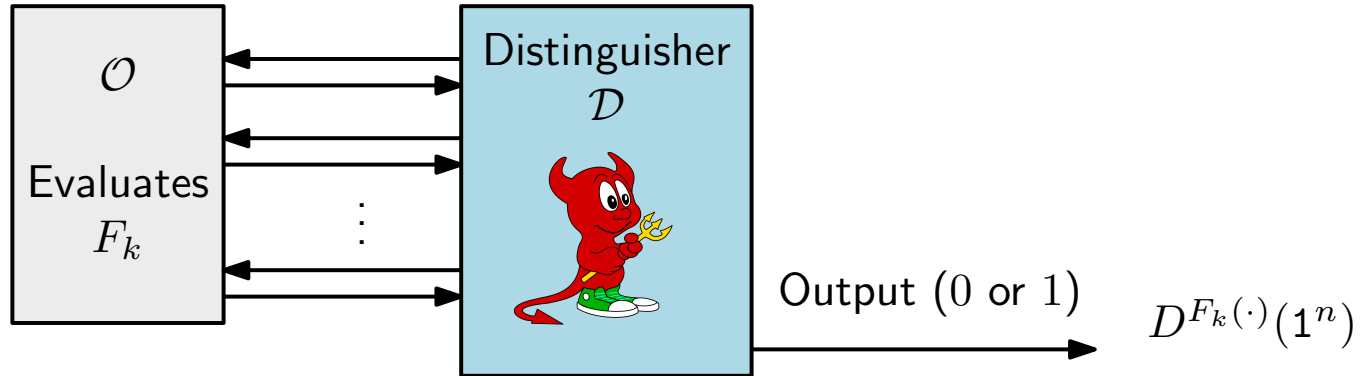
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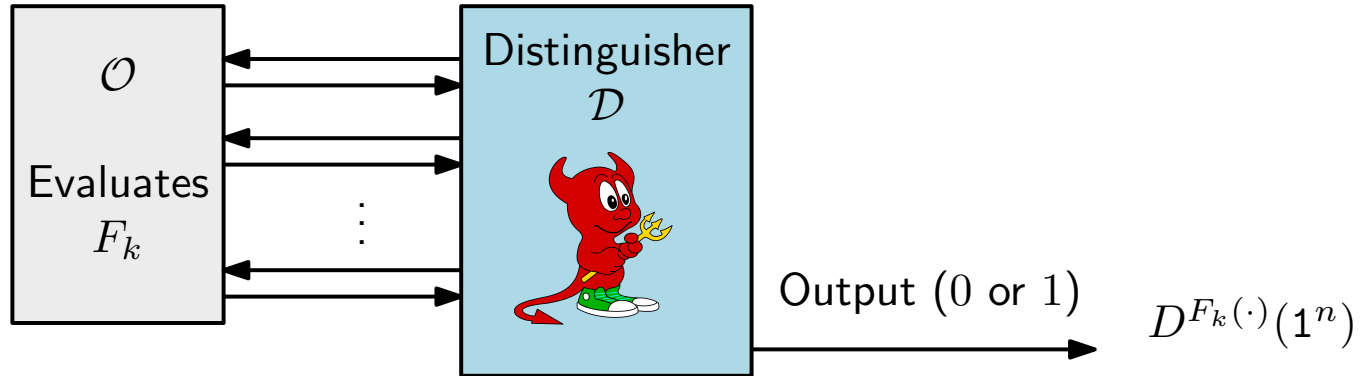
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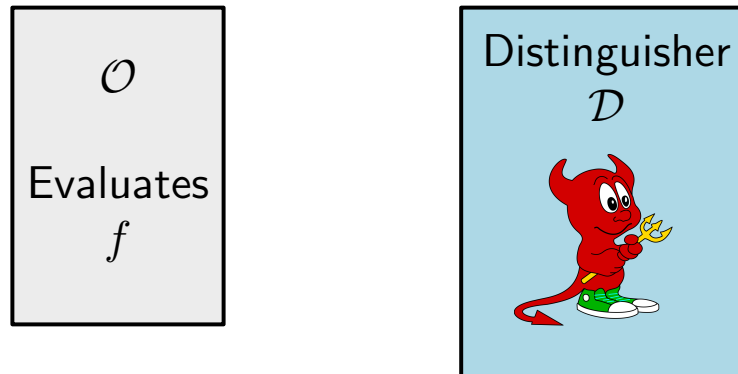
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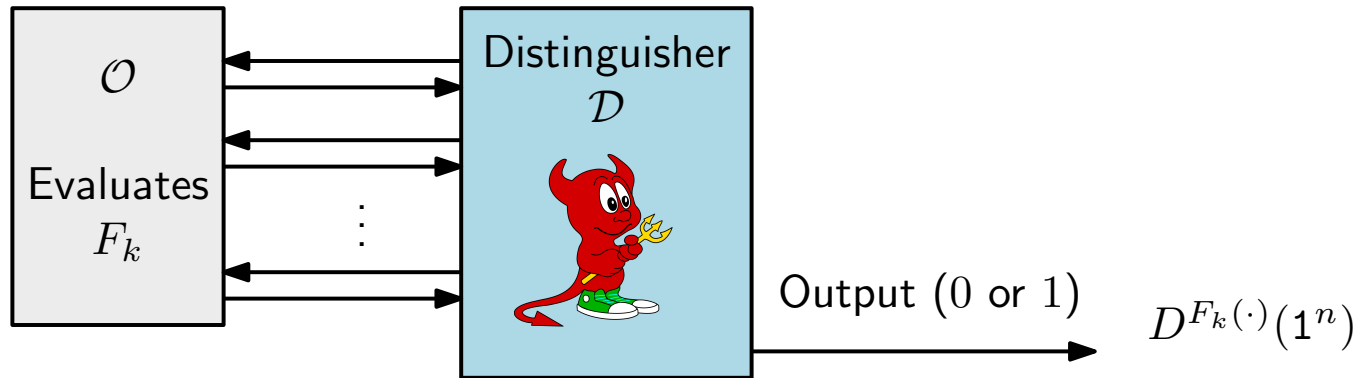
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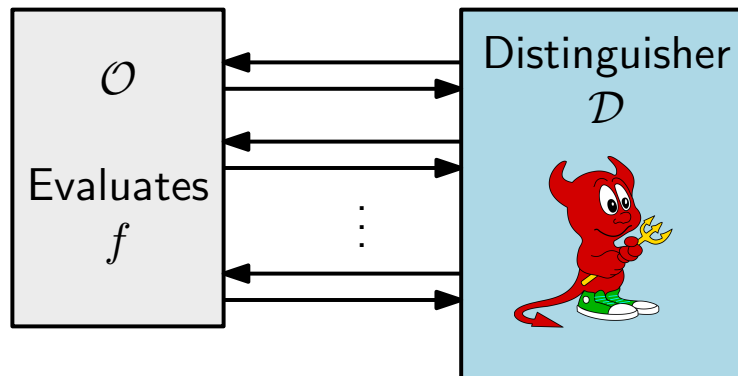
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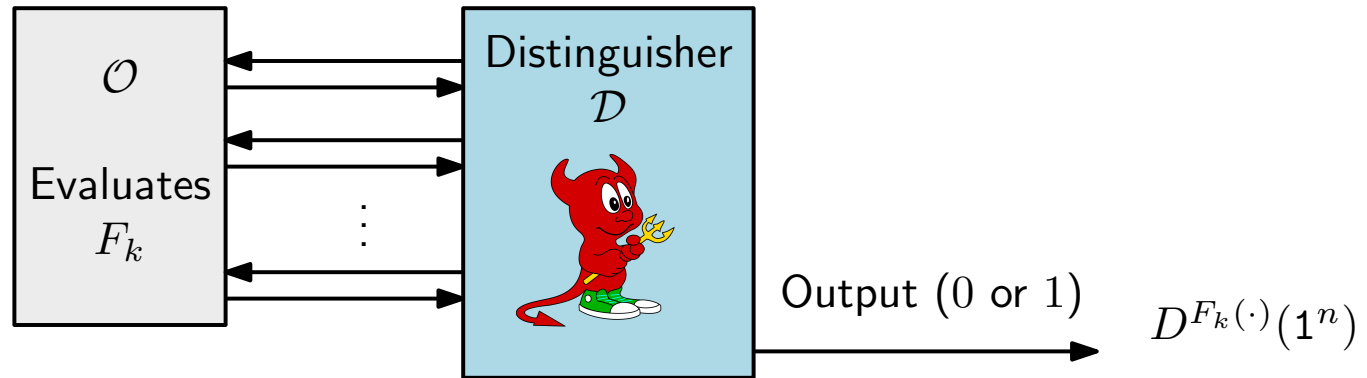
