

The Oblivious Transfer Protocol

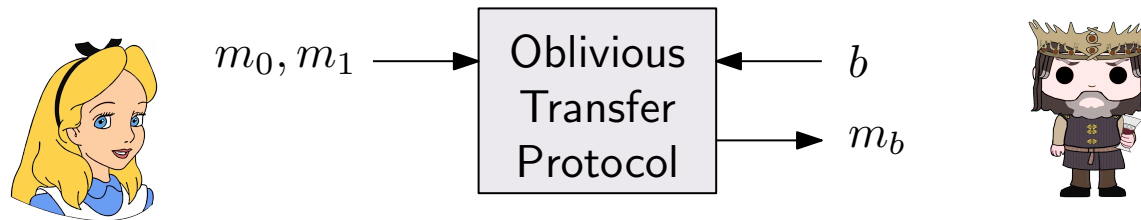
How does Bob learn the wire-labels corresponding to his input?

- He cannot just ask Alice, since this would reveal his inputs

Alice and Bob use a protocol known as **oblivious transfer protocol**

- In the oblivious transfer protocol Alice has two messages m_0, m_1 of length $\ell(n)$
- Bob wants to learn one of them, say m_b , without revealing which one he is interested in to Alice
- Alice wants to be sure that Bob learns exactly one of the two values

n is the security parameter



Reminder: DDH-Based Key Encapsulation Mechanism

Gen(1^n):

- Run $\mathcal{G}(1^n)$, where \mathcal{G} is a group generation algorithm, to obtain (G, q, g) where G is a group of order q and $g \in G$ is a generator
- Pick some key derivation function $H : G \rightarrow \{0, 1\}^{\ell(n)}$
- Choose a uniform x u.a.r. from $\{0, \dots, q - 1\}$
- Compute $h = g^x$
- Output (pk, sk) where $pk = (G, q, g, h, H)$ and $sk = (G, q, g, x, H)$.



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Encaps $_{pk}(1^n)$:

- Here $pk = (G, q, g, h, H)$
- Choose y u.a.r. from $\{0, \dots, q - 1\}$
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Decaps $_{sk}(c)$:

- Here $sk = (G, q, g, x, H)$
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Reminder: DDH-Based KEM & Hybrid Encryption

We can build a CPA-secure PKE scheme by combining a CPA-secure KEM with an EAV-secure DEM

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The resulting scheme is as follows:

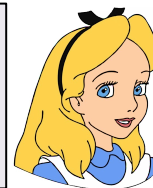
Gen(1^n):

- Pick a group G , its order q , a generator $g \in G$, a key-derivation function $H : G \rightarrow \{0, 1\}^{\ell(n)}$
We think of these as fixed public values agreed upon in advance between Alice and Bob.
- Pick a random $x \in G$, the **public-key** is $h = g^x$ and the **secret-key** is x



Enc $_h(m)$:

- Choose a uniform $y \in \{0, \dots, q - 1\}$. Return the pair $c = (g^y, H(h^y) \oplus m)$



Dec $_x((c, c'))$:

- Return $H(c^x) \oplus c'$



The Oblivious Transfer Protocol

Idea:

- We pick **two** public keys h_0, h_1 for Bob
- We ensure that Bob knows the secret key x_b corresponding to **exactly** one of the public keys (of his choice)
- Alice encrypts m_0 with h_0 and m_1 with h_1
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How can Bob “prove” to Alice that he knows exactly one private key?

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- Alice sends c_0 and c_1 to Bob
- Bob decrypts $c_b = (c, c')$ as $m_b = H(c^x) \oplus c'$

The Oblivious Transfer Protocol: Security (informal)

Can Bob learn m_{1-b} ?

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Secure under the Random Oracle model and the CDH assumption

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- Alice starts from a circuit that computes $f(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$

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- If Bob should also know the value of $f(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$, Alice shares it with Bob

Multiparty Computation

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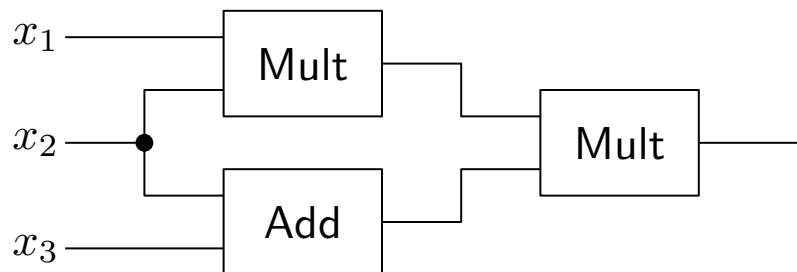
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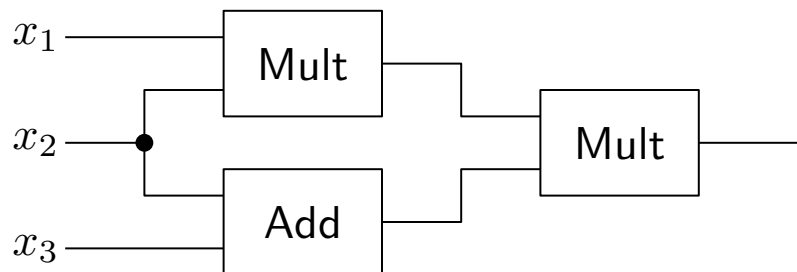


