

Reminder: Complexity Classes

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 - If $x \in L$, then $\Pr[T(x) \text{ accepts}] \geq \frac{2}{3}$
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The relation between BPP and NP is unknown

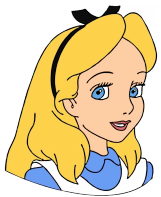
Zero Knowledge Proofs

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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3	4	5	2	8	6	1	7	9

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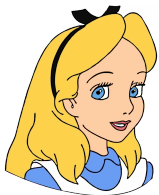


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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

I know a solution to this Sudoku instance

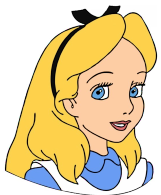


Zero Knowledge Proofs

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

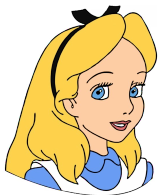
Really? Show it to me!



Zero Knowledge Proofs

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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	6					2	8	
			4	1	9			5
				8			7	9



I don't want to reveal it to you

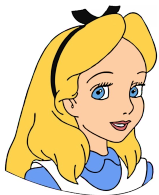


Zero Knowledge Proofs

5	3	4	6	7	8	9	1	2
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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3			7				
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8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Then I don't believe you really have a solution

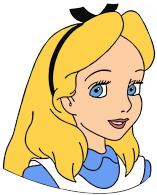


Zero Knowledge Proofs

5	3	4	6	7	8	9	1	2
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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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5	3			7				
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				8			7	9

I can prove to you I have a solution without revealing anything about it



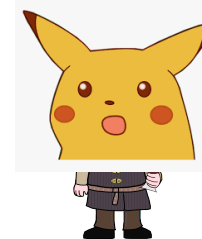
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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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5	3			7				
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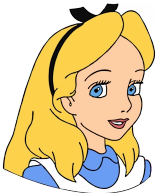
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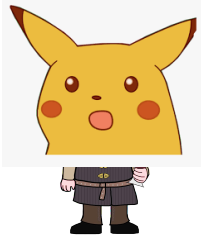
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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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5	3			7				
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I can prove to you I have a solution without revealing anything about it

Zero Knowledge protocol

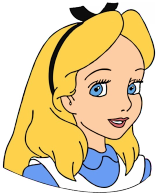


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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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8				6				3
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I am now convinced you have the solution



Zero Knowledge protocol



Zero Knowledge Proofs

5	3			7			
6			1	9	5		
	9	8					6
8				6			3
4			8		3		1
7				2			6
	6					2	8
			4	1	9		5
				8			7
							9

Hey, Charlie!
Alice has a solution to
this Sudoku instance



Zero Knowledge Proofs

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7				2			6
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			4	1	9		5
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							9



Prove it!



Zero Knowledge Proofs

5	3			7			
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	9	8					6
8				6			3
4			8		3		1
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	6					2	8
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						7	9

Even if I definitely know she has a solution, somehow I have no way of proving that



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There is some **claim** x known to both Alice and Bob

A **prover** (Alice) wants to convince a **verifier** (Bob) that the claim is true

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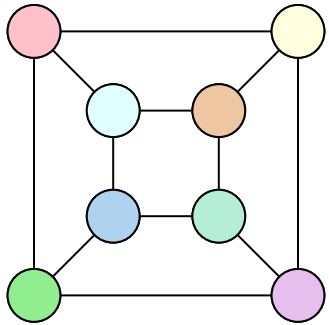
We can model the verifier as a polynomial-time algorithm $\mathcal{V}(x, w)$ that takes as input the claim x and the proof w , and outputs 1 if and only if it accepts w as a proof of x

Example: Graph Isomorphism

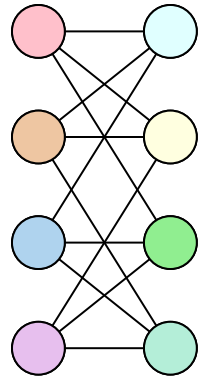
Graph isomorphism problem

G_1 is isomorphic to G_2 iff \exists bijection $\pi : V_1 \rightarrow V_2$ s.t. $(u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2$.