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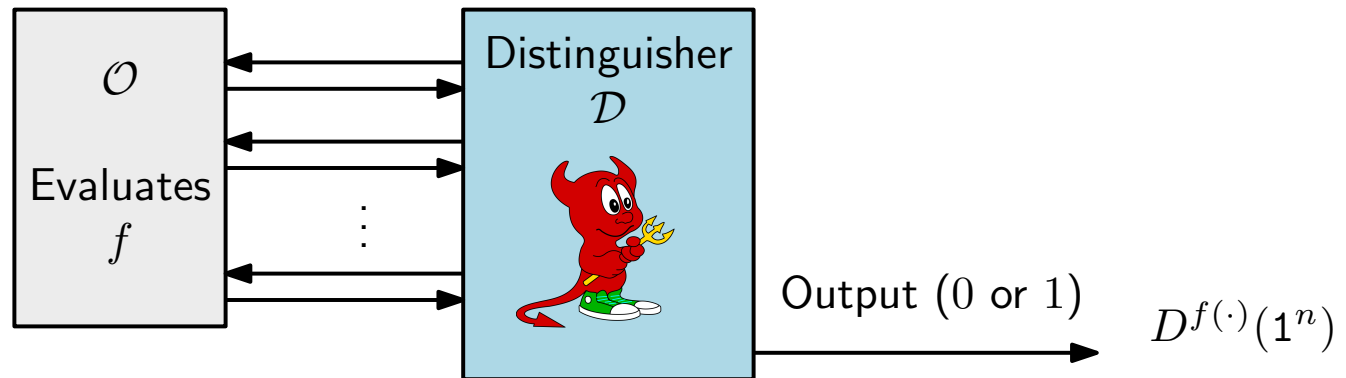
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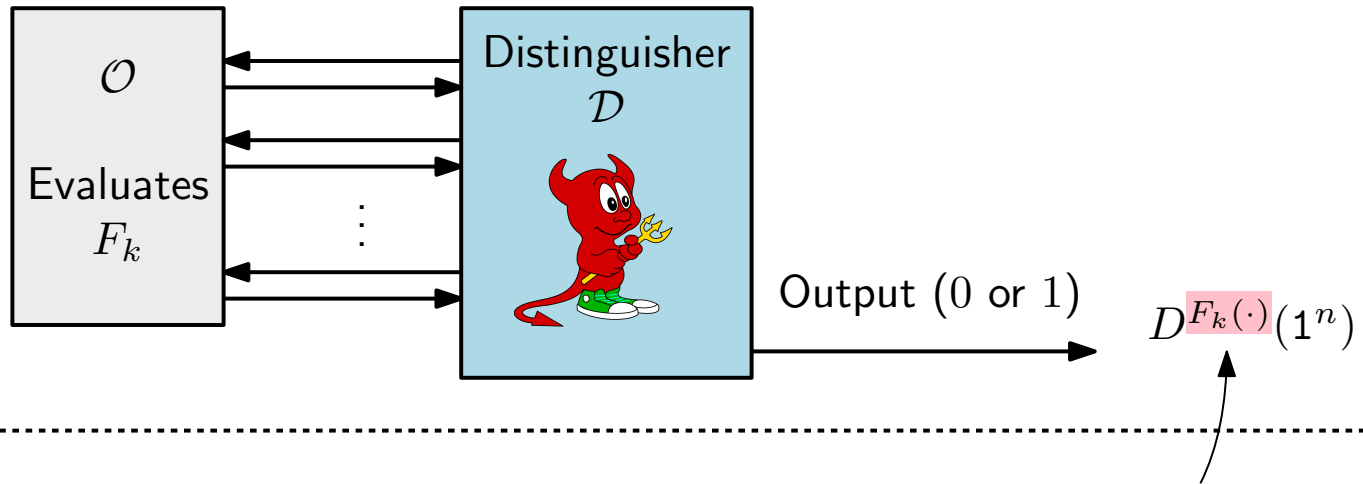
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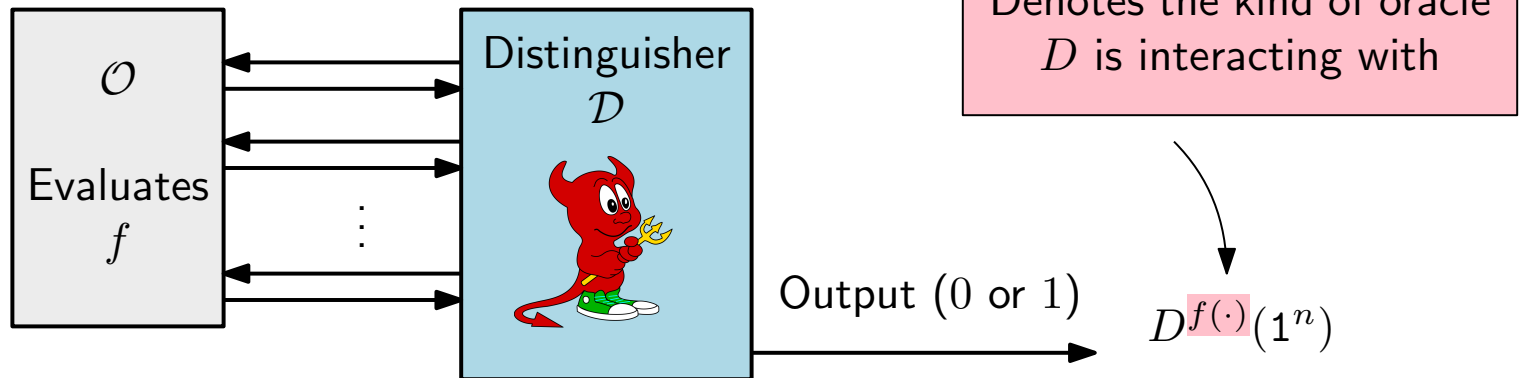
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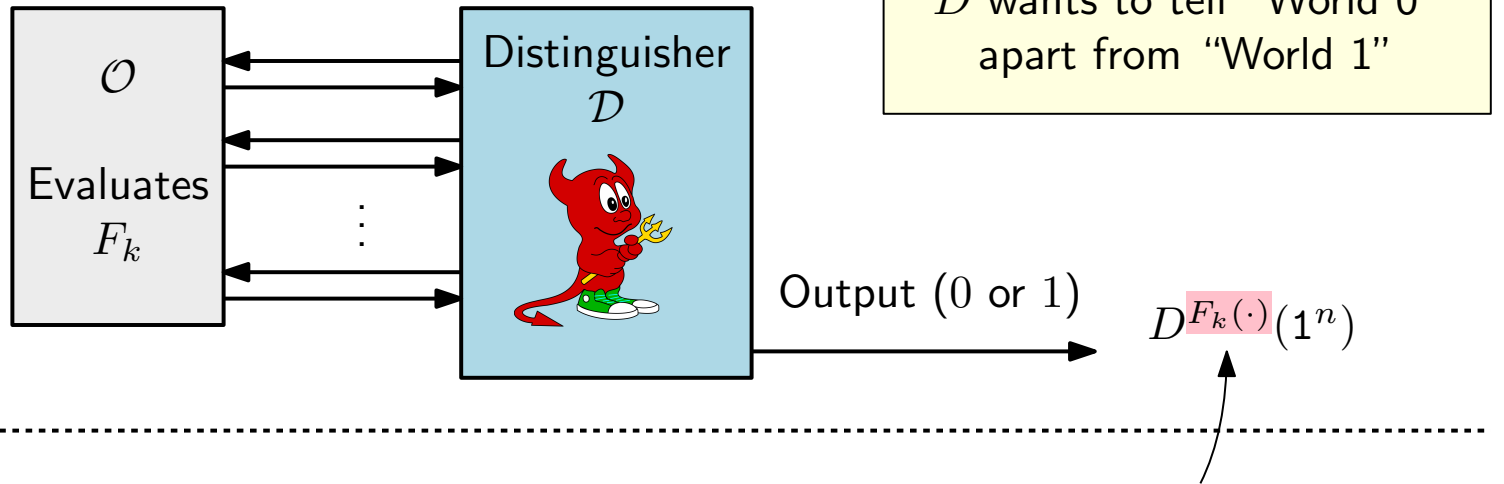
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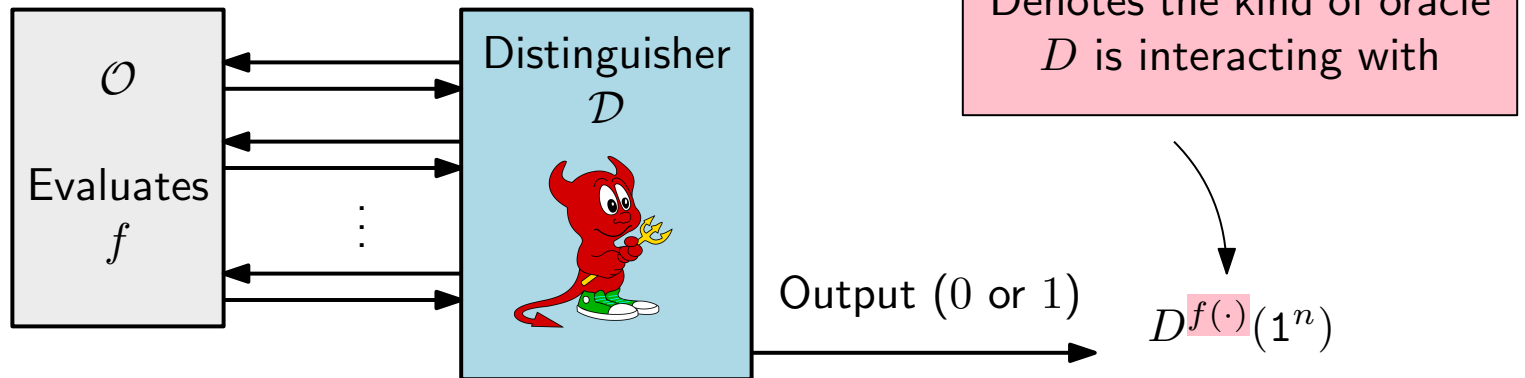
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Defining pseudorandom functions (formal)

