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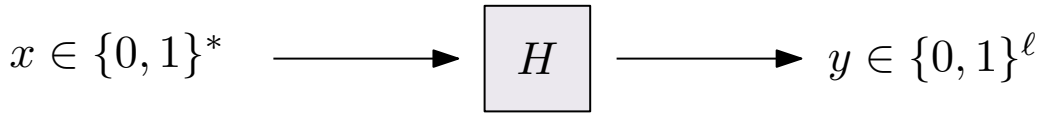

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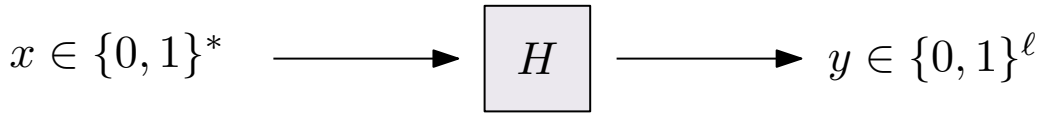

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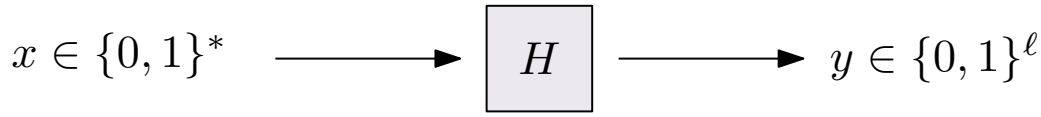

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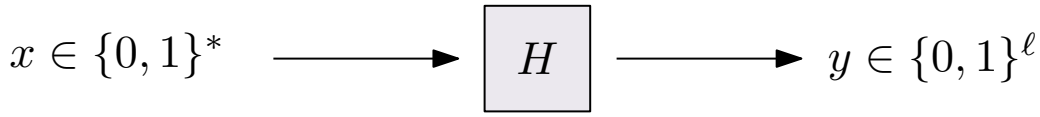

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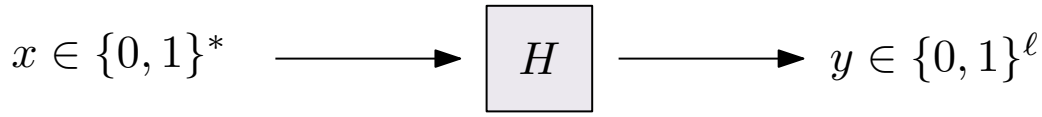

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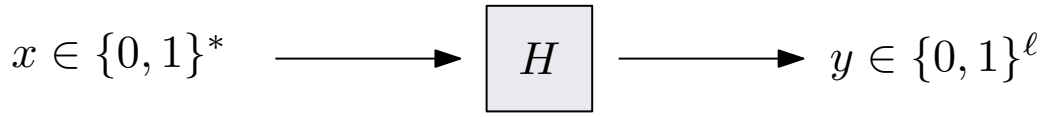

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  - **In cryptography, even few collisions are bad!**

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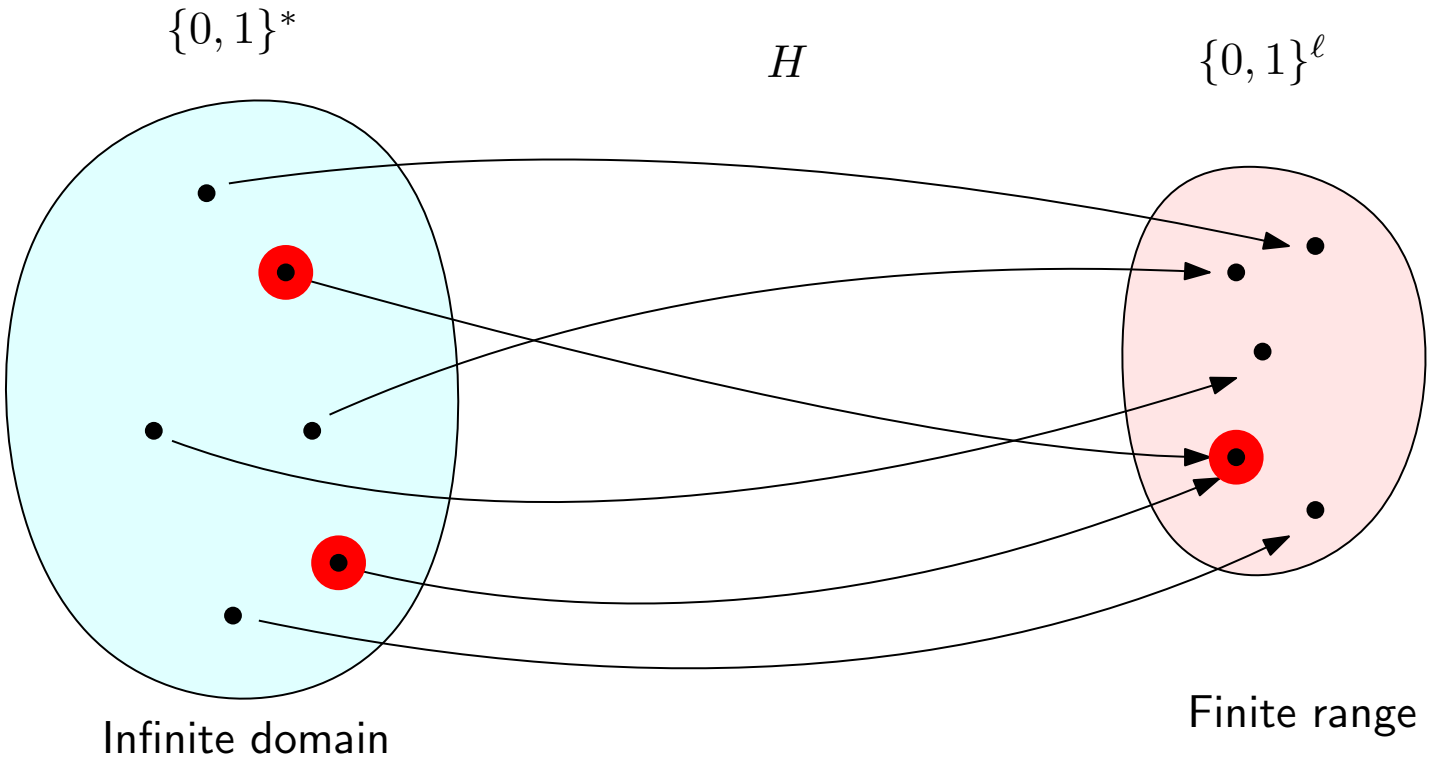

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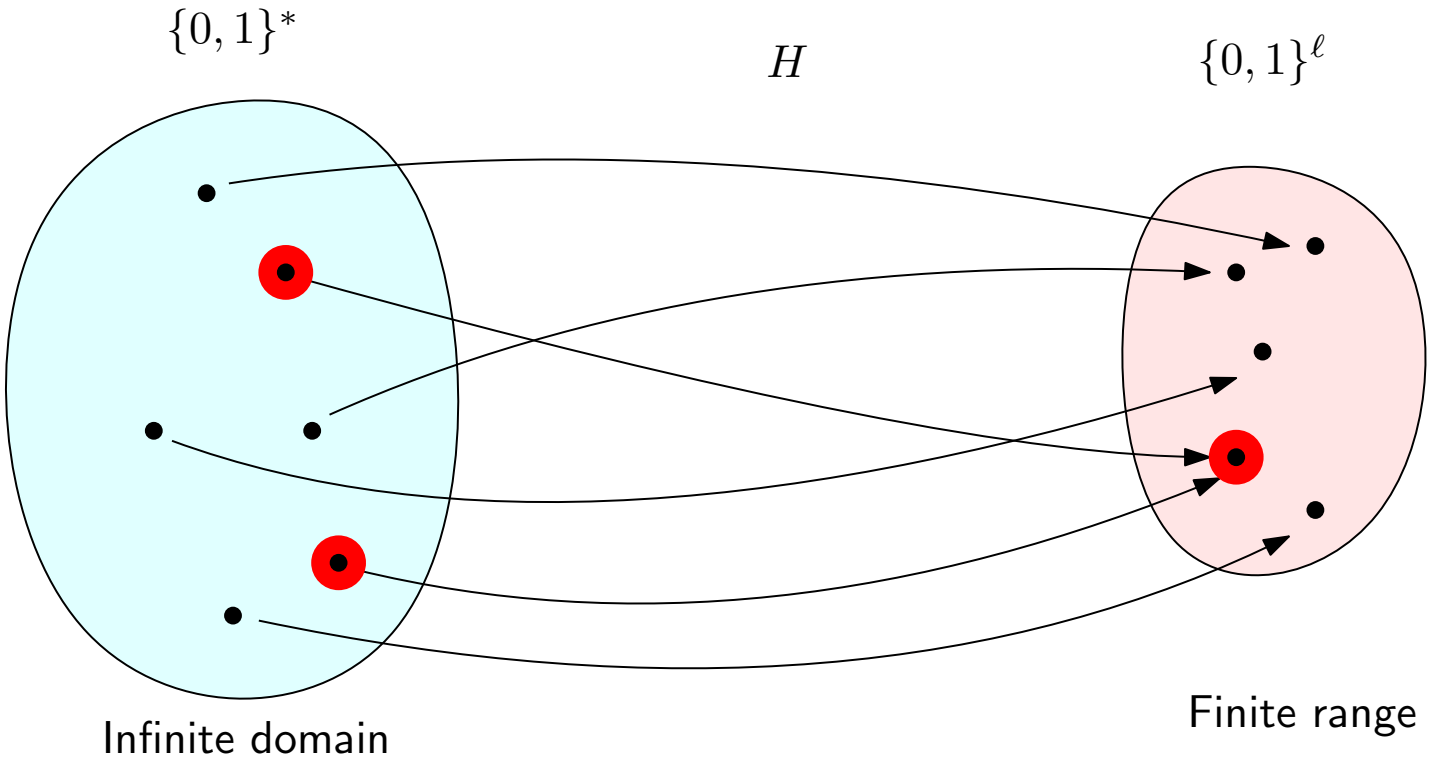

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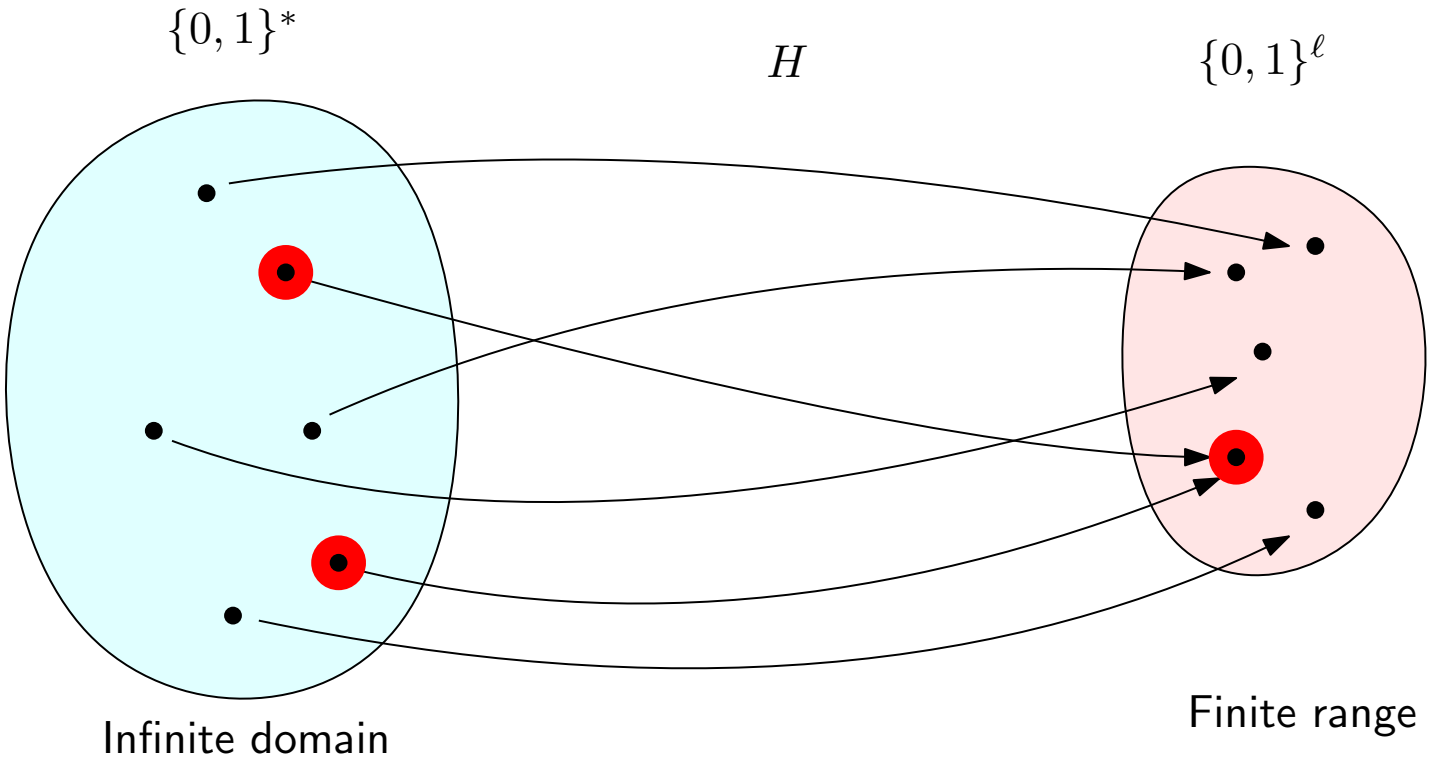

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Next best thing: Collisions are hard to find (by efficient adversaries)

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Small abuse of notation: when Gen is clear, we say that H is a hash function

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The Hash Collision experiment

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Definition: *A hash function $\mathcal{H} = (\mathsf{Gen}, H)$ is* **collision resistant** *if, for every probabilistic polynomial-time adversary $\mathcal{A}$, there is a negligible function ε such that:*

$$\Pr[\textit{Hash-coll}_{\mathcal{A},\mathcal{H}}(n) = 1] \leq \varepsilon(n)$$

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Collision resistance $\Longrightarrow$ **Second preimage resistance** $\Longrightarrow$ **Preimage resistance**

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Let $H^s : \{0,1\} \to \{0,1\}^\ell$ be **some** hash function.