$G_1 = (V_1, E_1)$



$G_2 = (V_2, E_2)$

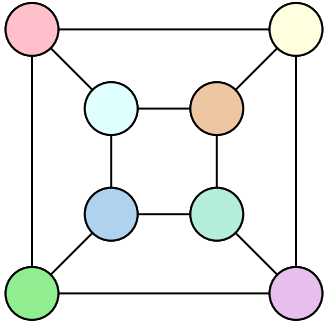


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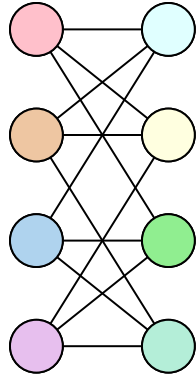
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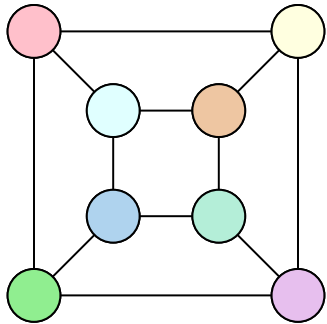
Claim: G_1 and G_2 are isomorphic.

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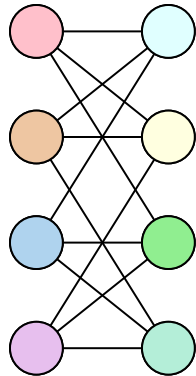
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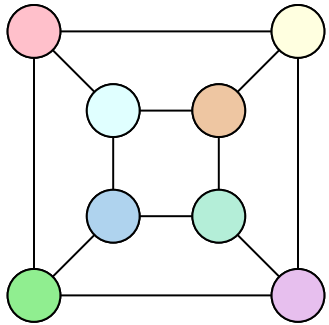
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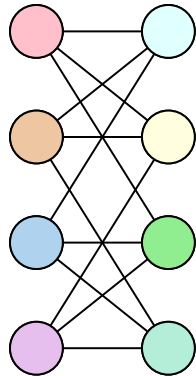
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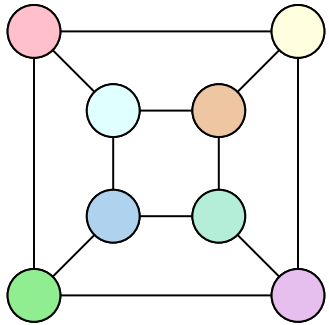
$x = (G_1, G_2)$

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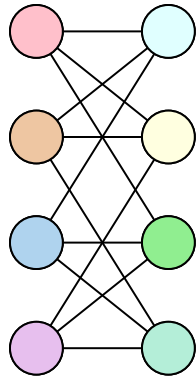
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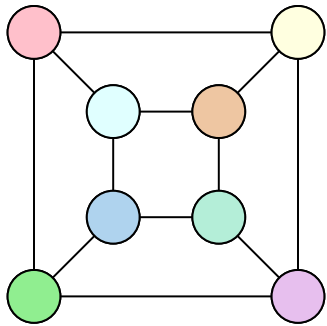
$w = \pi$

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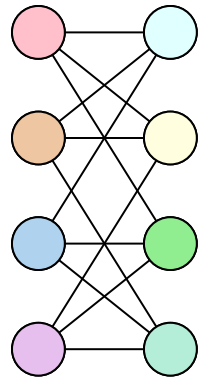
Graph isomorphism problem

G_1 is isomorphic to G_2 iff \exists bijection $\pi : V_1 \rightarrow V_2$ s.t. $(u, v) \in E_1 \iff (\pi(u), \pi(v)) \in E_2$.