Definition: An efficient, length preserving, keyed function $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ is a **pseudorandom function** if for all probabilistic polynomial-time distinguishers D , there is a negligible function ε such that:


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Probability over the randomness of the distinguisher and the uniform choice of $f \in \text{Func}_n$

Examples

What are some possible distinguishers from the following (failed attempts at) pseudorandom functions?

- $F(k, x) = 1^n$
- $F(k, x) = k$
- $F(k, x) = k \vee x$
- $F(k, x) = k \wedge x$
- $F(k, x) = k \oplus x$

PRFs and PRGs

If we have a PRF $F(k, x)$ we can use it to build a PRG G .

$G(s)$:

- Return $F_s(0\dots000) \parallel F_s(0\dots001)$

expansion factor $\ell(n) = 2n$

PRFs and PRGs

If we have a PRF $F(k, x)$ we can use it to build a PRG G .

$G(k)$:

- Return $F_k(\langle 0 \rangle) \parallel F_k(\langle 1 \rangle) \parallel \dots \parallel F_k(\langle L \rangle)$

$\langle x \rangle$ = binary encoding of x with n bits

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Proof that G is a PRG? Security reduction (“breaking G implies breaking F ”)

- Suppose that G is not a PRG, then there is some distinguisher D for G (with non negligible gap)
- Use D to build a distinguisher \mathcal{A} for F (with non negligible gap)
- This contradicts the fact that F is a PRF (i.e., no such D can exist)

PRFs and PRGs

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Random string
in $\{0, 1\}^{L \cdot n}$

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- Therefore F is not a PRF.



□

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Are PRFs a stronger cryptographic primitive than PRGs?

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Proof of security:

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- Suppose that F is not a PRF, then there is \mathcal{A} such that

$$\left| \Pr[\mathcal{A}^{F_k(\cdot)}(1^n) = 1] - \Pr[\mathcal{A}^{f(\cdot)}(1^n) = 1] \right| = \eta(n) \text{ for non-negligible } \eta(n)$$

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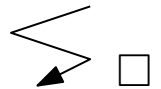
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$\implies G$ is not a PRG



The Goldreich-Goldwasser-Micali construction

Let G be a *length-doubling* PRG, i.e., $\ell(n) = 2n$.