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- Worst-case approach
- Model H as a random function

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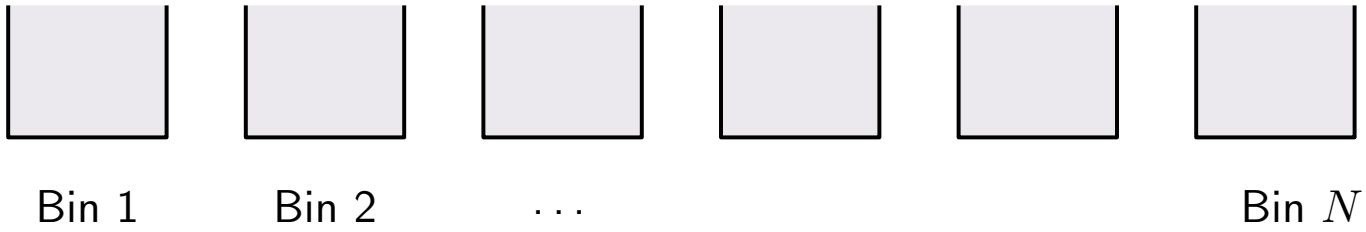
Can be thought of as a **balls into bins** experiment

Repeatedly throw a ball into a one out of N possible bins, chosen u.a.r.



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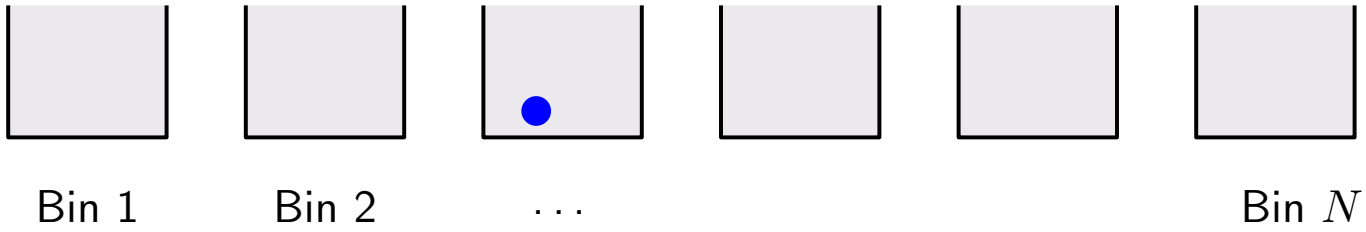

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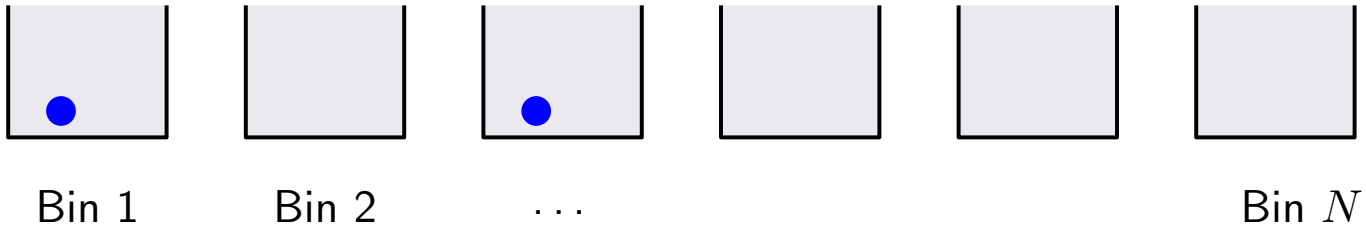

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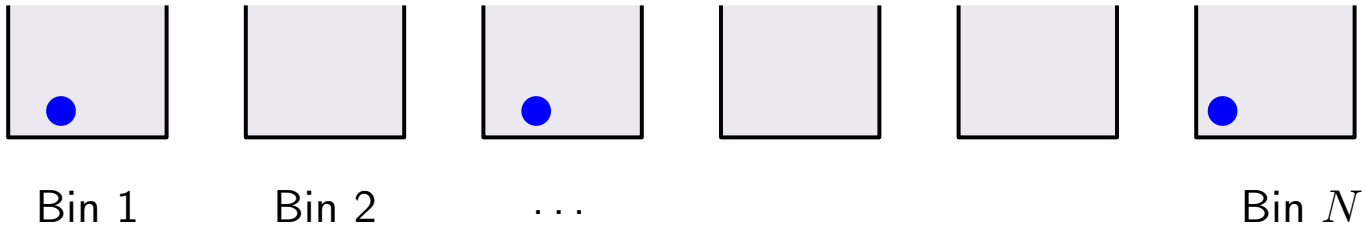

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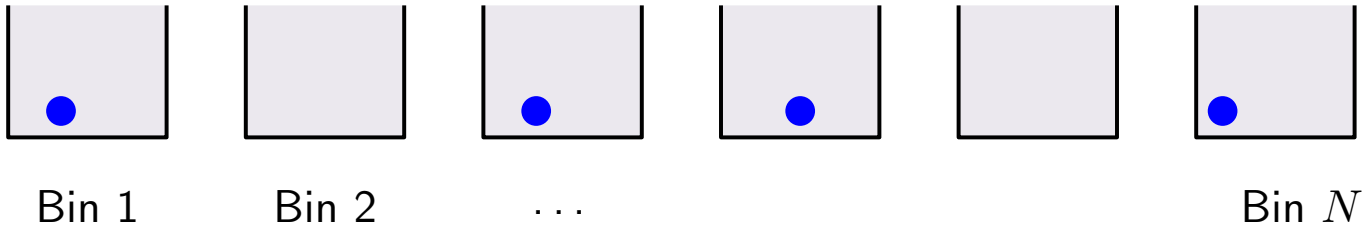

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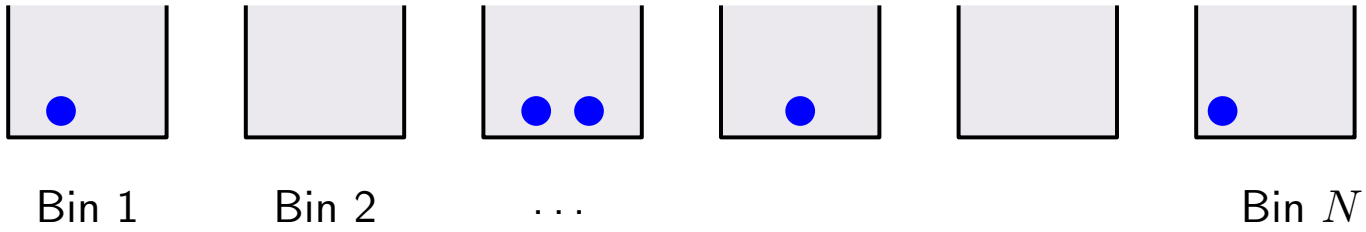

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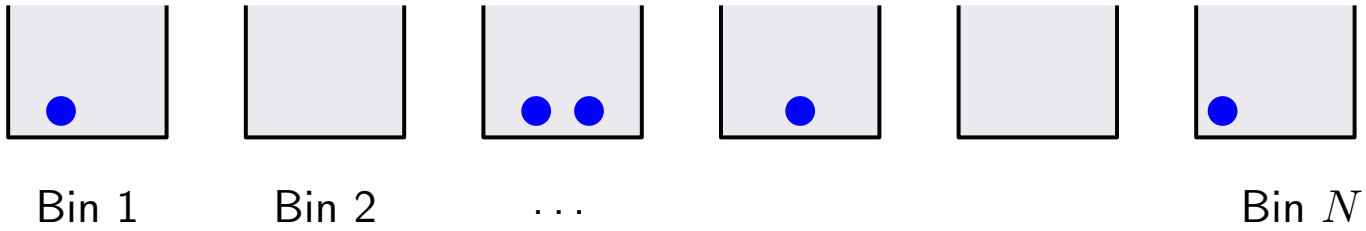

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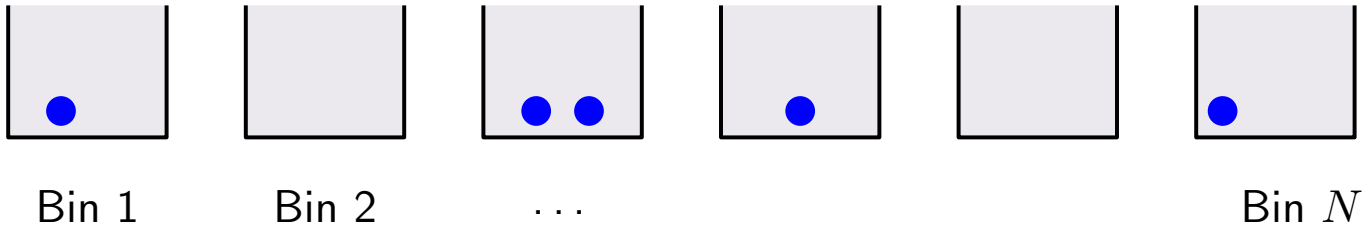

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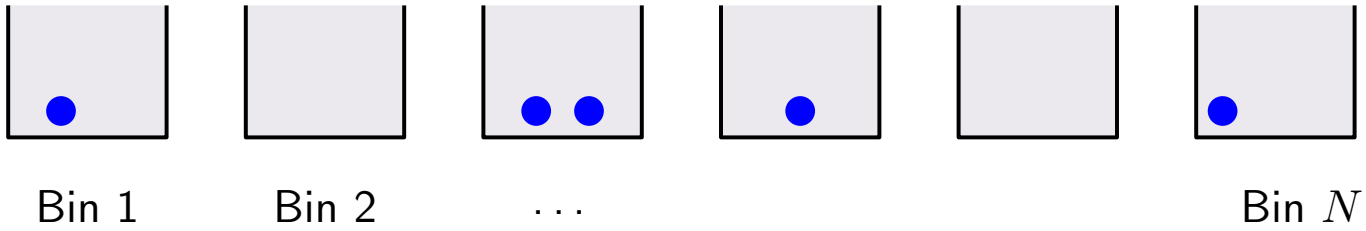

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We want to know: If we throw q balls, what's the chance that some bin contains at least 2 balls?

Probability of a collision

Theorem: Let Coll denote the event "at least one bin contains at least 2 balls".
If $q \leq \sqrt{2N}$, then $\frac{q(q-1)}{4N} \leq \Pr[\text{Coll}] \leq \frac{q(q-1)}{2N}$.

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- For block ciphers, if the key length was n, we wanted the best attack to take time $\approx 2^n$
- For hash functions, if we want to withstand attacks running in time $\approx 2^n$ we need $\ell \geq 2n$

Birthday Attack: Finding Meaningful Collisions

The collisions found by the birthday attack do not seem very useful

- The colliding inputs are random binary strings

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$A =$ {Today, This morning} I {took, went for} a {walk, stroll} in the city {center, park}. While there, I {had, drank} {a coffee, an espresso} and ate a {cream, sweet} {doughnut, donut}. $|A| = 2^8$

$B =$ This is to {inform, notify} you that I am {resigning, quitting} from my {job, position} {effective immediately, at once}. Please {give, send} me my {final, last} paycheck as {soon, quickly} as possible. {Goodbye, Regards}. $|B| = 2^8$

(Collision Resistant) Hash Functions... do they even exist?