$G_1 = (V_1, E_1)$



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Claim: G_1 and G_2 are isomorphic.

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Algorithm $\mathcal{V}((G_1, G_2), \pi)$:

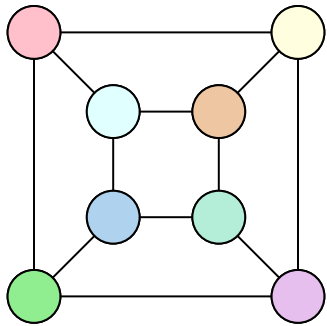
- For all $u, v \in V_1^2$:
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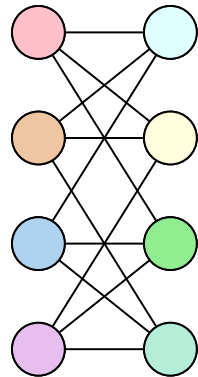
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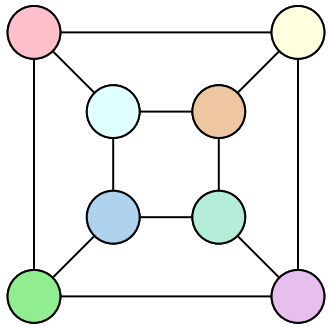
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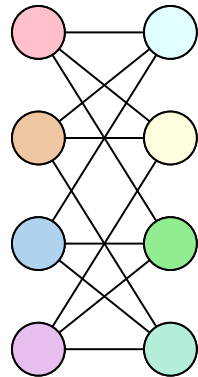
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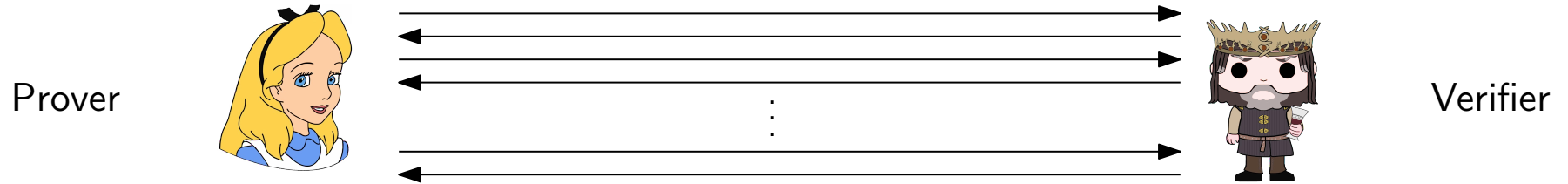
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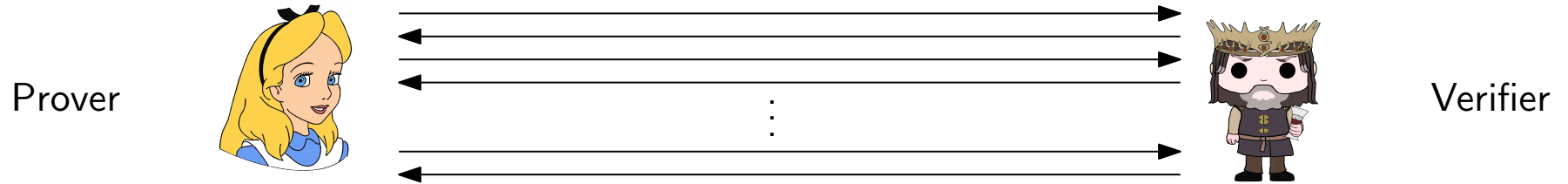
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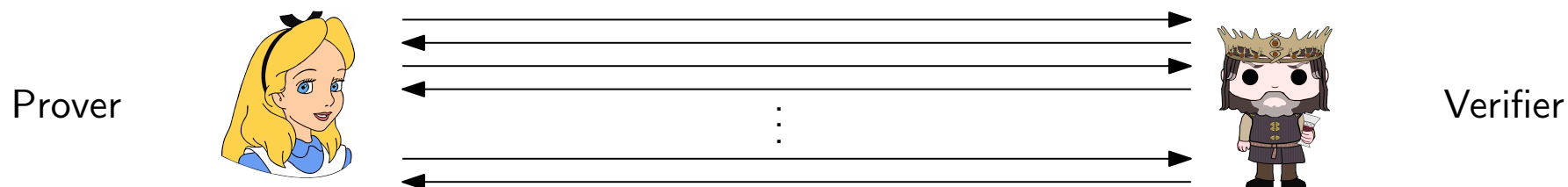