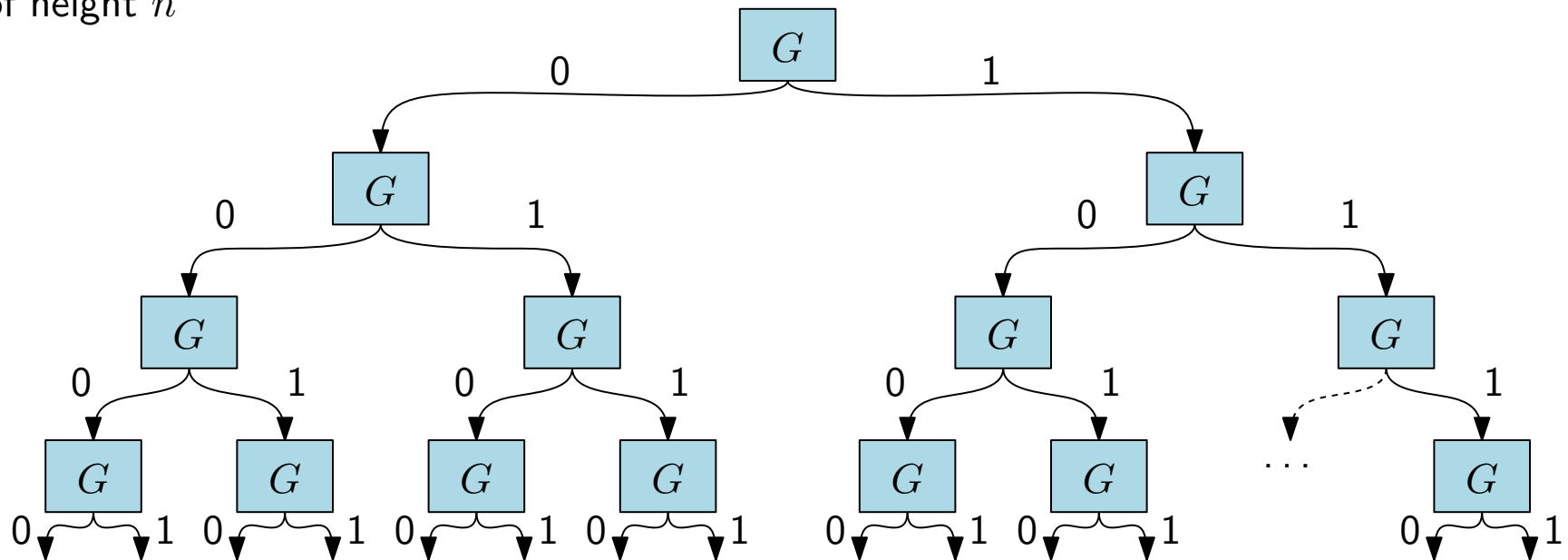
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Imagine the following complete binary tree of height n