Let H^s be **any** function that is computable in polynomial-time

Consider the following decision problem $C^s(\alpha, \beta)$:

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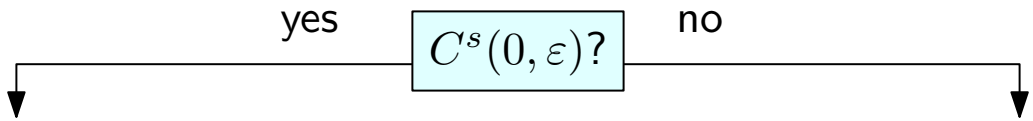

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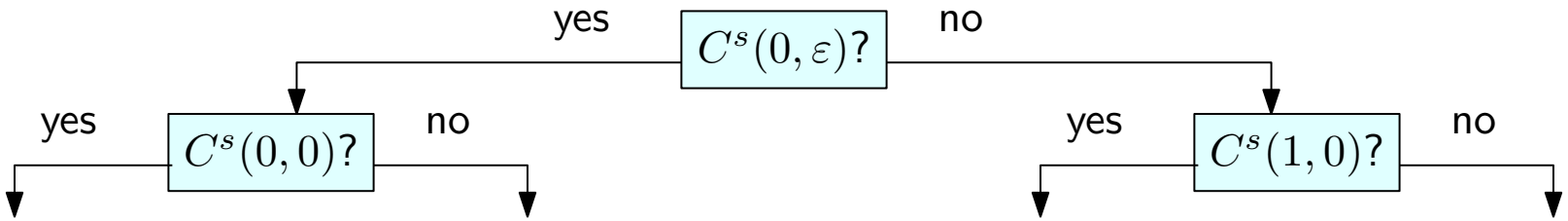

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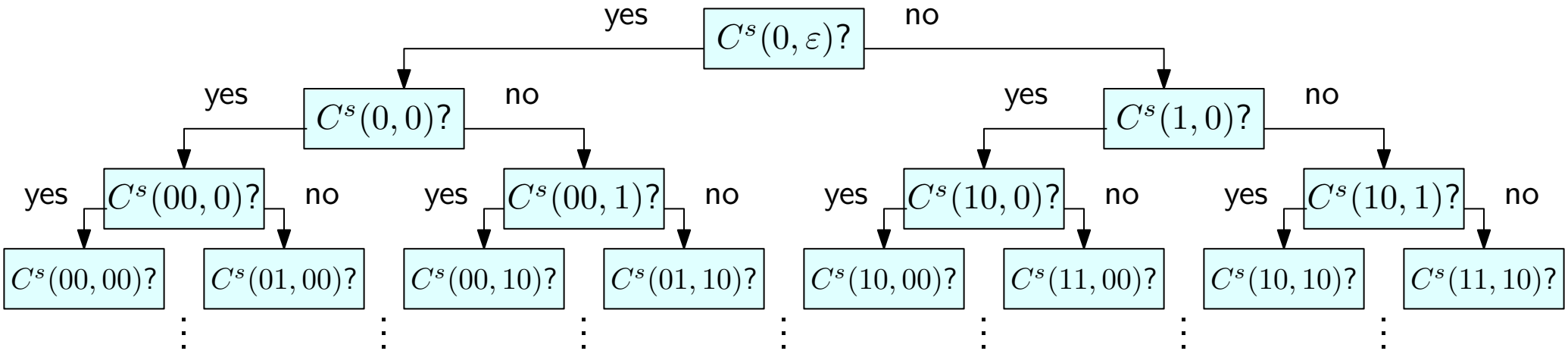

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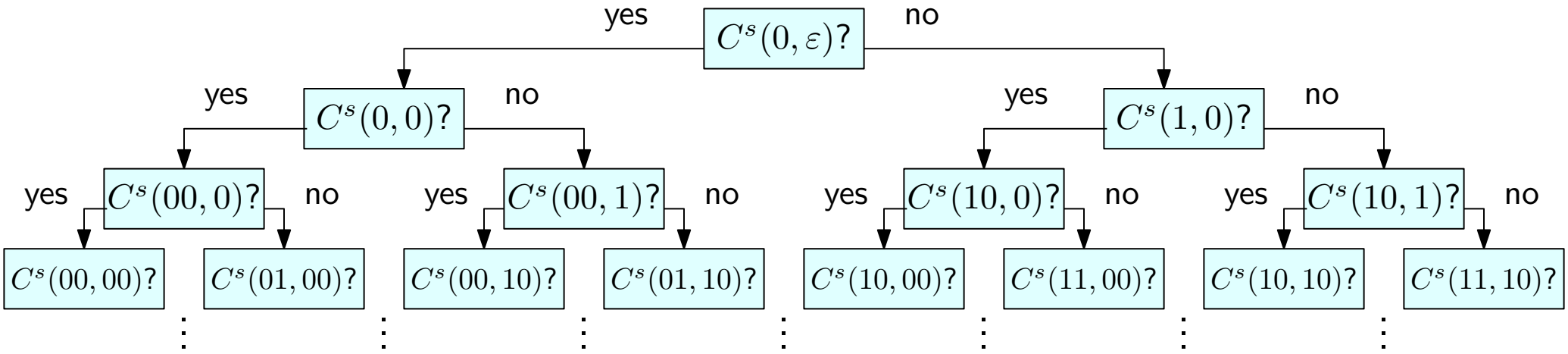

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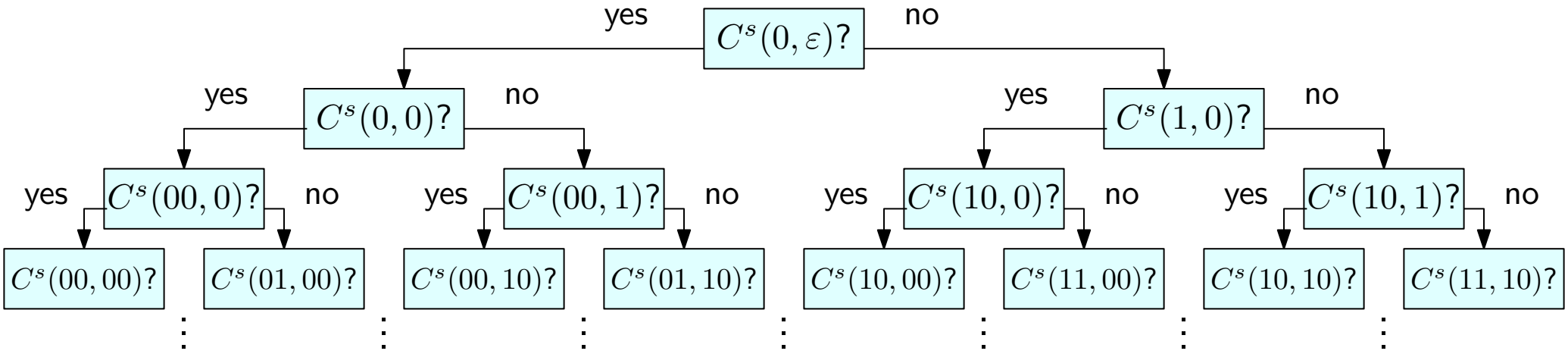

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Pragmatic approach: Pretend that Hash functions exist & use practical constructions

Constructing a Hash Function

Step 1: Start with a collision-resistant compression function for short, fixed-length, inputs

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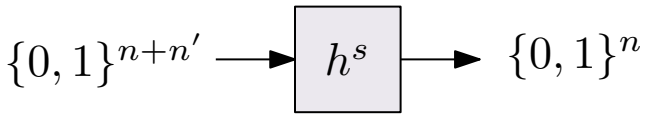

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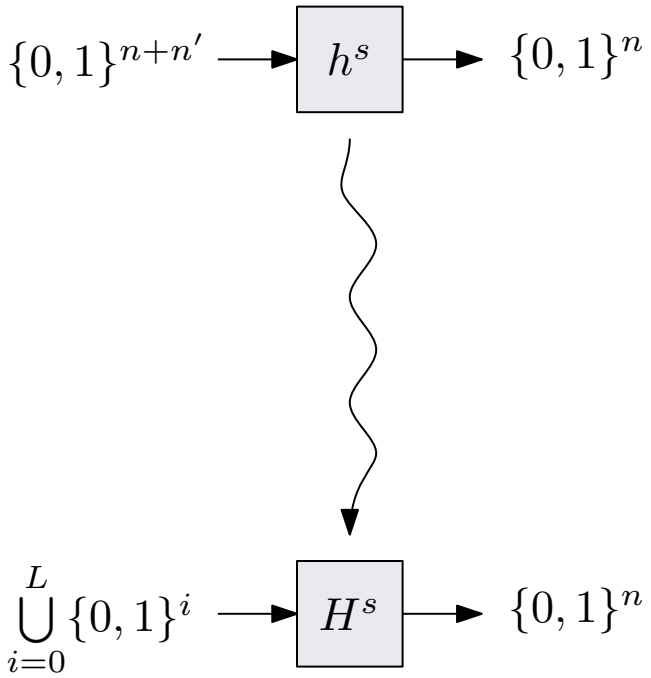

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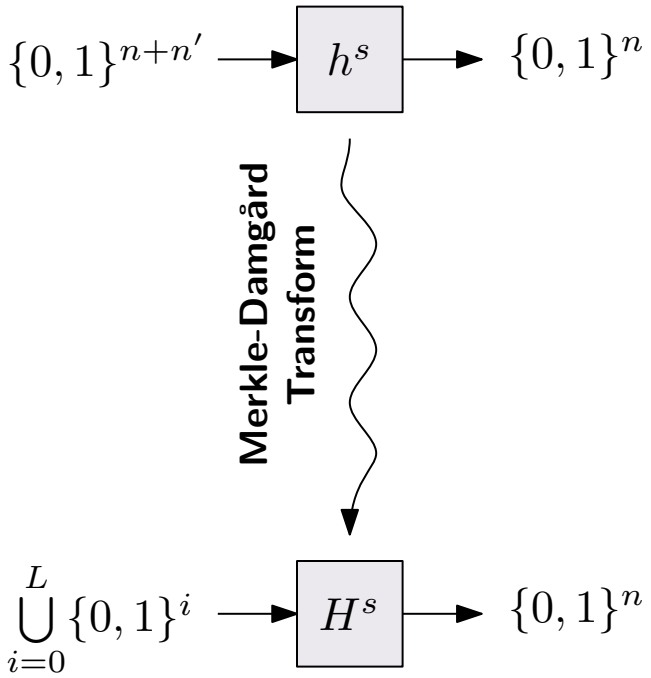

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Merkle-Damgård Transform

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- Compute $H^s(x)$ by repeatedly evaluating h^s

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Pick some fixed parameter $\lambda \leq n'$, IV $\in \{0,1\}^n$. For $x \in \{0,1\}^*$ with $|x| < 2^\lambda$ (and key s) compute $H^s(x)$ as follows:

- Pad x so that it also encodes $|x|$ and the new length is a multiple of n'

  — Append a 1 to x followed by as many 0s as needed to obtain a length that is λ less than a multiple of n'

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  — Parse the resulting string as a concatenation of B blocks $x_1 \,\|\, x_2 \,\|\, \ldots \,\|\, x_B$ where $|x_i| = n'$.

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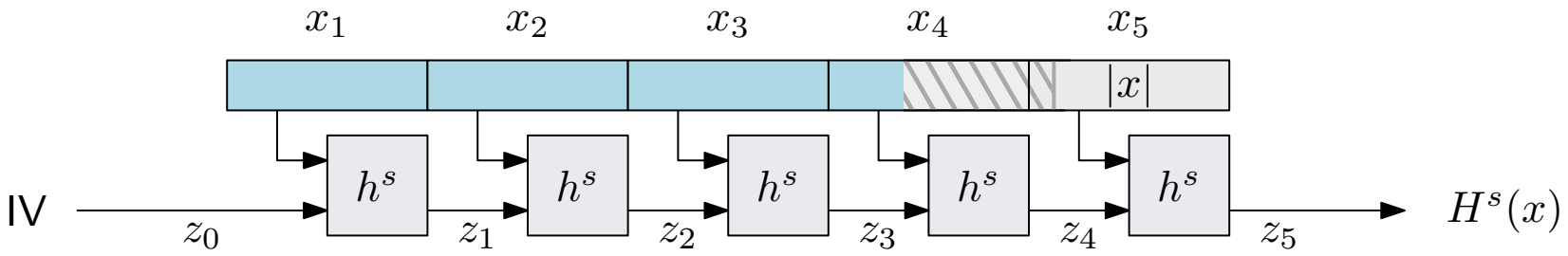

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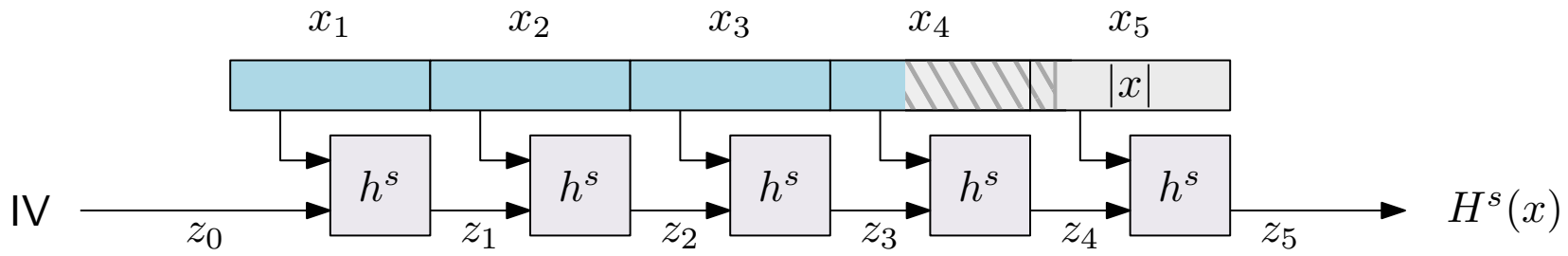

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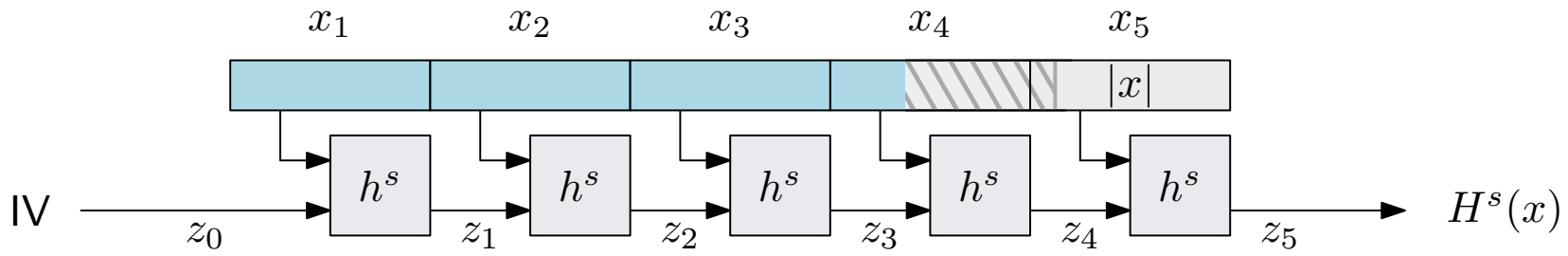