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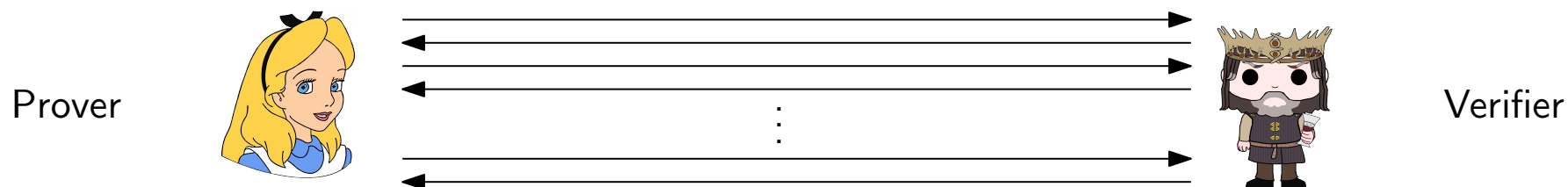
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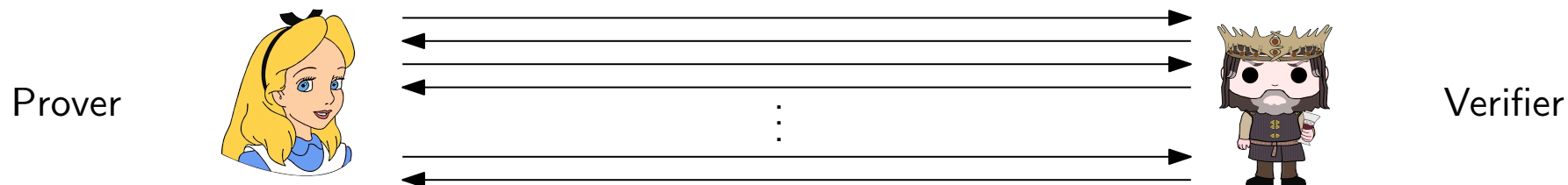
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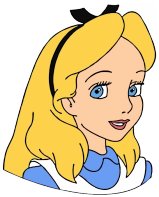
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We relax the above requirements by allowing the verifier to commit errors (with a small probability).

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Suppose that the prover needs to convince a colorblind verifier that this sheet of paper contains multiple colors:

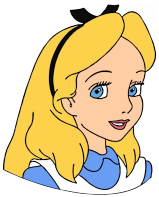
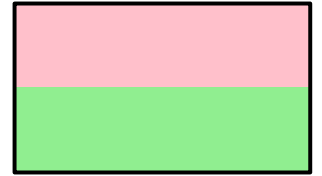


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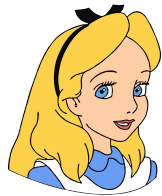
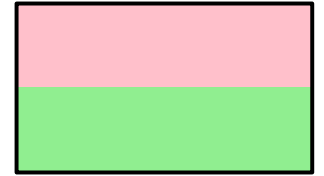
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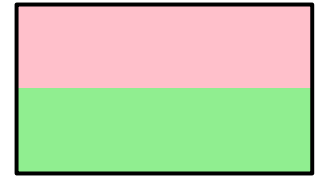