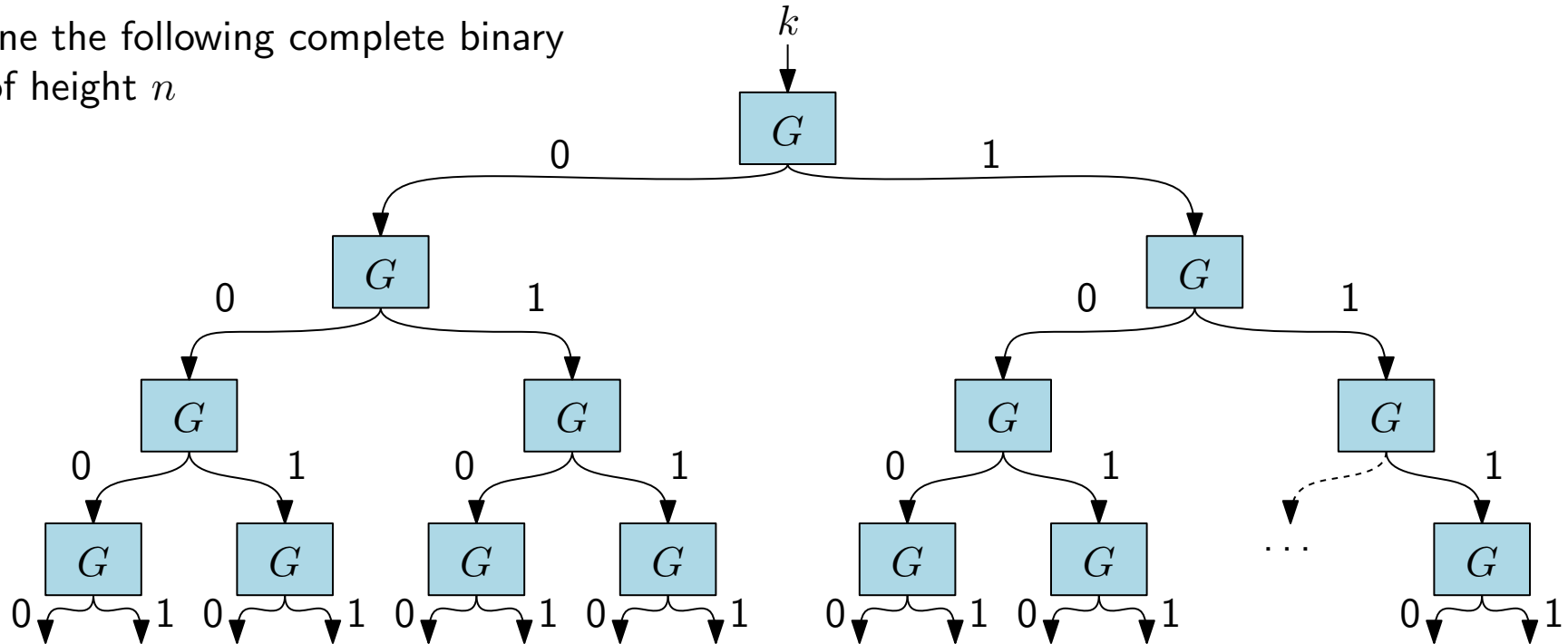


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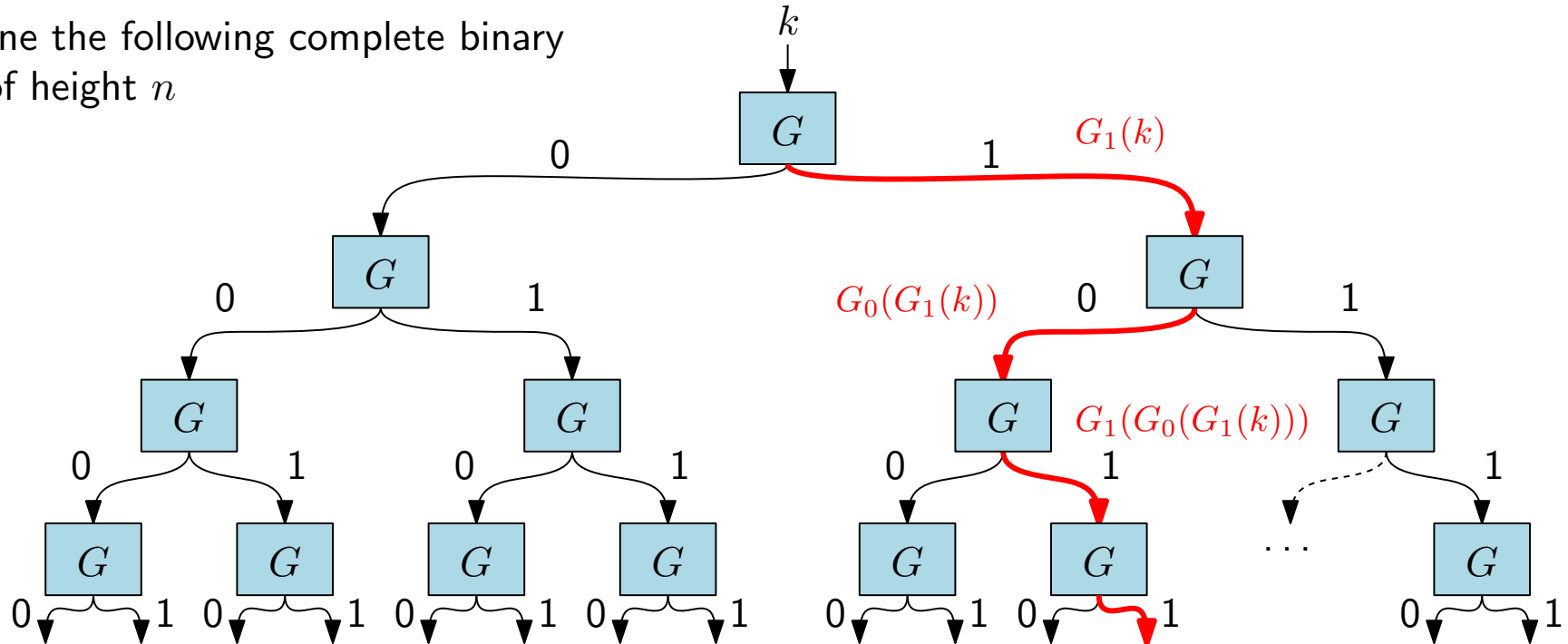
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Interpret the binary digits of x as a path in the tree

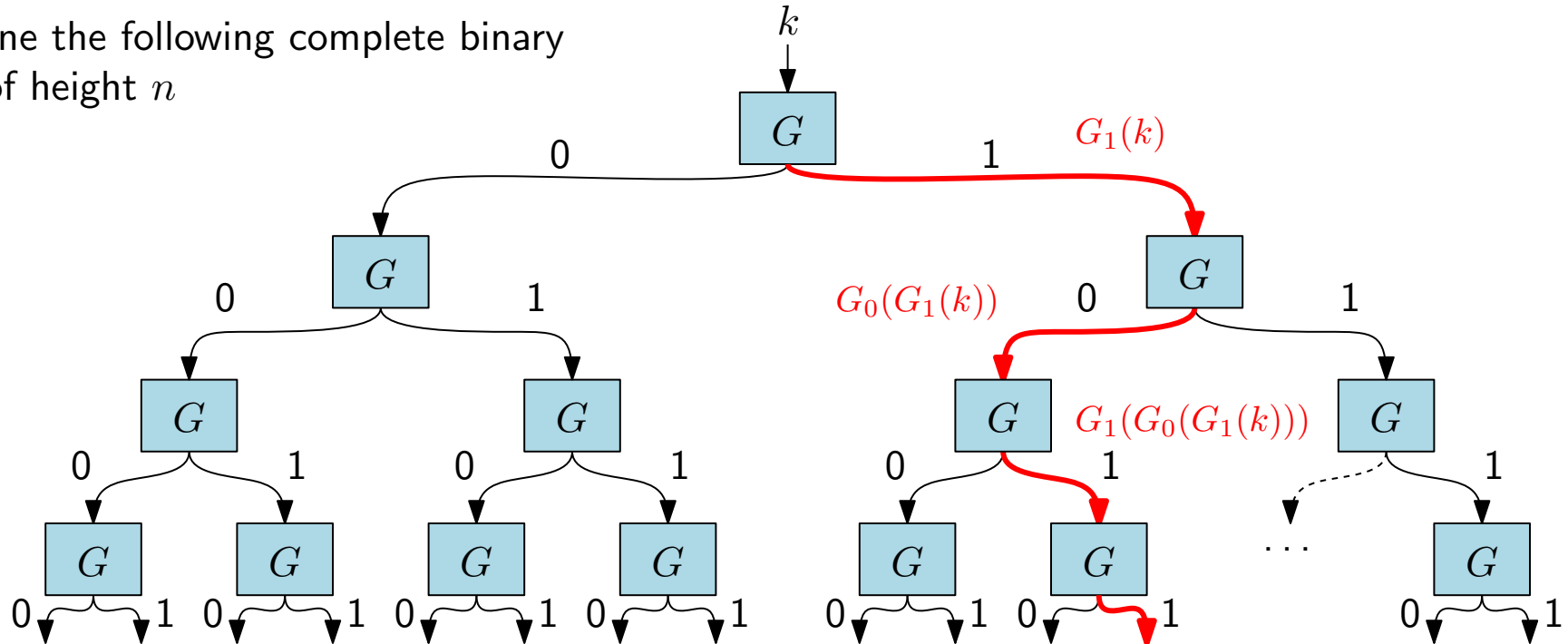
$$F(k, 1011)$$

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Imagine the following complete binary tree of height n



Interpret the key k of $F(k, x)$ as the seed of the root of the tree

Interpret the binary digits of x as a path in the tree

Interpret the output of the leaf as the output of $F(k, x)$

$$F(k, 1011) = G_1(G_1(G_0(G_1(k))))$$

The Goldreich-Goldwasser-Micali construction

If G is a secure length-doubling PRG, then the Goldreich-Goldwasser-Micali construction is a PRF

We won't see a proof of this fact (see Section 8.5 of the textbook if interested).

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What if don't have a length-doubling PRG?