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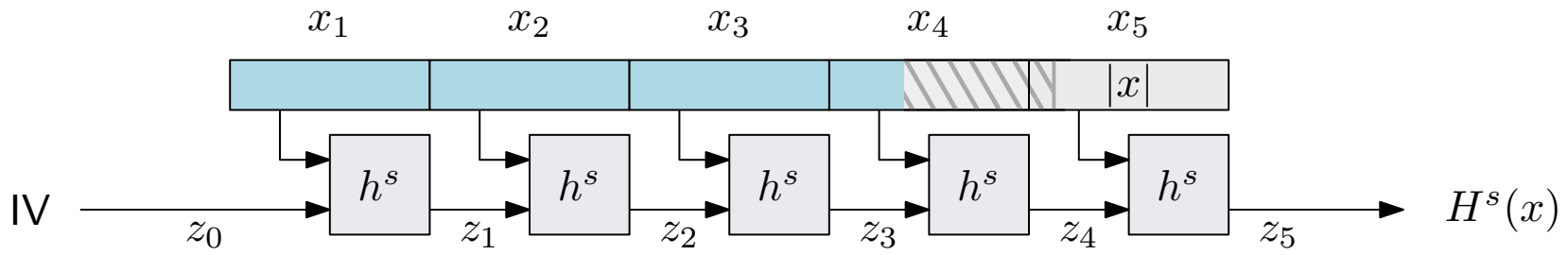

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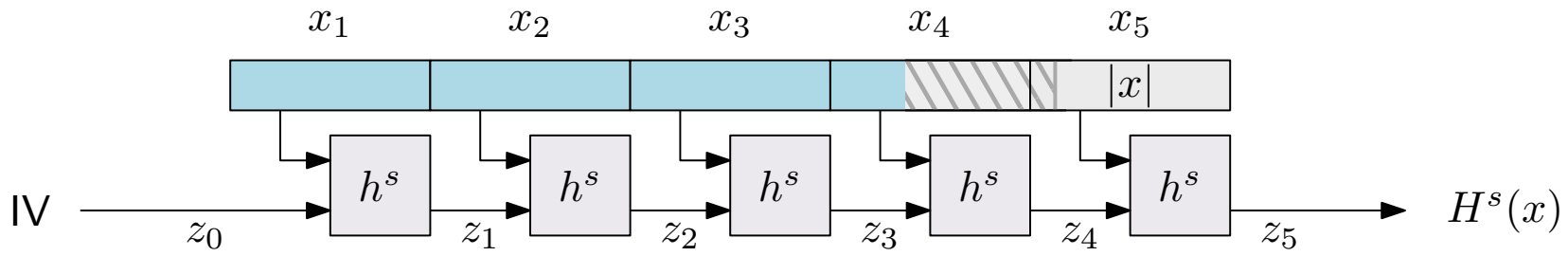

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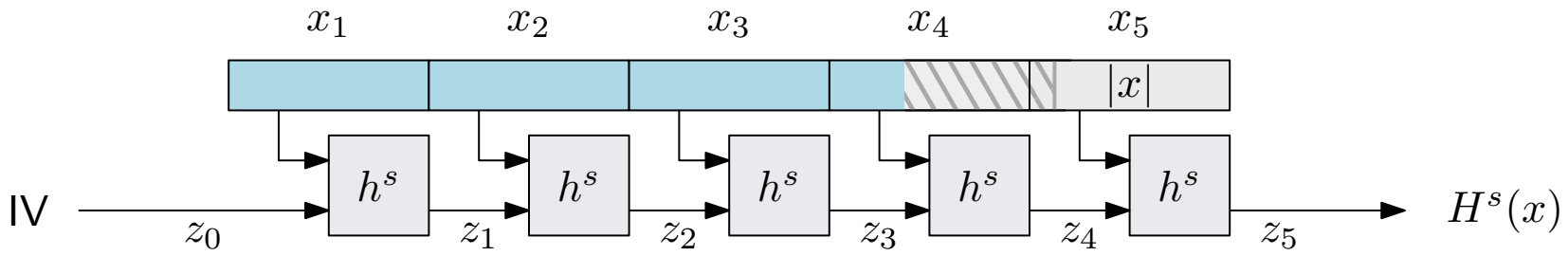

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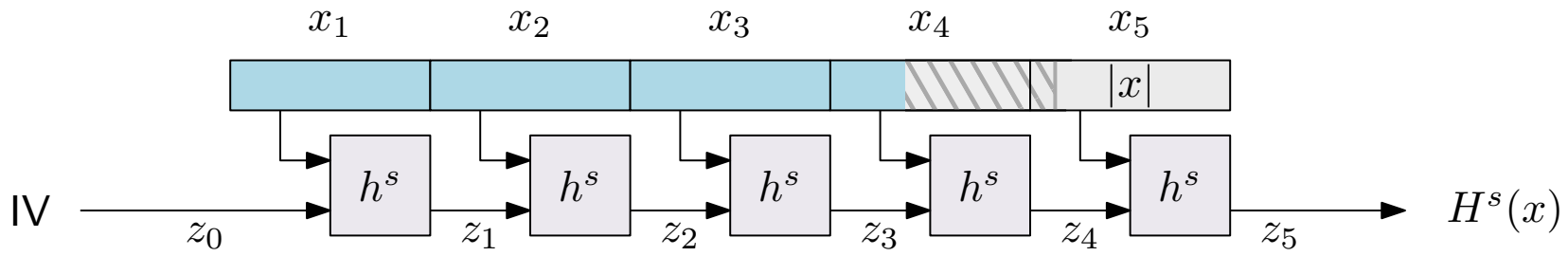

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We have $h^s(z_{B-1} \| x_B) = h^s(z'_{B-1} \| x'_{B'})$, and $z_{B-1} \| x_B \neq z'_{B-1} \| x'_{B'}$ (since $x_B \neq x'_B$)

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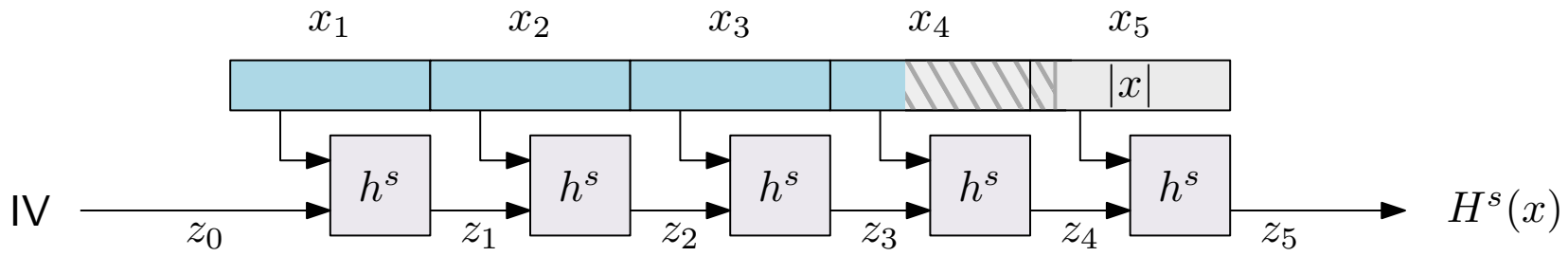

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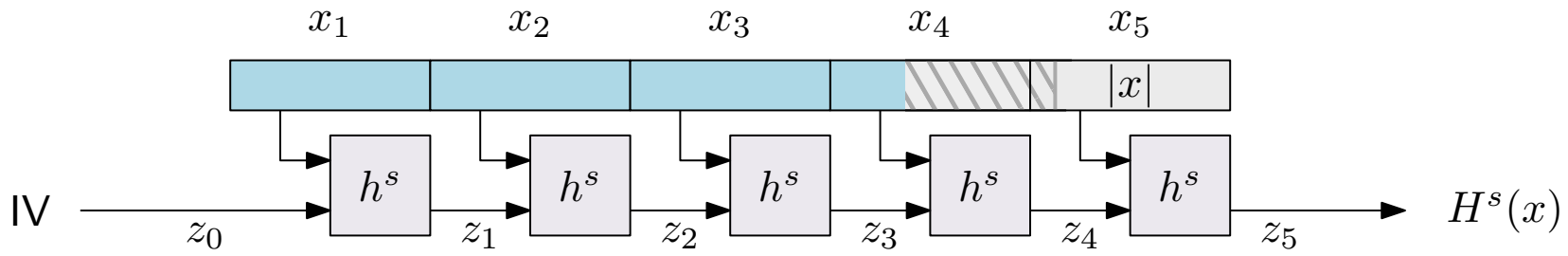

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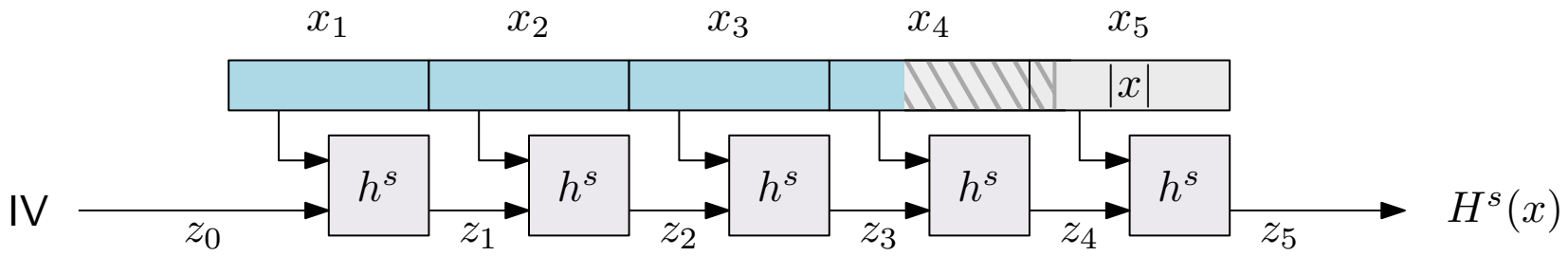

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Then $h^s(z_{i-1}\|x_i) = z_i = z'_i = h^s(z'_{i-1}\|x'_i)$

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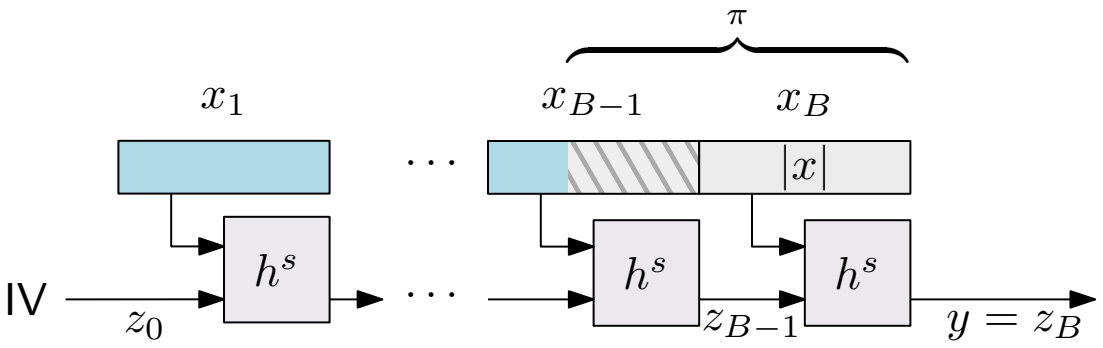

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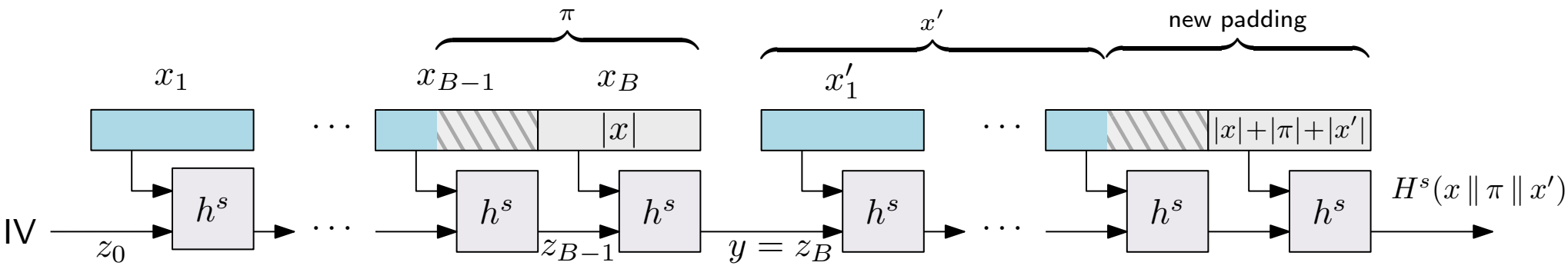

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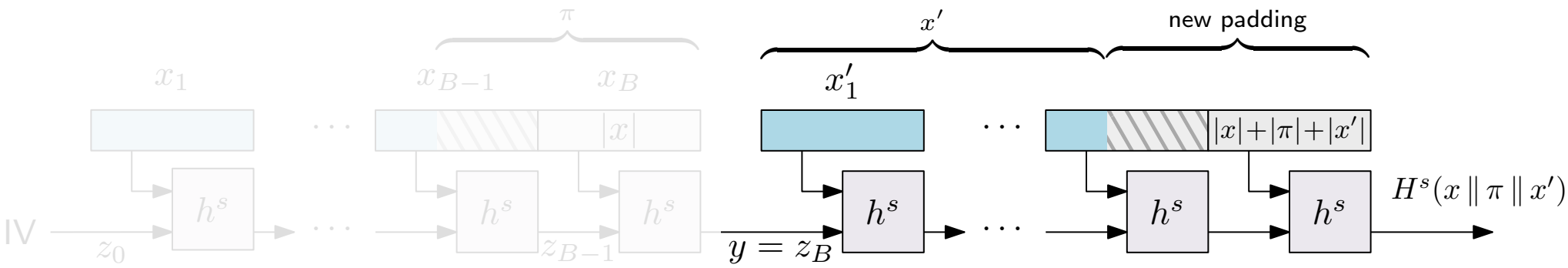

Hash Function in Practice

Practical construction of hash functions are unkeyed...