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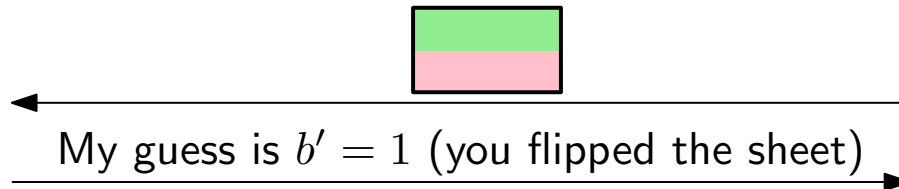
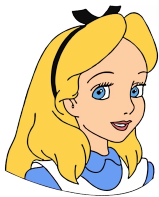
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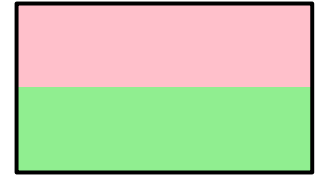
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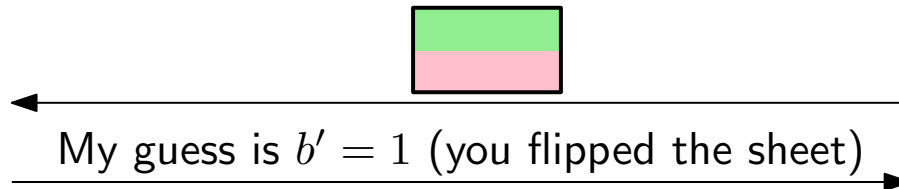
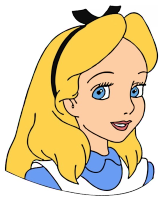


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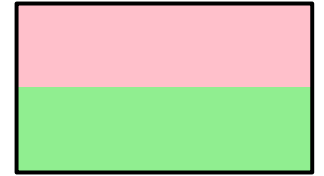


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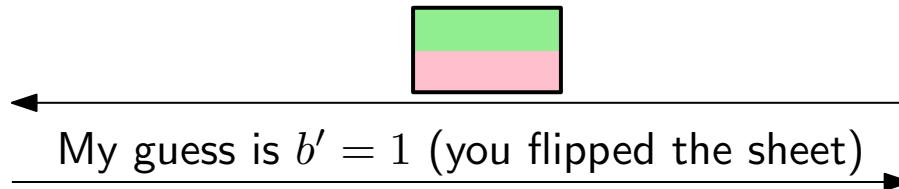
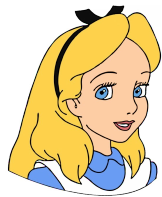