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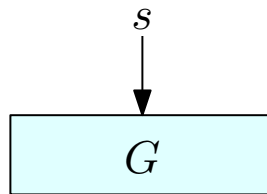
We can build one from any PRG, even if the expansion factor is just $\ell(n) = n + 1$

In fact, we can build a PRG with expansion factor $n + p(n)$ **for any polynomial** $p(n)$

Increasing the expansion factor

An easy case: increasing the expansion factor by 1

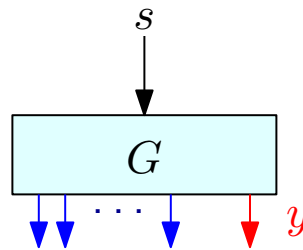
- Start from a PRG G with expansion factor $\ell(n) = n + 1$



Increasing the expansion factor

An easy case: increasing the expansion factor by 1

- Start from a PRG G with expansion factor $\ell(n) = n + 1$
- Call $G(s)$ and interpret the first n bits $x_1x_2 \dots x_n$ of the output as a new seed
- Let the last bit of $G(s)$ be y

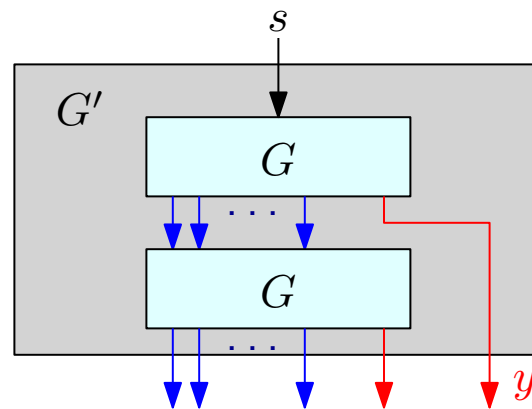


$$G(s) = x_1x_2x_3 \dots x_ny$$

Increasing the expansion factor

An easy case: increasing the expansion factor by 1

- Start from a PRG G with expansion factor $\ell(n) = n + 1$
- Call $G(s)$ and interpret the first n bits $x_1x_2 \dots x_n$ of the output as a new seed
- Let the last bit of $G(s)$ be y
- Return $G(x_1x_2 \dots x_n) \parallel y$

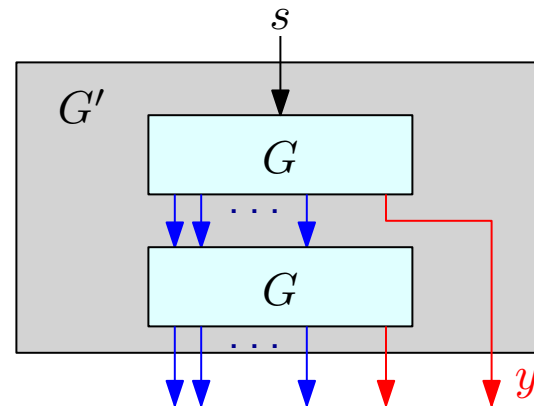


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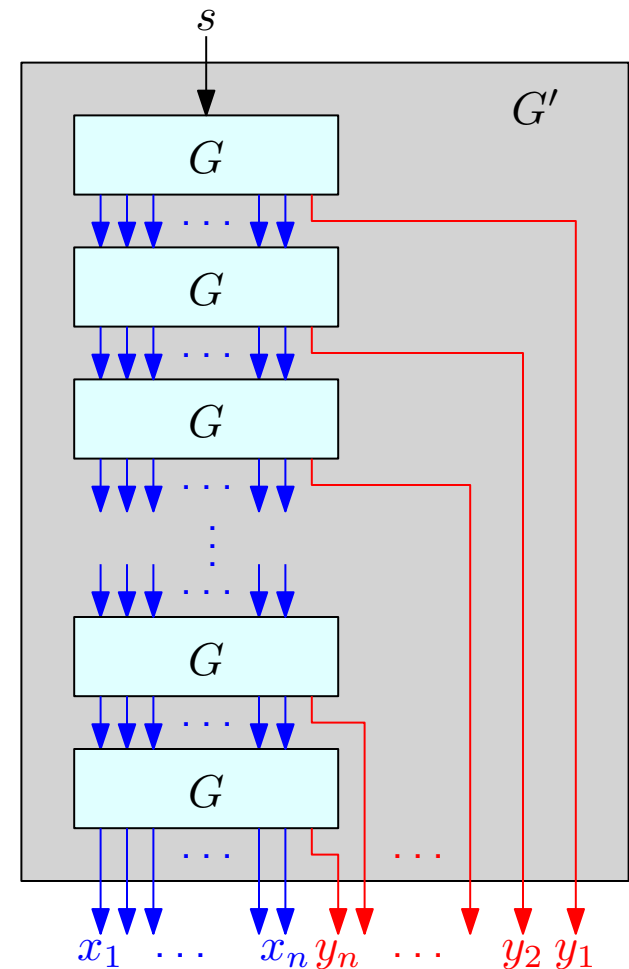
$$\text{Overall expansion factor } \ell(n) = n + 2$$

Increasing the expansion factor (length-doubling)

Increasing the expansion factor from $n + 1$ to $2n$

- Start from a PRG G with expansion factor $\ell(n) = n + 1$
- Repeat the previous idea for n levels
- The i -th intermediate level outputs $n + 1$ bits
 - n bits are used as a seed for the next level
 - The $(n + 1)$ -th bit y_i will be part of the output of the whole construction
- The last level outputs $n + 1$ bits $x_1 x_2 \dots x_n y_n$
- The final output is $x_1 x_2 \dots x_n y_n y_{n-1} \dots y_1$

Overall expansion factor: $\ell(n) = n + n = 2n$

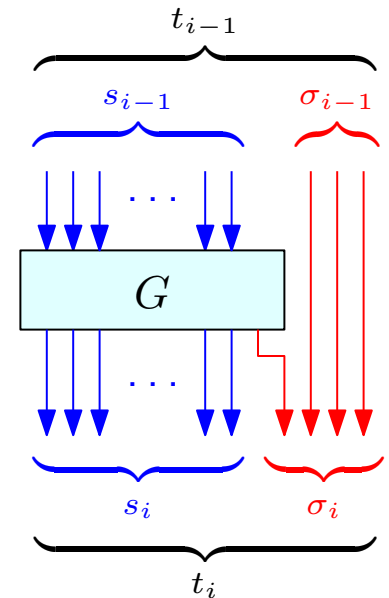


Increasing the expansion factor to $n + p(n)$

Repeat the previous idea $p(n)$ times

Algorithm $\widehat{G}(s)$: (here $s \in \{0, 1\}^n$)

- $t_0 \leftarrow s$
- For $i = 1, 2, \dots, p(n)$:
 - Interpret t_{i-1} as $s_{i-1} \parallel \sigma_{i-1}$ where $|s_{i-1}| = n$ and $|\sigma_{i-1}| = i - 1$
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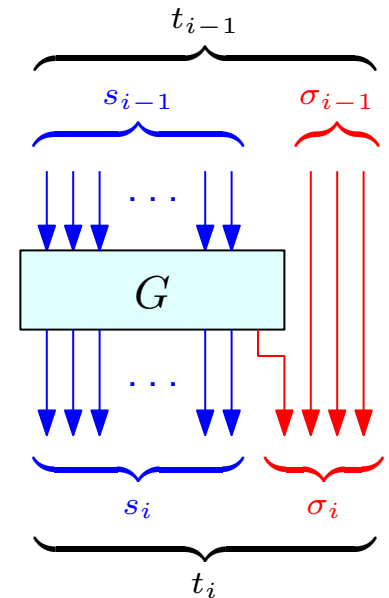