MD4
- 128 bit digest
- Birthday attack (μs), Preimage attack (theoretical)

MD5
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SHA1
- 160 bit digest
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- Actually a family of algorithms: 224, 256, 384, and 512 bit digests
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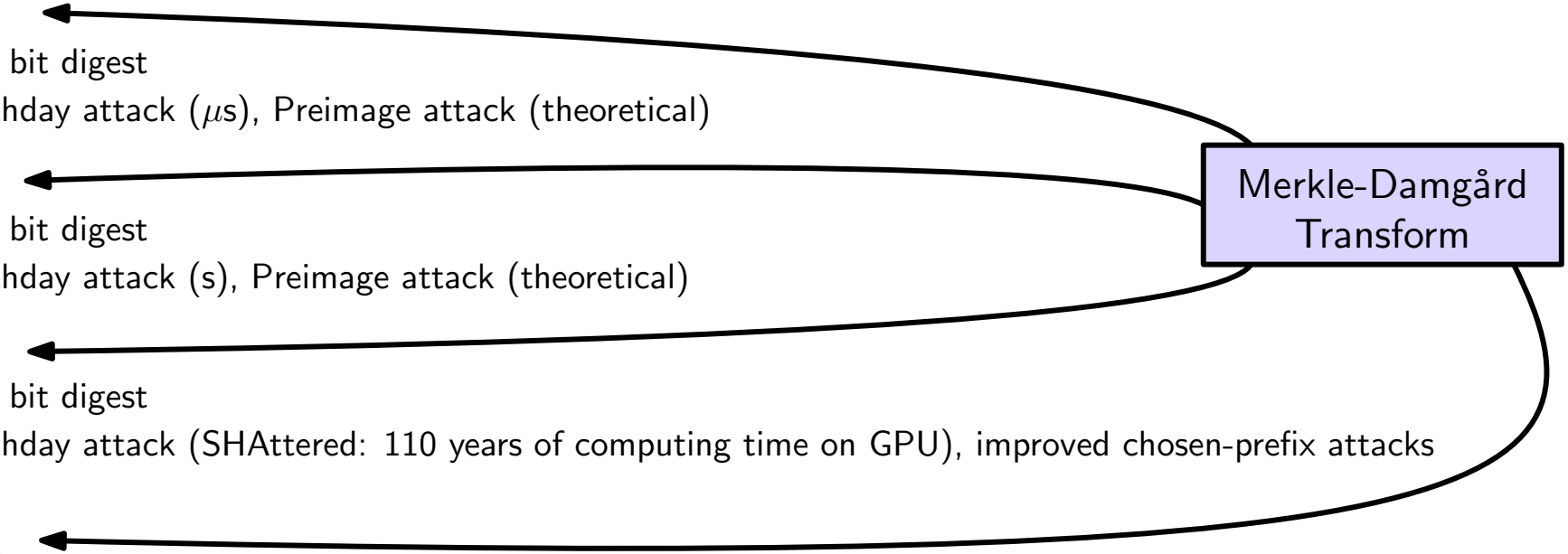

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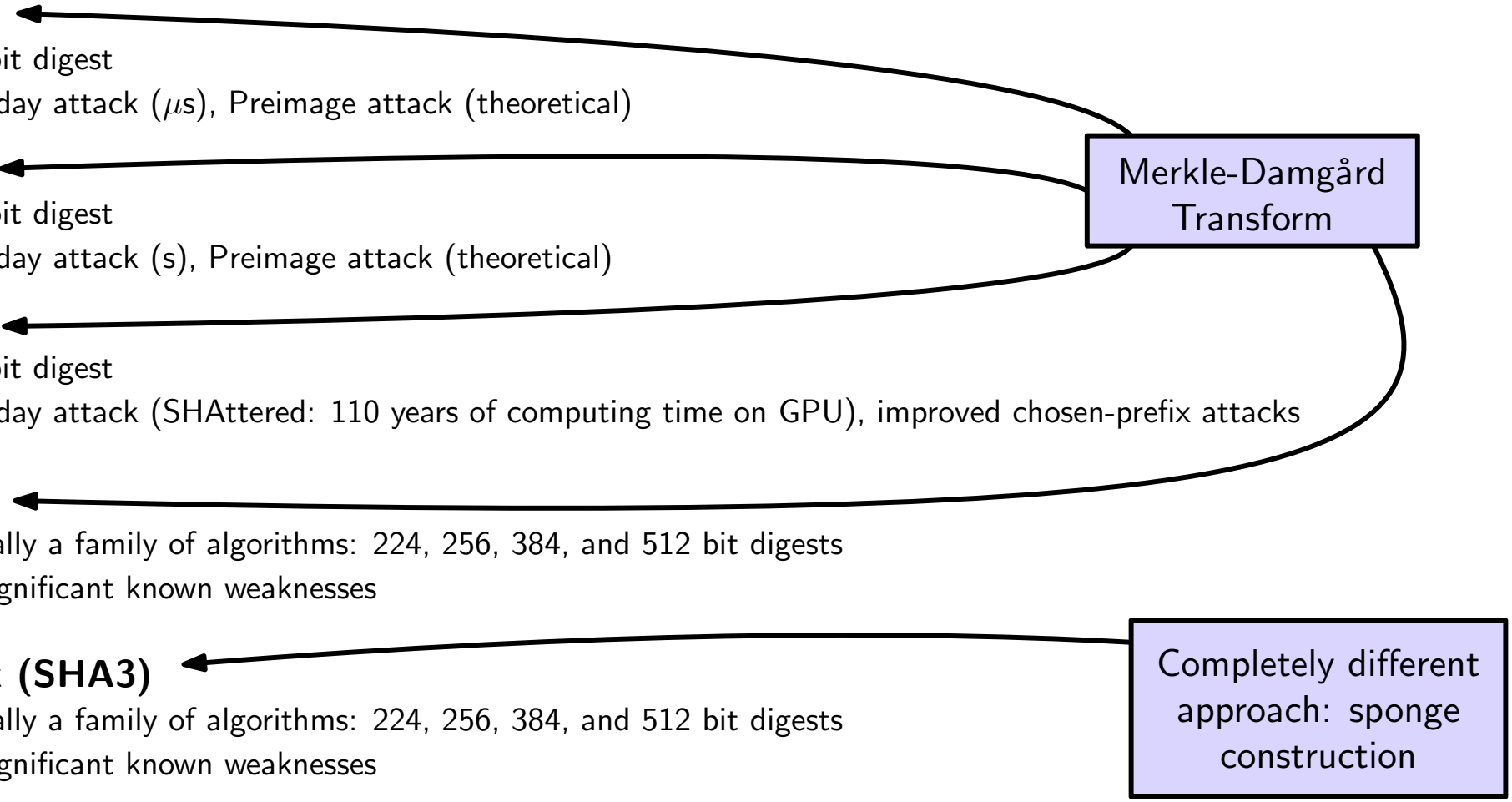

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Merkle-Damgård Transform

Completely different approach: sponge construction

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Hash-and-Mac

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Theorem: if Π' is a secure MAC for messages of length ℓ and $\mathcal{H}$ is collision resistant, then the hash-and-mac construction Π is a secure MAC.

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We will show that an adversary $\mathcal{A}$ that breaks the security of Π can be used to either break the security of Π' or to find a collision in $\mathcal{H}$ (possibly both).

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At least one of the summands is non-negligible

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This contradicts the collision resistance of $\mathcal{H}$!

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$\Pr[\text{Mac-forge}_{\mathcal{A}'',\Pi'}(n)] \geq \Pr[\text{Mac-forge}_{\mathcal{A},\Pi}(n) \wedge \overline{\text{coll}}]$ This contradicts the unforgeability of Π'. □

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In practice:

- Prove security in the Random Oracle model
- Replace the Random Oracle with a concrete hash function
- Cross your fingers...

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- There are no known "natural" schemes that have been attacked while proven secure in the Random Oracle model
- A security proof is in the random oracle model is better than no security proof at all... maybe?

Applications of Hash Functions: Fingerprinting & Deduplication

If H is a collision-resistant hash function, and x is a (part of) a file, then we can think of $H(x)$ as a unique identifier of that (part of the) file

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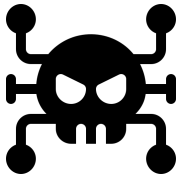

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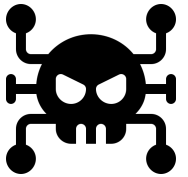

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Peer-to-peer file sharing: Hashes are used to uniquely identify files (and chunk of files) in peer-to-peer file-sharing networks.

`magnet:?xt=urn:btih:C9A337562CB0360FD6F5AB40FD2B1B81D5325DBD`

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Storing a password as a plaintext is dangerous!

- We can instead store a hash $y = H(x)$ of the password x.
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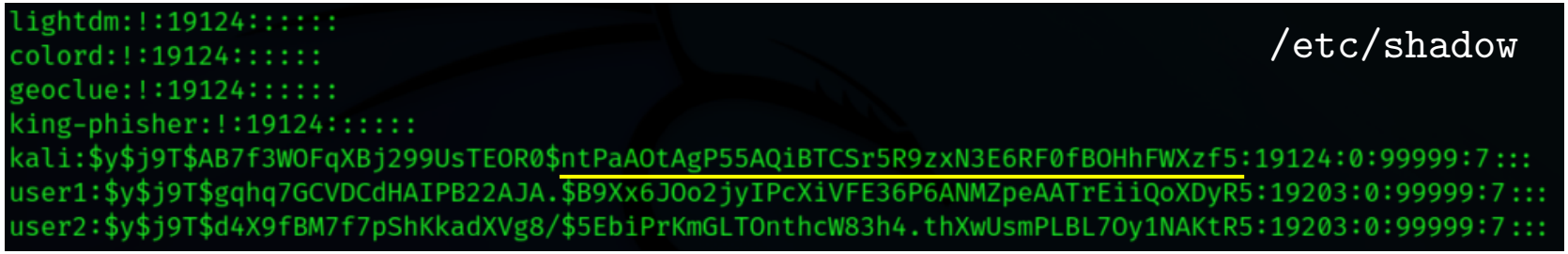

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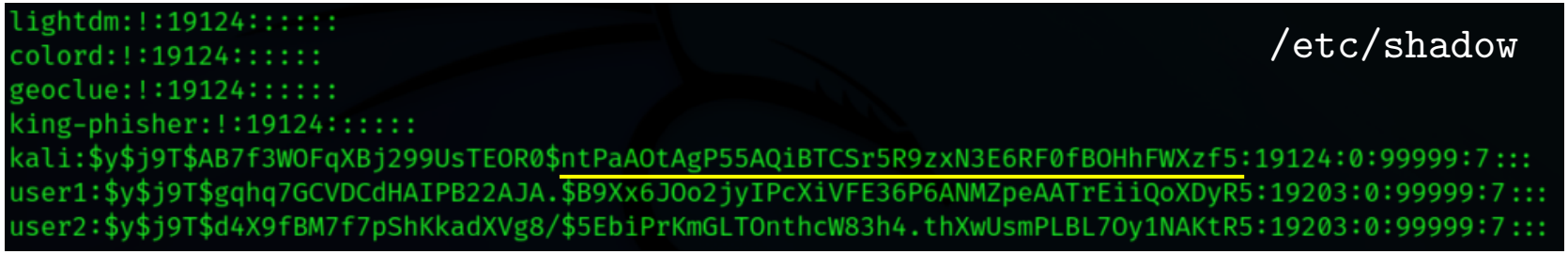

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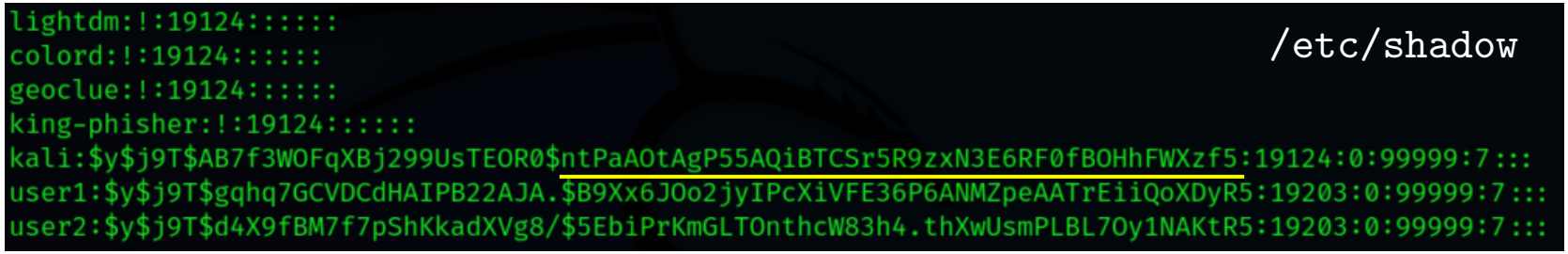

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/etc/shadow

```
lightdm:!:19124::::::
colord:!:19124::::::
geoclue:!:19124::::::
king-phisher:!:19124::::::
kali:$y$j9T$AB7f3WOFqXBj299UsTEOR0$ntPaAOtAgP55AQiBTCSr5R9zxN3E6RF0fBOHhFWXzf5:19124:0:99999:7:::
user1:$y$j9T$gqhq7GCVDCdHAIPB22AJA.$B9Xx6JOo2jyIPcXiVFE36P6ANMZpeAATrEiiQoXDyR5:19203:0:99999:7:::
user2:$y$j9T$d4X9fBM7f7pShKkadXVg8/$5EbiPrKmGLTOnthcW83h4.thXwUsmPLBL7Oy1NAKtR5:19203:0:99999:7:::
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king-phisher:!:19124::::::
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user1:$y$j9T$gqhq7GCVDCdHAIPB22AJA.$B9Xx6JOo2jyIPcXiVFE36P6ANMZpeAATrEiiQoXDyR5:19203:0:99999:7:::
user2:$y$j9T$d4X9fBM7f7pShKkadXVg8/$5EbiPrKmGLTOnthcW83h4.thXwUsmPLBL7Oy1NAKtR5:19203:0:99999:7:::
```

- If an attacker learns y, it still cannot efficiently recover x... assuming that x is a good password!
- What if x is bad password? E.g., what if x is an English word?
- We can easily check $H(x') = y$ for all English words x'.
- In fact, we can **store** all $H(x')$ in a **rainbow table**, to recover x in seconds!