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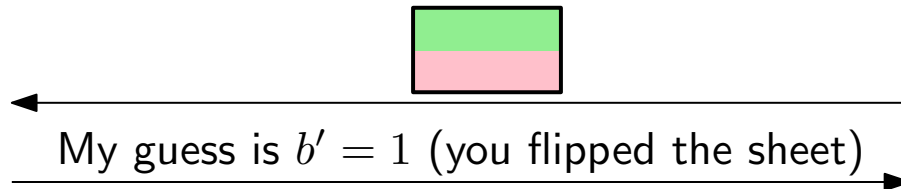
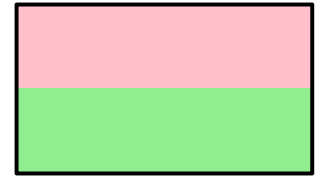
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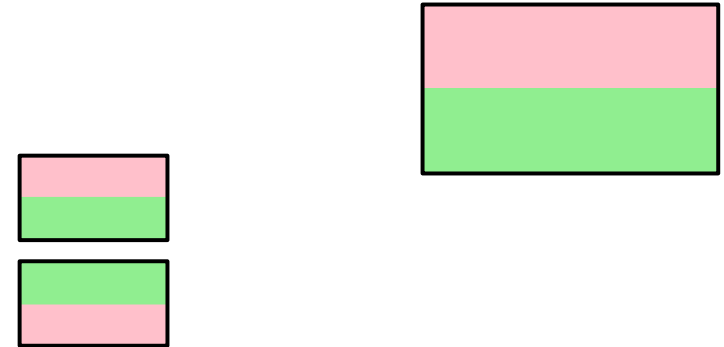
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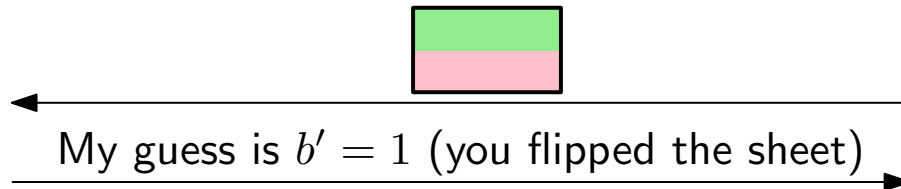
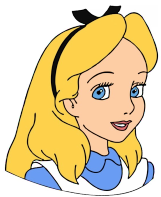
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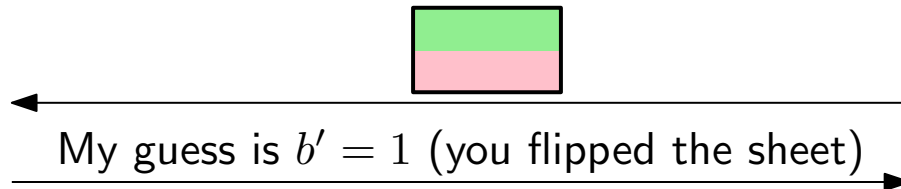
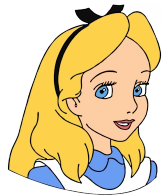


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Repeat the experiment k times. Accept iff all trials succeed.

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IP is the class of all languages that admit an interactive proof system

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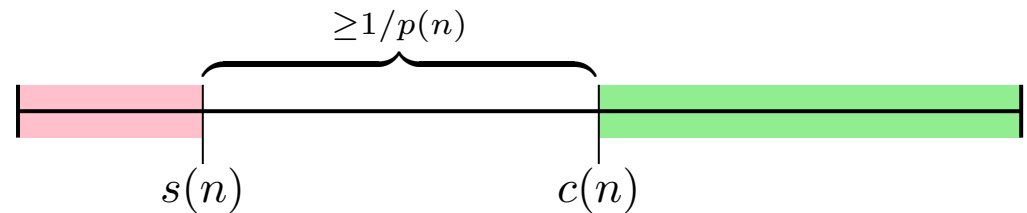
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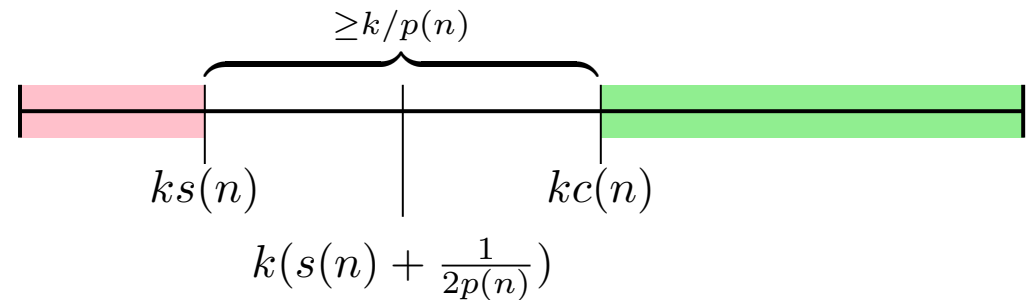


