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If $\langle g \rangle$ contains all m elements, then g is a **generator** of G.

- We can obtain all elements in G (in **some** order) by exponentiating g.

Cyclic Groups

If G has a generator, then G is called a **cyclic group**

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Notice that the elements are generated in a different order

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$\square$

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Notice that the order of $\mathbb{Z}_p^*$ is $\phi(p) = p - 1$, which is not prime (for $p > 3$)

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Definition: the **discrete logarithm** of h with respect to g (in the group G of order m) is denoted by $\log_g h$ and is the unique value $x \in \{0, 1, \ldots, m-1\}$ such that $g^x = h$.

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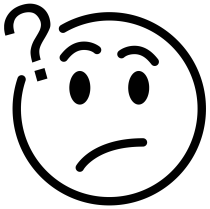

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How do we formalize this?

Let $\mathcal{G}$ be a polynomial-time group-generation algorithm that takes 1^n as input, and outputs:

- (a description of) a cyclic group G;
- the order q of G with $\log q \geq n$;
- a generator g of G.

The Discrete Logarithm Assumption

For a group-generation algorithm $\mathcal{G}$ and an algorithm $\mathcal{A}$, define the experiment $\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n)$ as:

- Run $\mathcal{G}(1^n)$ to obtain (G, q, g), where G is a cyclic group of order q (and q is an n-bit integer), and g is a generator of G.
- Choose a uniform $h \in G$.
- G, q, g and h are given to $\mathcal{A}$
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We need two more related (but not equivalent) assumptions:

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Relating the Discrete Logarithm and the DH Problems

The discrete-logarithm problem is hard relative to $\mathcal{G}$.

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Conjectured: $\not\Downarrow$

That there are groups for which the the CDH problem is believed to be hard but the DDH problem is known not to be hard

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Hardness of CDH $\Longrightarrow$ Hardness of DL

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Proof: We show that a polynomial-time algorithm $\mathcal{A}$ that solves the discrete-logarithm problem (i.e., wins the DLog experiment with non-negligible probability) can be used to solve the CDH problem

Suppose that discrete-logarithm problem is not hard w.r.t. G and consider an algorithm $\mathcal{A}$ such that

$$\Pr[\mathsf{DLog}_{\mathcal{A},\mathcal{G}}(n) = 1] = \eta(n) \quad \text{where } \eta(n) \text{ is not negligible.}$$

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Choice of Groups

The cryptographic schemes can be described in terms of a generic group

- We can focus on the key idea of the construction, ignoring the details of the specific group
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Choice of Groups

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- Finding a generator in such groups is trivial (pick any element except for the identity)

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The group $\mathbb{Z}_p^*$, for prime p has several nice properties:

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However

- The order of the group $q = p - 1$ is not a prime number
- The DDH problem is known **not to be hard** in such groups

Choice of Groups: the group of r-th residues modulo p

Solution:

- Pick two prime numbers p, q such that $p = qr + 1$ for some r
- Consider the set of *r-th residues modulo* p, defined as:

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- There is a polynomial-time algorithm to find a generator of G

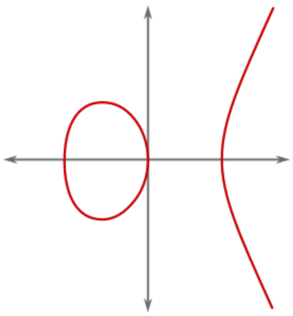
Choice of Groups: other options

- Subgroups of finite fields when using the polynomial representation for elements

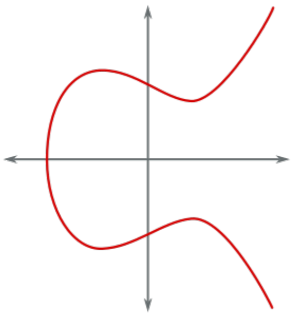
- Elliptic curves
    - Consider cubic equations modulo p with two variables x, y of the form

      $$y^2 = x^3 + Ax + B \pmod{p}$$

    - Let $E(\mathbb{Z}_p)$ be the set of points $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p$ that satisfy the equation, plus a special *point at infinity* $\mathcal{O}$
    - It is possible to define a suitable *addition* operation over $E(\mathbb{Z}_p)$
    - The set $E(\mathbb{Z}_p)$ is a group under the addition operation, and the identity element is $\mathcal{O}$



$y^2 = x^3 - x$



$y^2 = x^3 - x + 1$