- **Solution:** pick a random string z called **salt**.
  Compute $y = H(z\|x)$ and store the pair (z, y).

Applications of Hash Functions: Key Derivation

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Choose $k = H(x)$

- If H is a random oracle, then $H(x)$ is uniform as long as the attacker does not query H with x.
- If an attacker makes q queries to $H(\cdot)$, it will query H with x with probability at most $q \cdot 2^{-m}$.

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A commitment scheme allows a party to

- Commit to a value m
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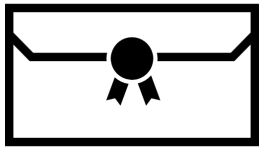

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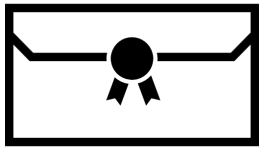

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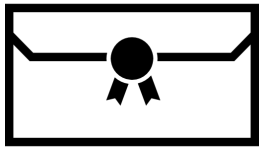

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- Pick a random string r and compute com $= H(\langle m, r \rangle)$

To open the commitment:

- Send m and r. Given some m' and r' one can easily check whether com $\stackrel{?}{=} H(\langle m', r' \rangle)$

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Solution 3: Merkle trees

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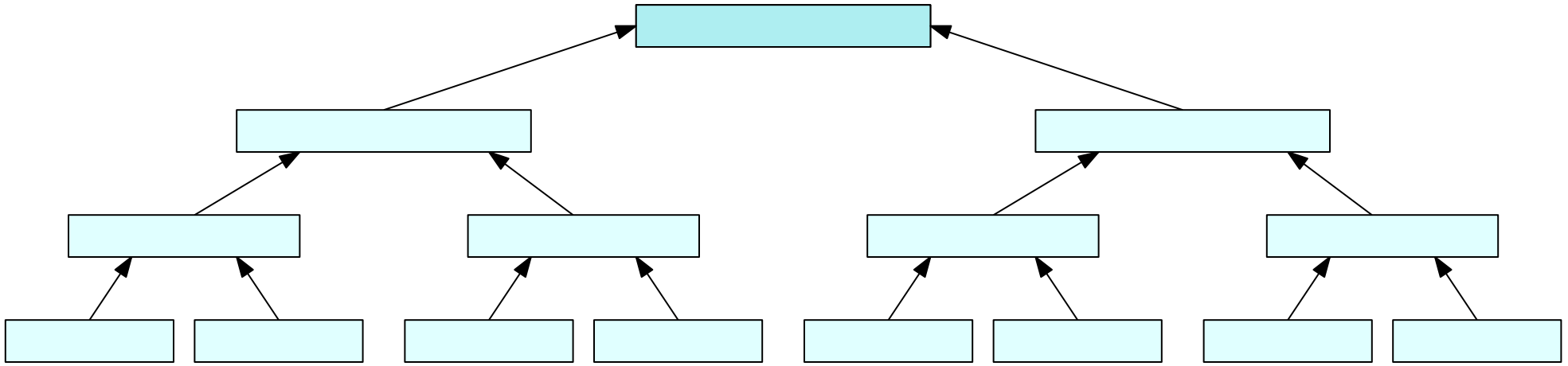
- Build a complete binary tree with t leaves
- Each node u stores a hash



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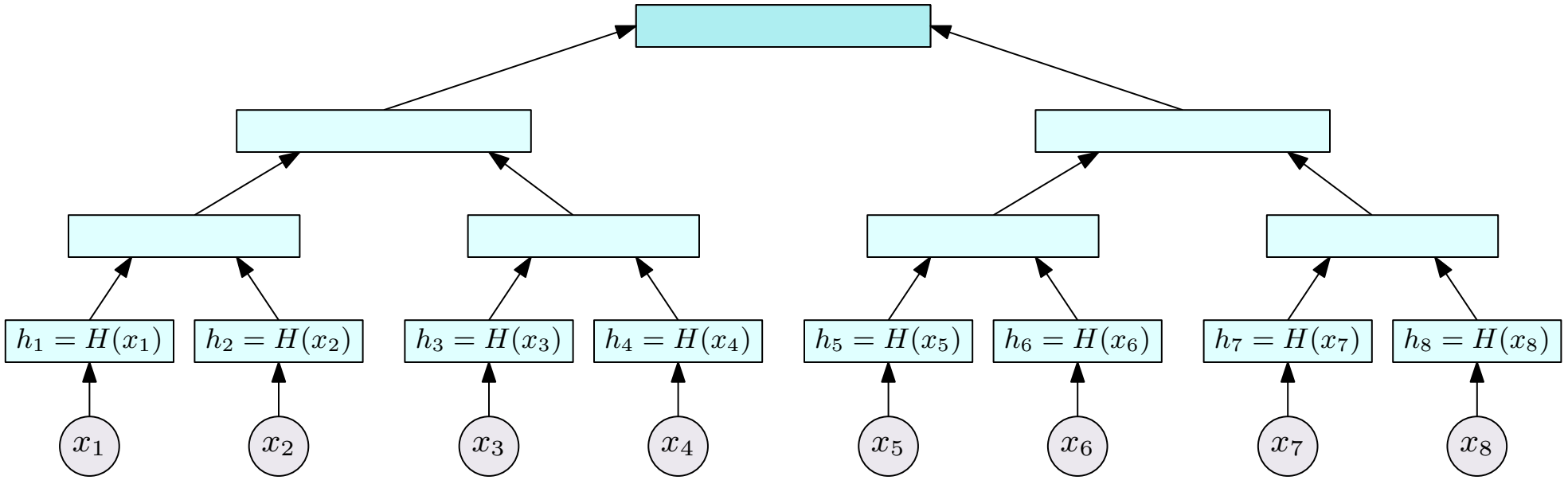

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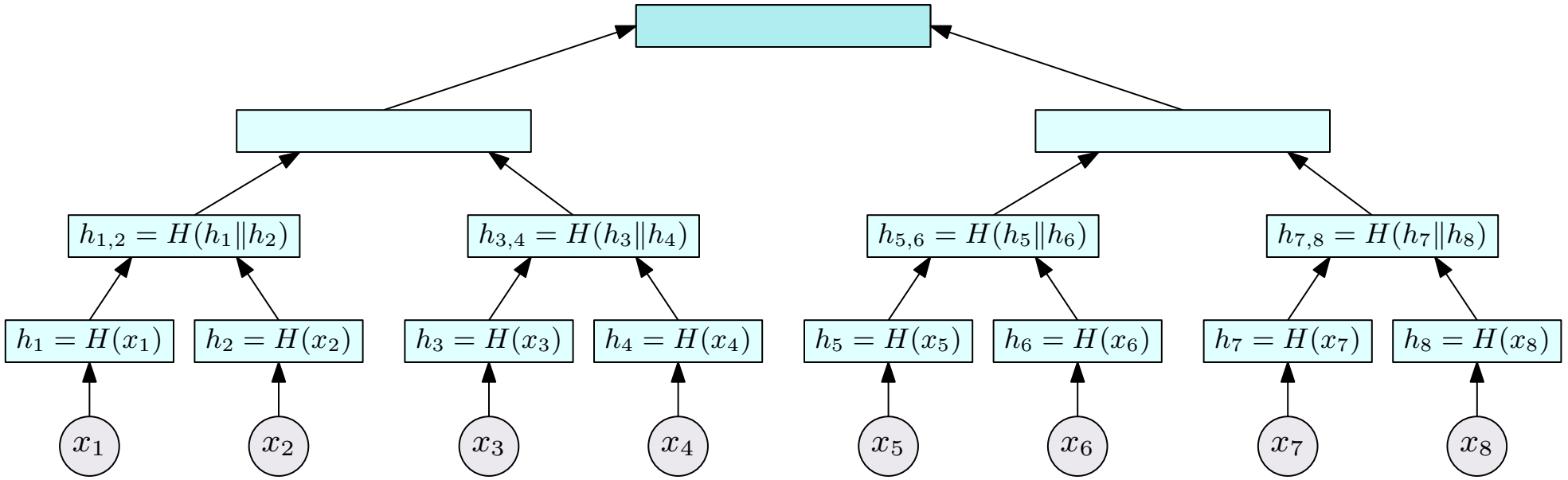

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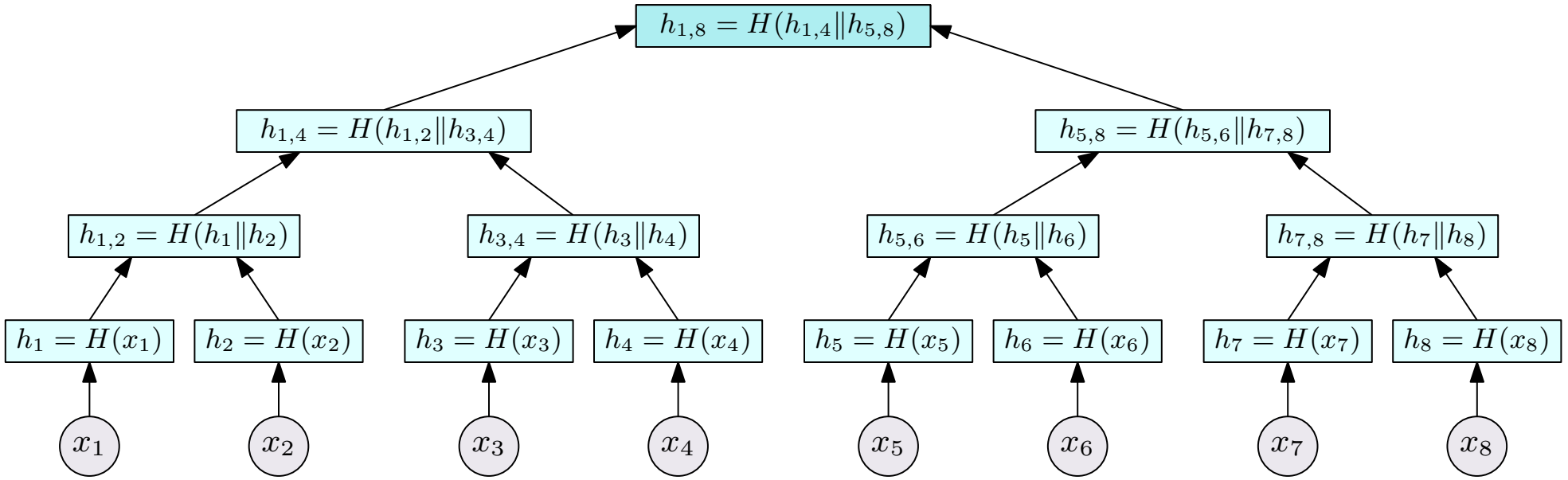
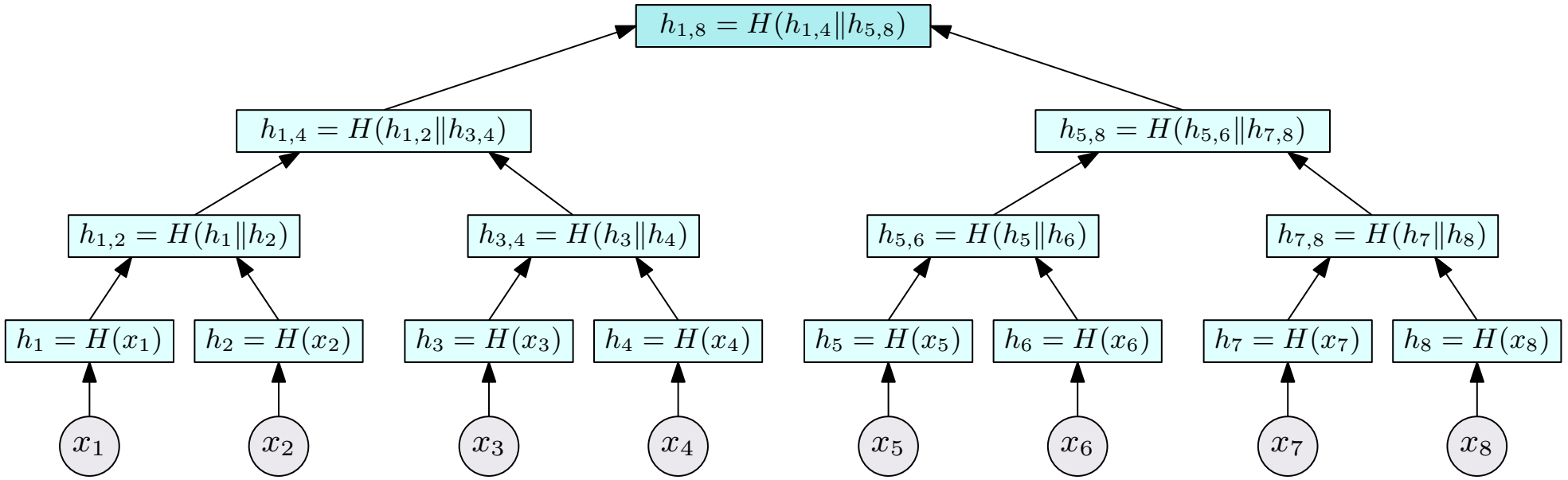


Applications of Hash Functions: Merkle Trees

- Build a complete binary tree with t leaves
- Each node u stores a hash
- Alice only saves (locally) only the hash stored in the root
- The hash stored in the i-th leaf is $H(x_i)$
- The hash stored in an internal node with u and v as children is $H(h_u \| h_v)$