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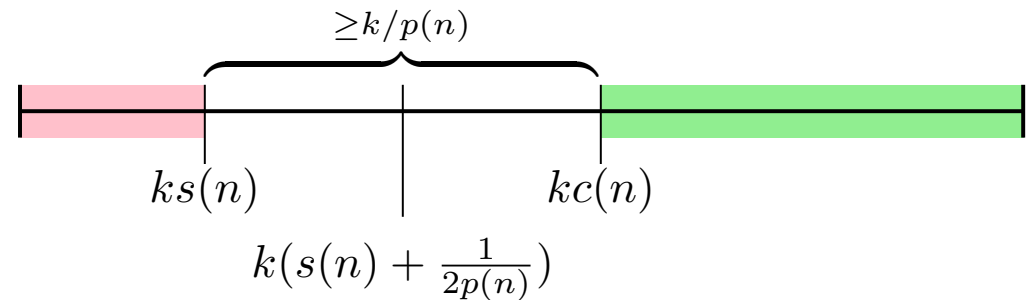
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Chernoff bound: Let $X = X_1 + X_2 + \dots$ where the X_i s are independent binary random variables and let $\mu = \mathbb{E}[X]$. Then, for any $\delta > 0$:

$$\Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu / 2}$$

$O(p(n)^3)$ repetitions suffice

Efficiently verifiable proofs

The class NP can be thought of as the set of all claims that admit a short, efficiently verifiable proof

A language $L \subseteq \{0, 1\}^*$ is in NP iff there exists a non-deterministic polynomial-time Turing machine T such that $x \in L$ iff (at least one computation path of) $T(x)$ accepts

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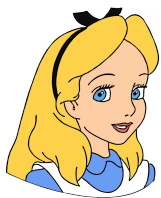
Proof sketch:

\implies : A (polynomial-length) accepting computation path of $T(x)$ is a witness w
(\mathcal{V} simulates T but already know the “right” non-deterministic choices from w)

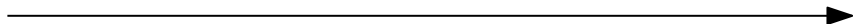
\impliedby : A non-deterministic Turing machine T can “guess” the witness w , and then check if $\mathcal{V}(x, w) = 1$