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Define H_n^j to be the distribution on strings of length $n + p(n)$ output by the following process:

- Choose t_j u.a.r. from $\{0, 1\}^{n+j}$
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Let \widehat{D} be a distinguisher such that:

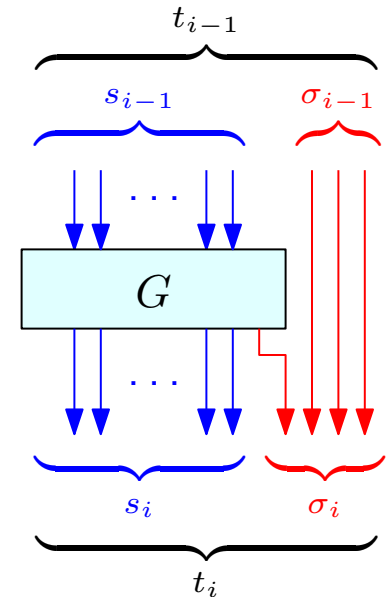
$$\left| \Pr_s[\widehat{D}(\widehat{G}(s))] - \Pr_r[\widehat{D}(r)] \right| = \eta(n) \text{ for some non-negligible } \eta(n)$$

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Algorithm $D(w)$: (here $w \in \{0, 1\}^{n+1}$)

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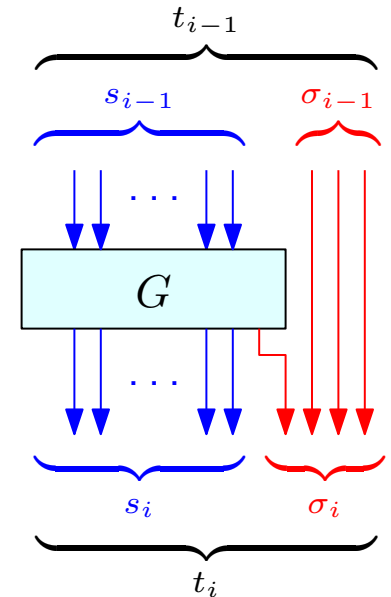
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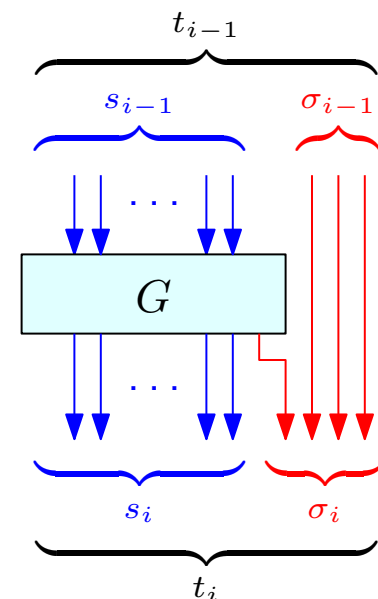
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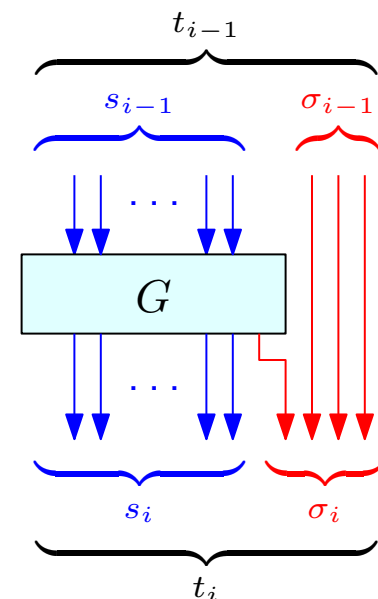


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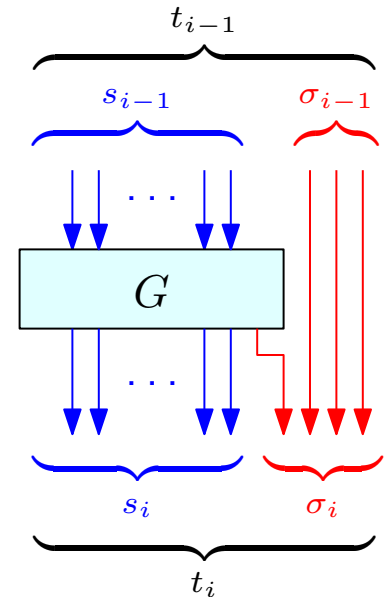
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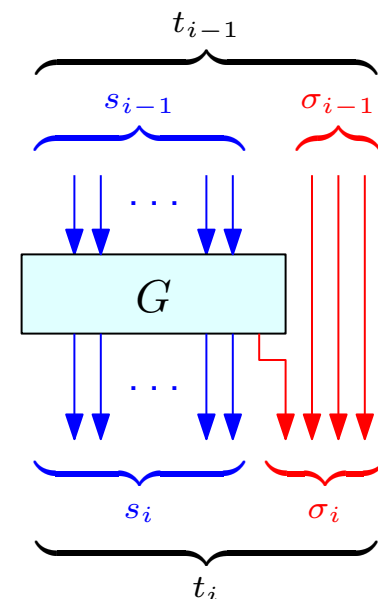
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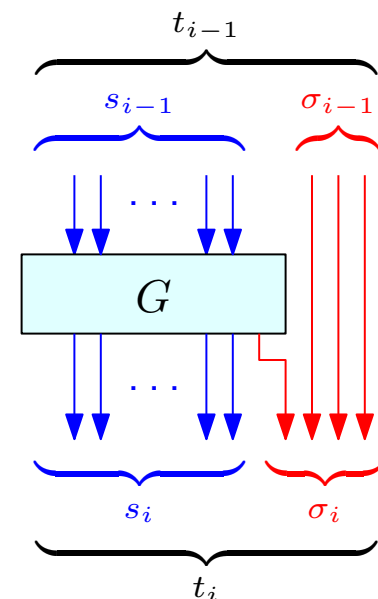


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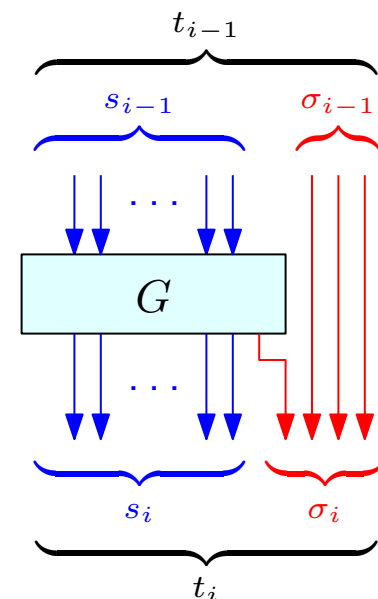
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Increasing the expansion factor to $n + p(n)$

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Perm_n

|

$= 2^n \cdot (2^n - 1) \cdot \dots \cdot 1$
 $= (2^n)!$

Number of Permutations vs Number of Functions

Since a function $F \in \text{Perm}_n$ is bijective, it must be **invertible**

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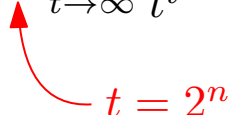
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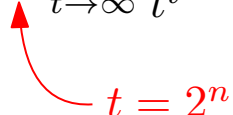
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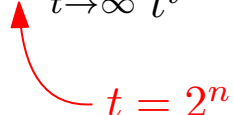
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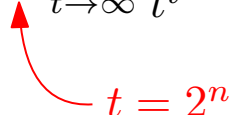
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Asymptotically, almost no function in Func_n is a permutation!