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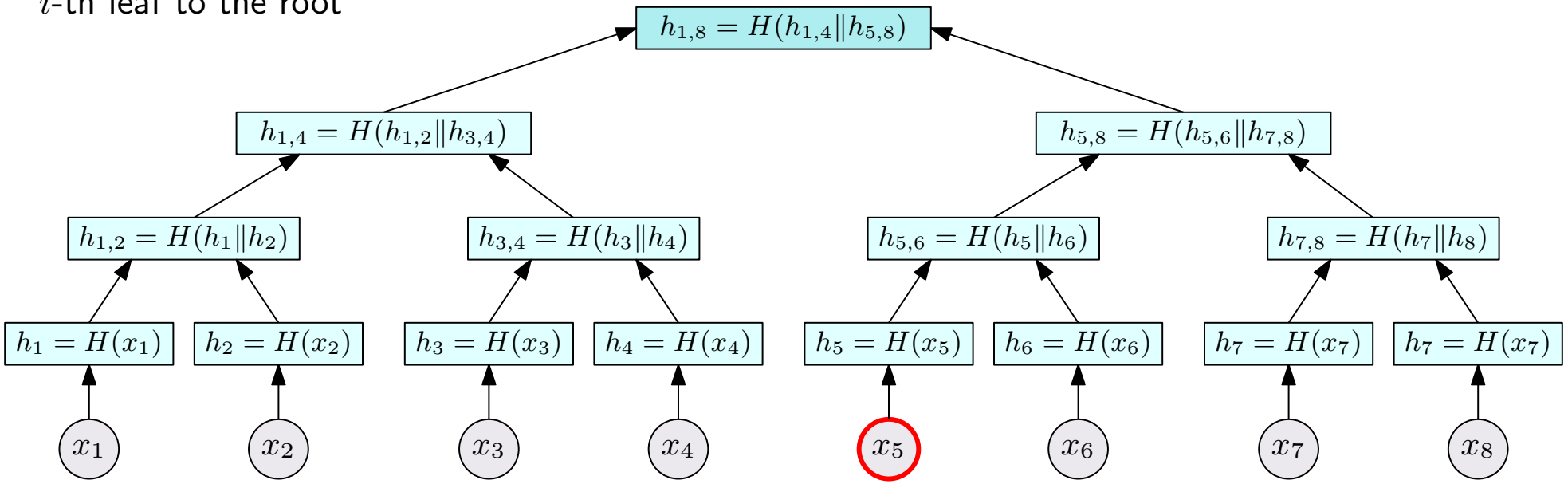
To convince Alice that x_i has not been altered the server sends a **proof of inclusion**:

- The proof consists of i, x_i and of the hashes of all the siblings of the vertices in the path from the i-th leaf to the root



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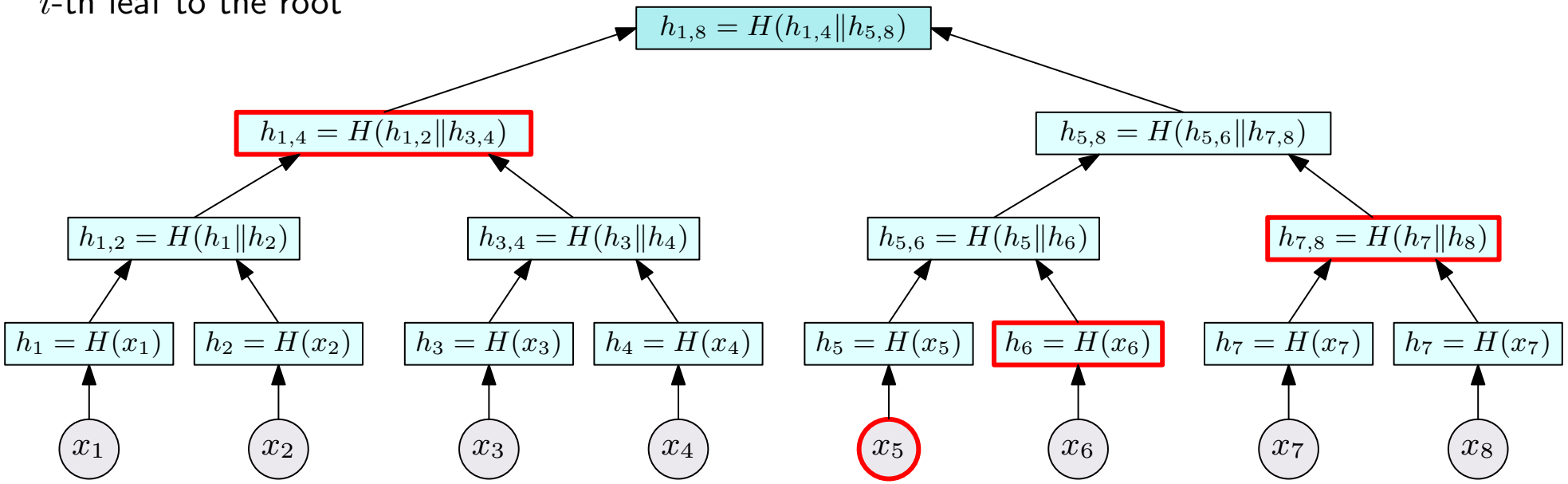

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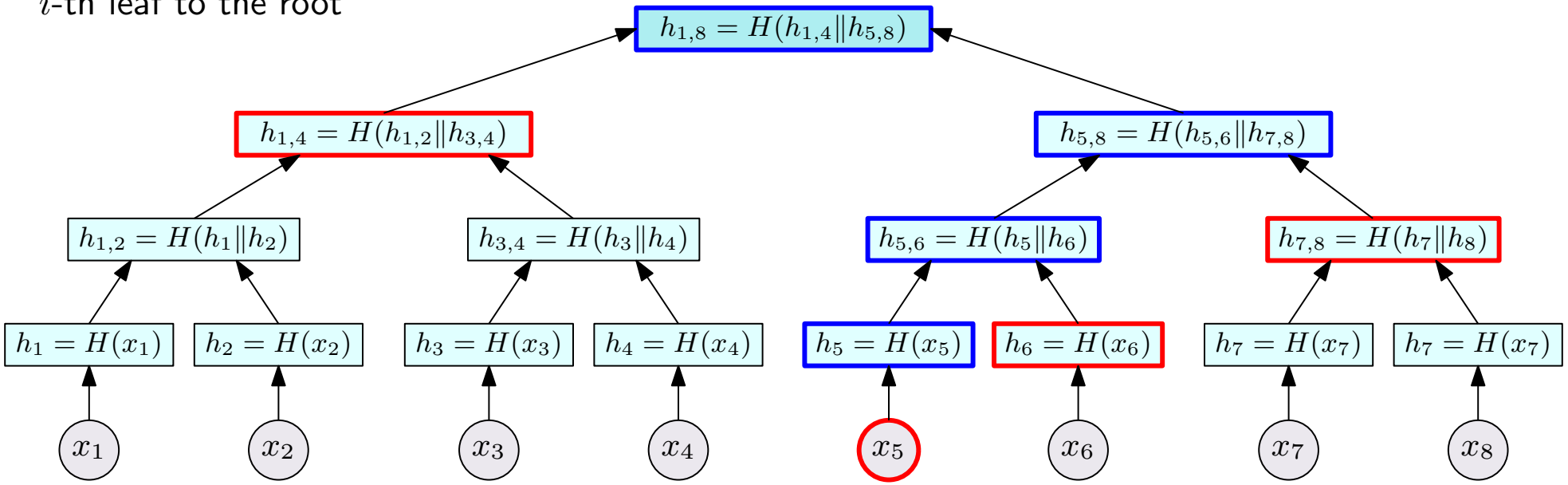

- Alice can recompute the root hash, and check that it matches the one saved locally

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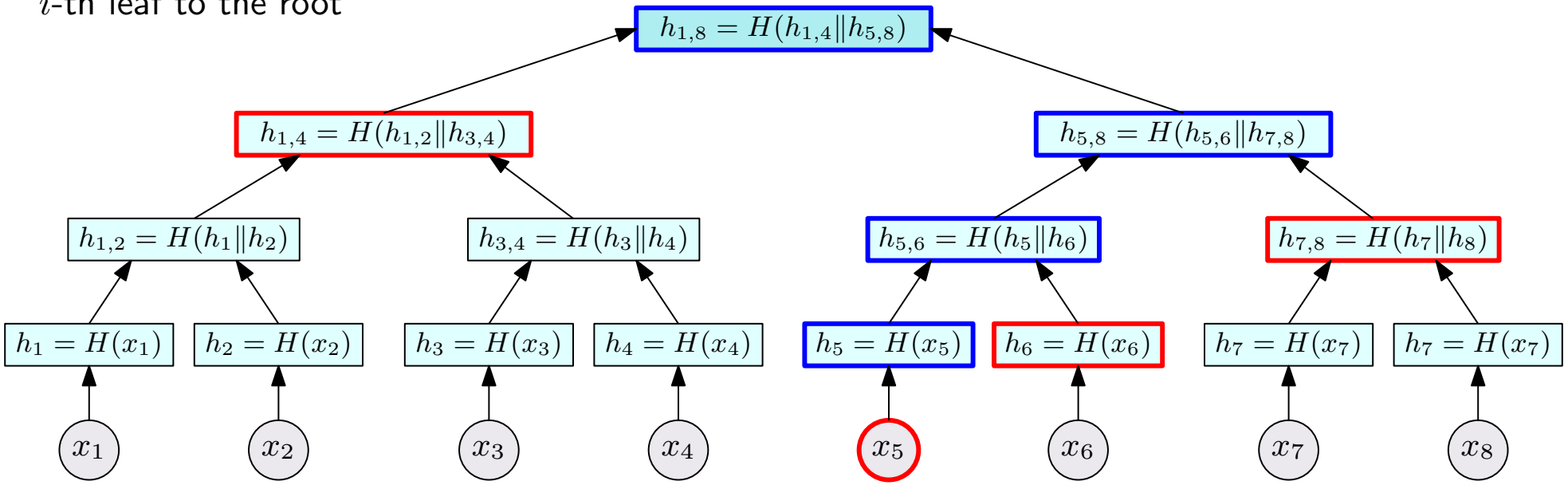

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- Short messages: the server only sends x_i plus i and $O(\log t)$ short hashes

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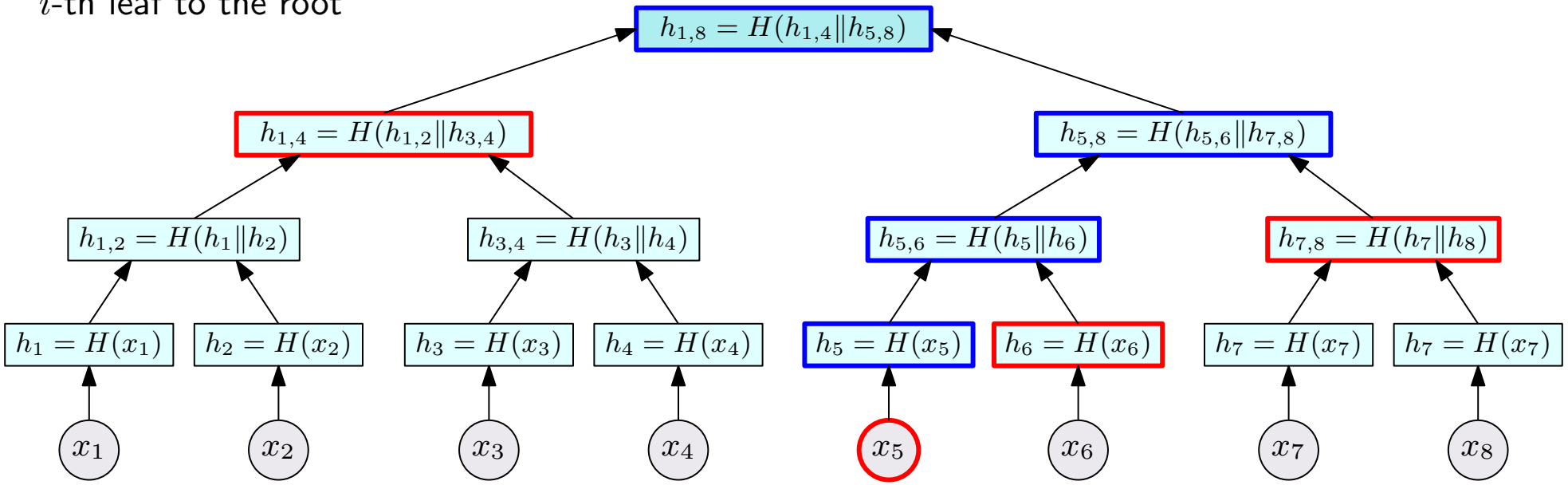

Advantages:

- Short messages: the server only sends x_i plus i and $O(\log t)$ short hashes
- Small local storage: Alice only needs to remember a single hash