Interactive Proofs for NP and BPP

We immediately have an interactive proof for all languages $L \in \text{NP}$



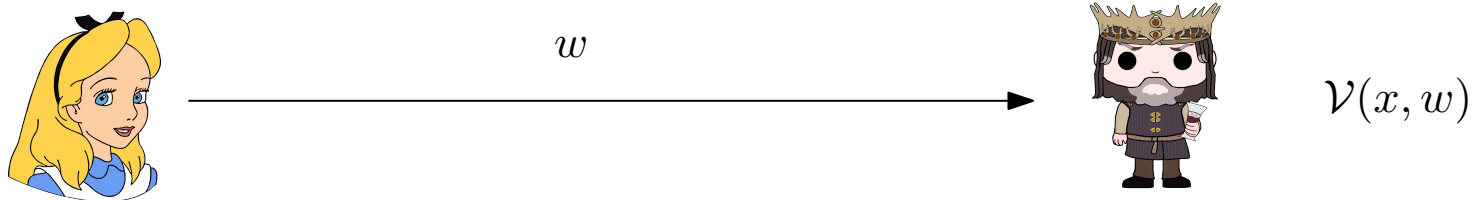
w



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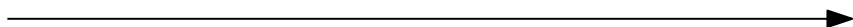
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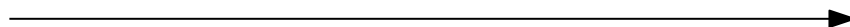
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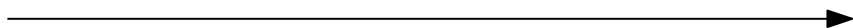
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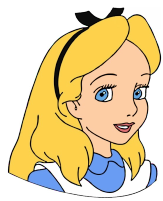
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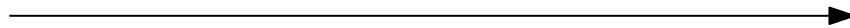
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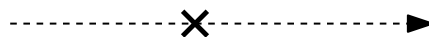
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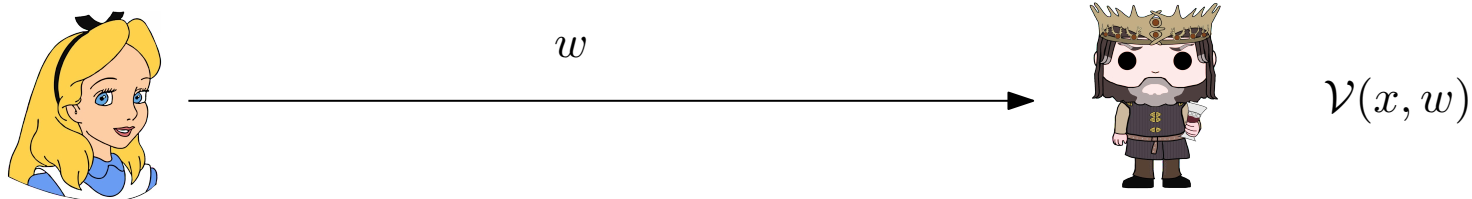


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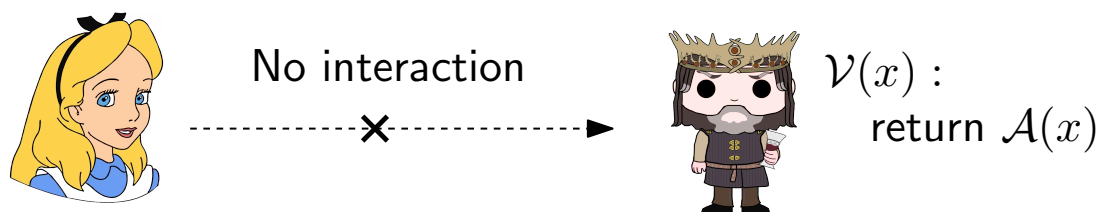


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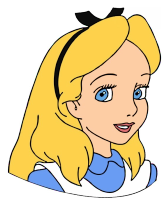
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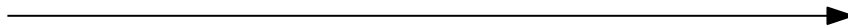
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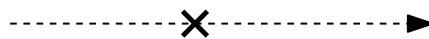
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An interactive proof in which the verifier never talks to the prover is **degenerate**

An Interactive Proof for Graph Non-Isomorphism

Let's look at a non-degenerate interactive proof for a problem that is not known to be in $NP \cup BPP$

The language L contains all pairs of graphs (G_1, G_2) such that G_1 and G_2 are **not** isomorphic

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Common input: $x = (G_1, G_2)$ where $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and $V = \{1, \dots, n\}$

- The verifier chooses b u.a.r. in $\{1, 2\}$
- The verifier picks a random permutation $\pi : V \rightarrow V$ and sends the graph $G' = \pi(G_b)$ to the prover
- The prover checks whether G' is isomorphic to G_1 . If so it replies with $b' = 1$, otherwise it replies with $b' = 2$.
- If $b' = b$, the verifier accepts. Otherwise it rejects

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The verifier accepts with probability:

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If G_1 and G_2 , are isomorphic and π is a random permutation then $\Pr[\pi(G_1) = H] = \Pr[\pi(G_2) = H]$

For any graph G' isomorphic to G_1 (and G_2):

$$\Pr[b = 1 \mid \pi(G_b) = G'] = \Pr[b = 2 \mid \pi(G_b) = G'] = \frac{1}{2}$$

Let R be the (possibly randomized) process used by the prover to compute the reply to the verifier

The verifier accepts with probability:

$$\Pr[R(G_b) = b] = \sum_{G'} \Pr[R(G') = b \mid \pi(G_b) = G'] \cdot \Pr[\pi(G_b) = G'] = \sum_{G'} \frac{1}{2} \cdot \Pr[\pi(G_b) = G'] = \frac{1}{2}$$

$$\Pr[R(G') = b \mid \pi(G_b) = G'] = \frac{1}{2}$$

Use probability amplification

□