Keyed permutations

A **keyed permutation** is a keyed function $F : \{0, 1\}^{\ell_{key}(n)} \times \{0, 1\}^{\ell_{in}(n)} \rightarrow \{0, 1\}^{\ell_{out}(n)}$ such that:

- $\ell_{in}(n) = \ell_{out}(n)$ (this quantity is called the **block length**); and
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A keyed permutation is **efficient** if:

- There is a polynomial-time algorithm that computes $F(x)$ given x ; and
- There is a polynomial-time algorithm that computes $F^{-1}(y)$ given y

Pseudorandom permutations, formal definition

Definition: An efficient, length preserving, keyed permutation $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ is a **pseudorandom permutation** if for all probabilistic polynomial-time distinguishers D , there is a negligible function ε such that:

$$\left| \Pr[D^{F_k(\cdot)}(\mathbf{1}^n) = 1] - \Pr[D^{f(\cdot)}(\mathbf{1}^n) = 1] \right| \leq \varepsilon(n)$$

Probability over the randomness of the distinguisher and the choice of k

Probability over the randomness of the distinguisher and the uniform choice of $f \in \text{Perm}_n$

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Intuition: a keyed permutation is pseudorandom permutation if no polynomial-time algorithm can distinguish it from a random permutation

Pseudorandom permutations

Recall that (asymptotically) almost no function in Func_n is a permutation

Nevertheless:

- As soon as $\ell_{in}(n) \geq n$, a PRP is indistinguishable (in polynomial time, with non-negligible gap) from a PRF
- Since a PRF is indistinguishable from a random function, this implies that PRPs with $\ell_{in}(n) \geq n$ are also indistinguishable from random functions!

Strong pseudorandom permutations

Sometimes we need even even “stronger” functions than pseudorandom permutation

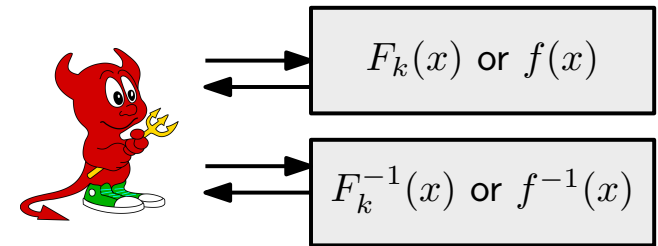
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We define **strong** pseudorandom permutations that are indistinguishable from random permutation even if the adversary has oracle access to **both** the permutation and its inverse

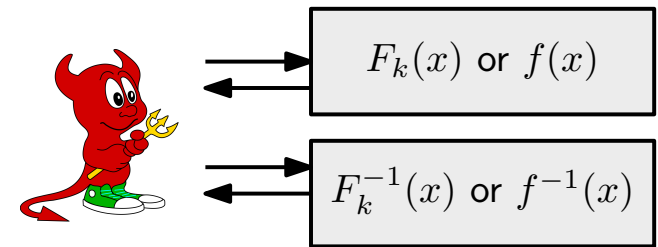


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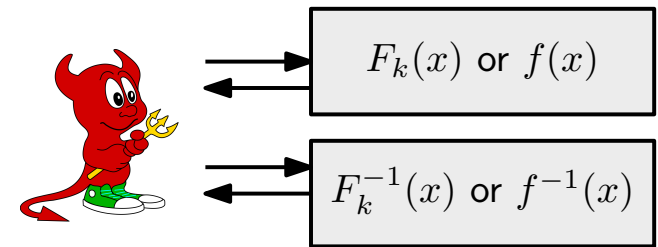
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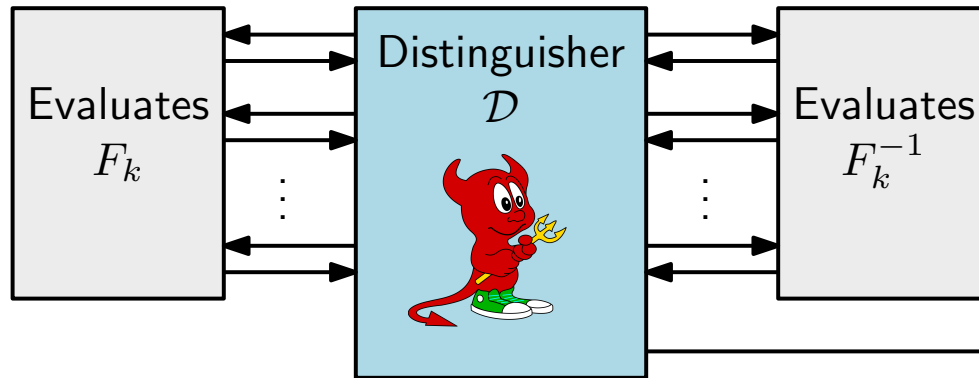


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Strong pseudorandom permutations

“World 1”: k is chosen u.a.r. in $\{0, 1\}^n$

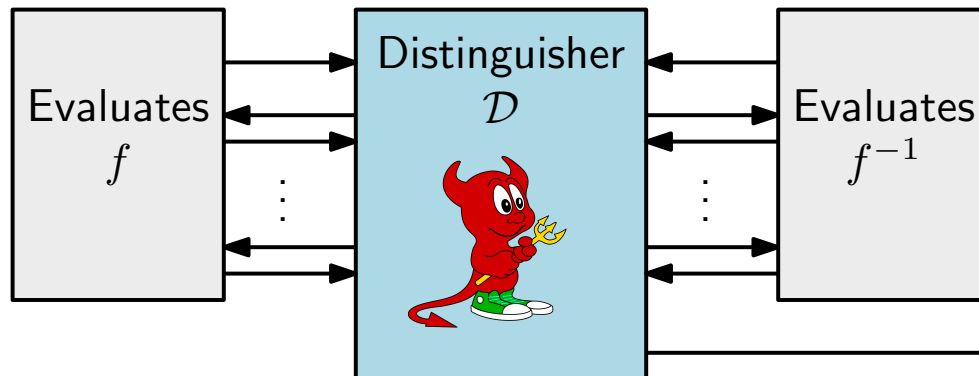


D wants to tell “World 0” apart from “World 1”

$$D^{F_k(\cdot), F_k^{-1}(\cdot)}(\mathbf{1}^n)$$

Output (0 or 1)

“World 0”: f is chosen u.a.r. in Perm_n



Denotes the kind of oracle D is interacting with

$$D^{f(\cdot), f^{-1}(\cdot)}(\mathbf{1}^n)$$

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