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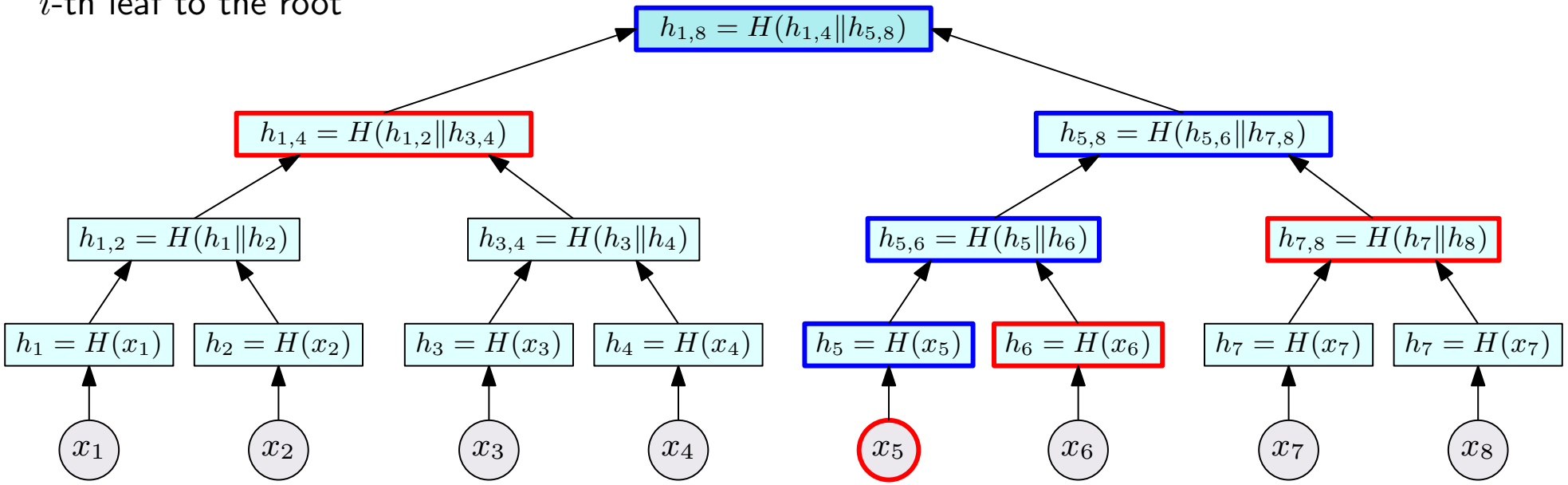


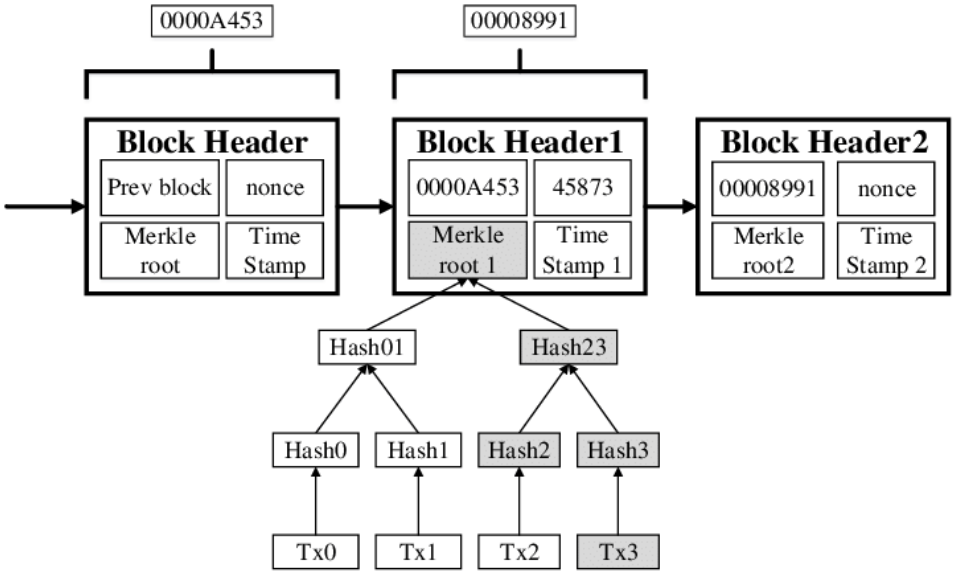
If H is collision resistant, then the hash function computed by the above Merkle tree construction is collision resistant for any fixed t.

The construction can be generalized to handle nonconstant t.

Merkle Trees: Bitcon & SPV

In Bitcoin, Merkle tree are used to provide proofs of inclusion for the transactions in a block:

- Each block of the blockchain contains a list of transactions $x_1, \ldots, x_t$
- The header of the block contains a hash of $x_1, \ldots, x_t$ computed using a Merkle tree



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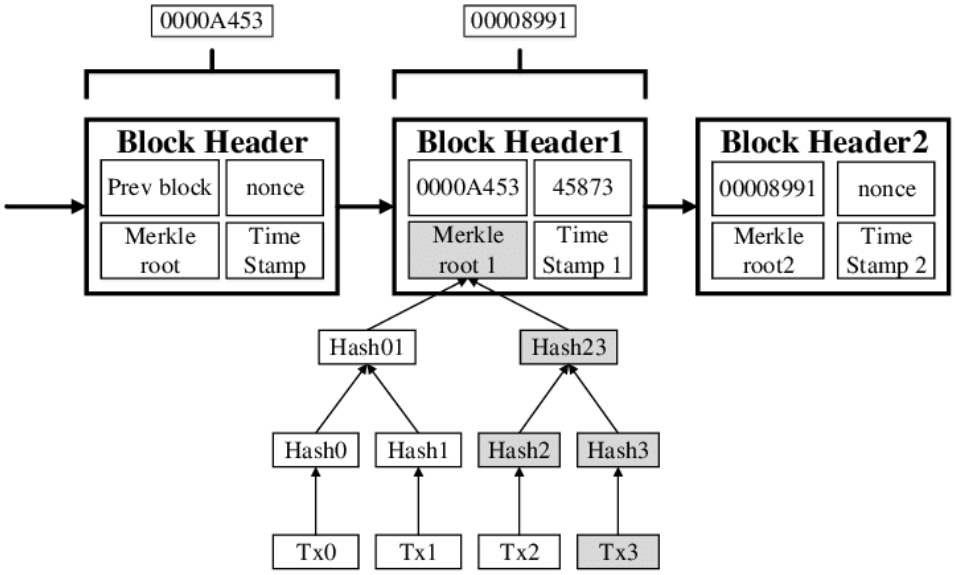

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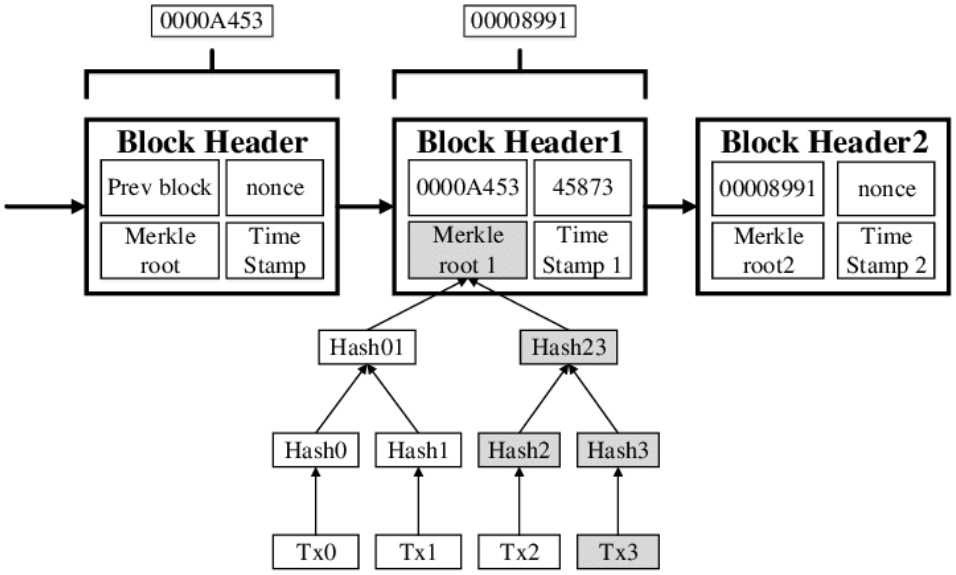
- It is to convince a SPV node that a given transaction belongs to a block in the blockchain via a proof of inclusion



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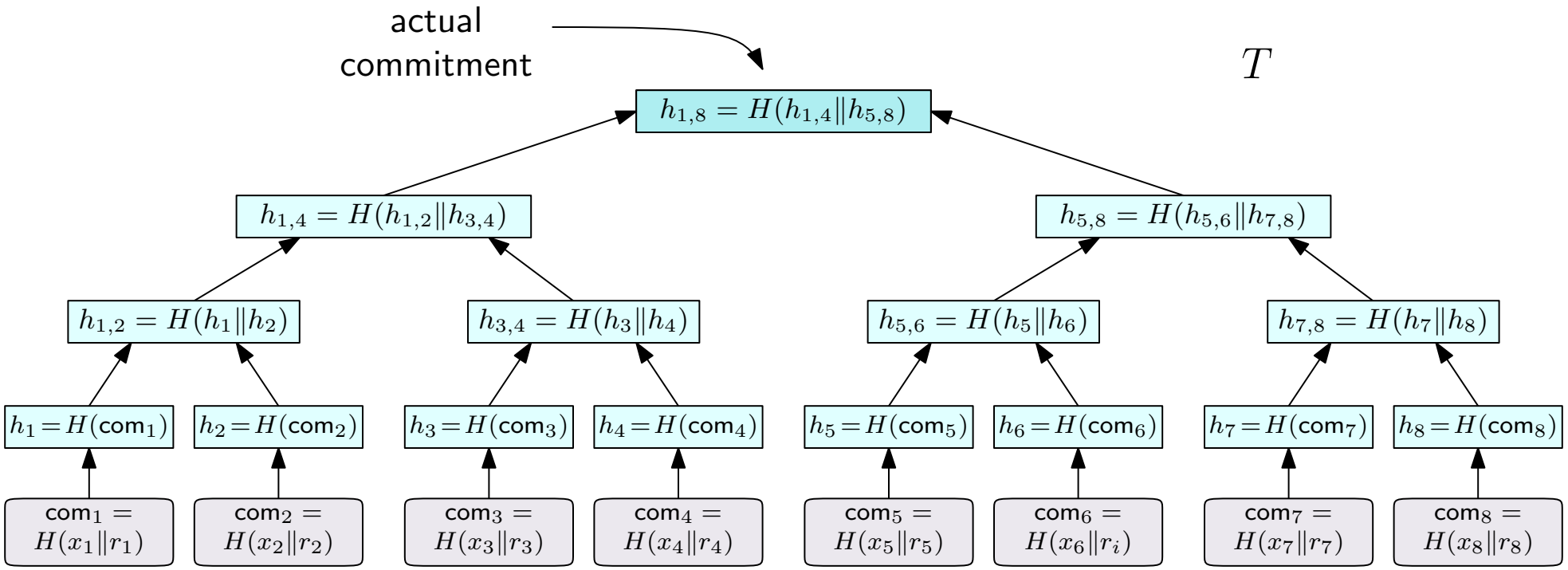
Idea: Combine Merkle trees with commitment schemes

- Build a Merkle tree T of $\mathsf{com}_1, \ldots, \mathsf{com}_t$
- The final commitment is the hash stored in the root of the Merkle tree
- To reveal x_i: send a proof of inclusion of com_i into T, open com_i by sending x_i and r_i
- Short commitment, short opening messages

Commitment Schemes with Partial Reveal



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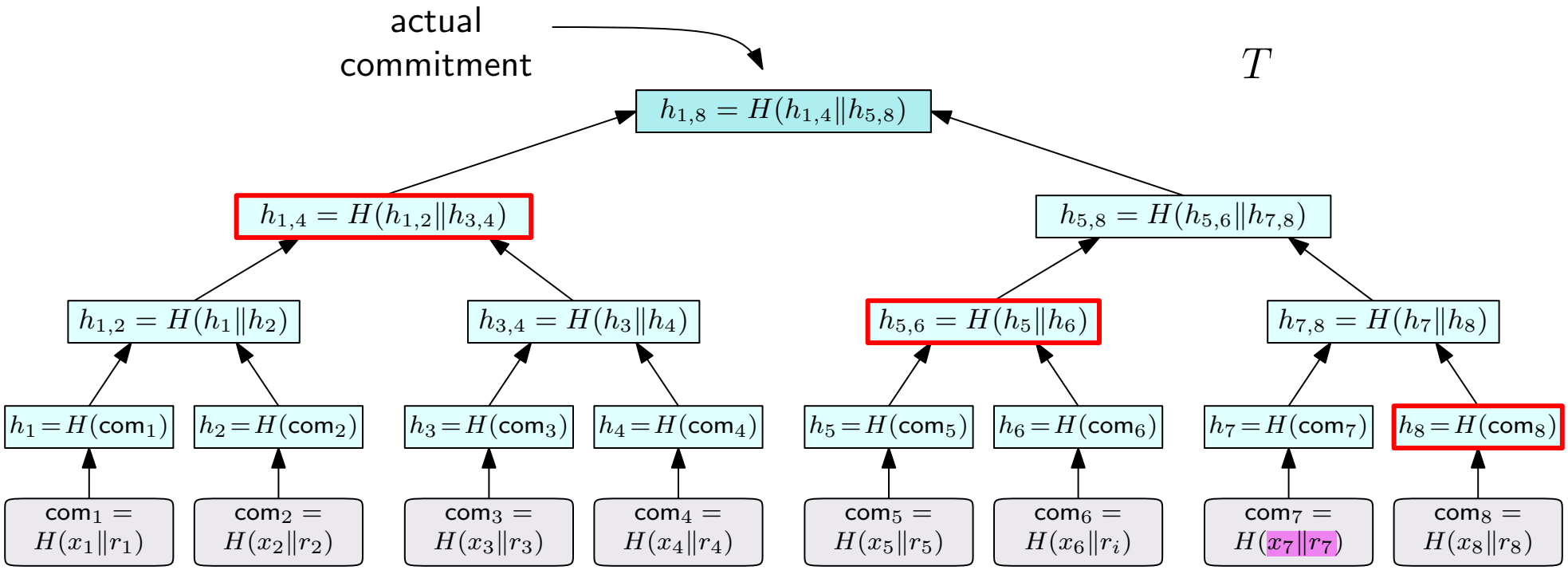

E.g.: x_7 can be reveled by sending x_7, r_7, and a proof of inclusion of com_7 into T.