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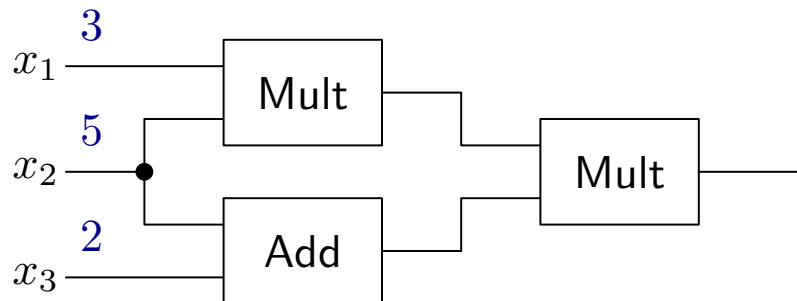
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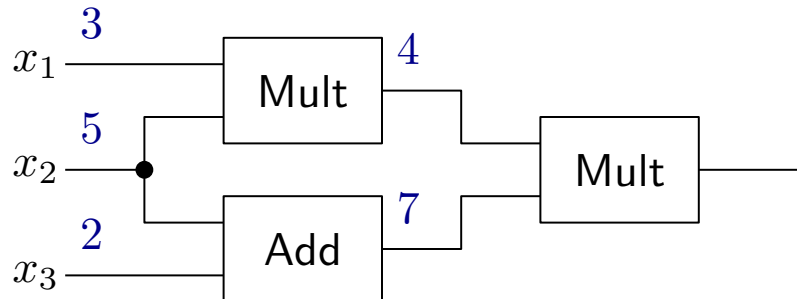
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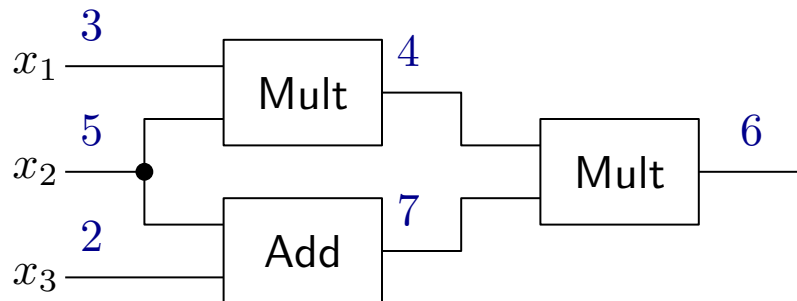
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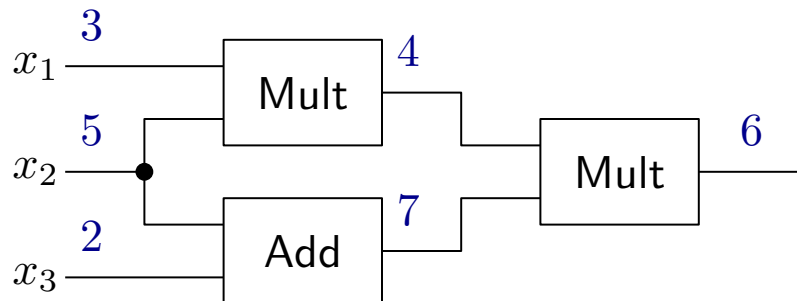
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- Replace each wire of the arithmetic circuit with $\lceil \log p \rceil$ Boolean wires
- Replace each Addition/Multiplication gate with a Boolean circuit that computes the Sum/Product of the inputs modulo p

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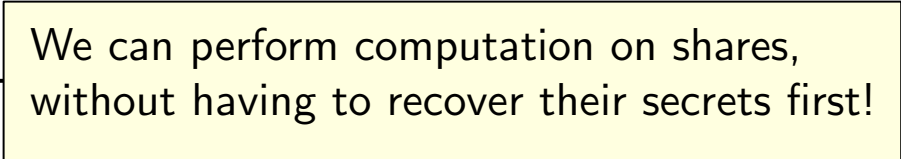
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- The parties combine their output shares and recover the value of $f(x_1, \dots, x_n)$

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- Let $f_a(x)$ and $f_b(x)$ be the polynomials (of degree at most $k - 1$) used to share a and b

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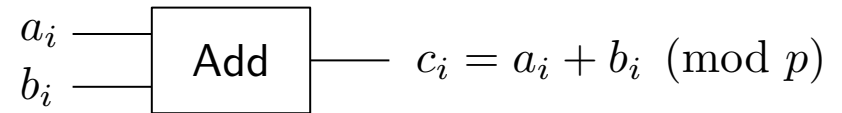
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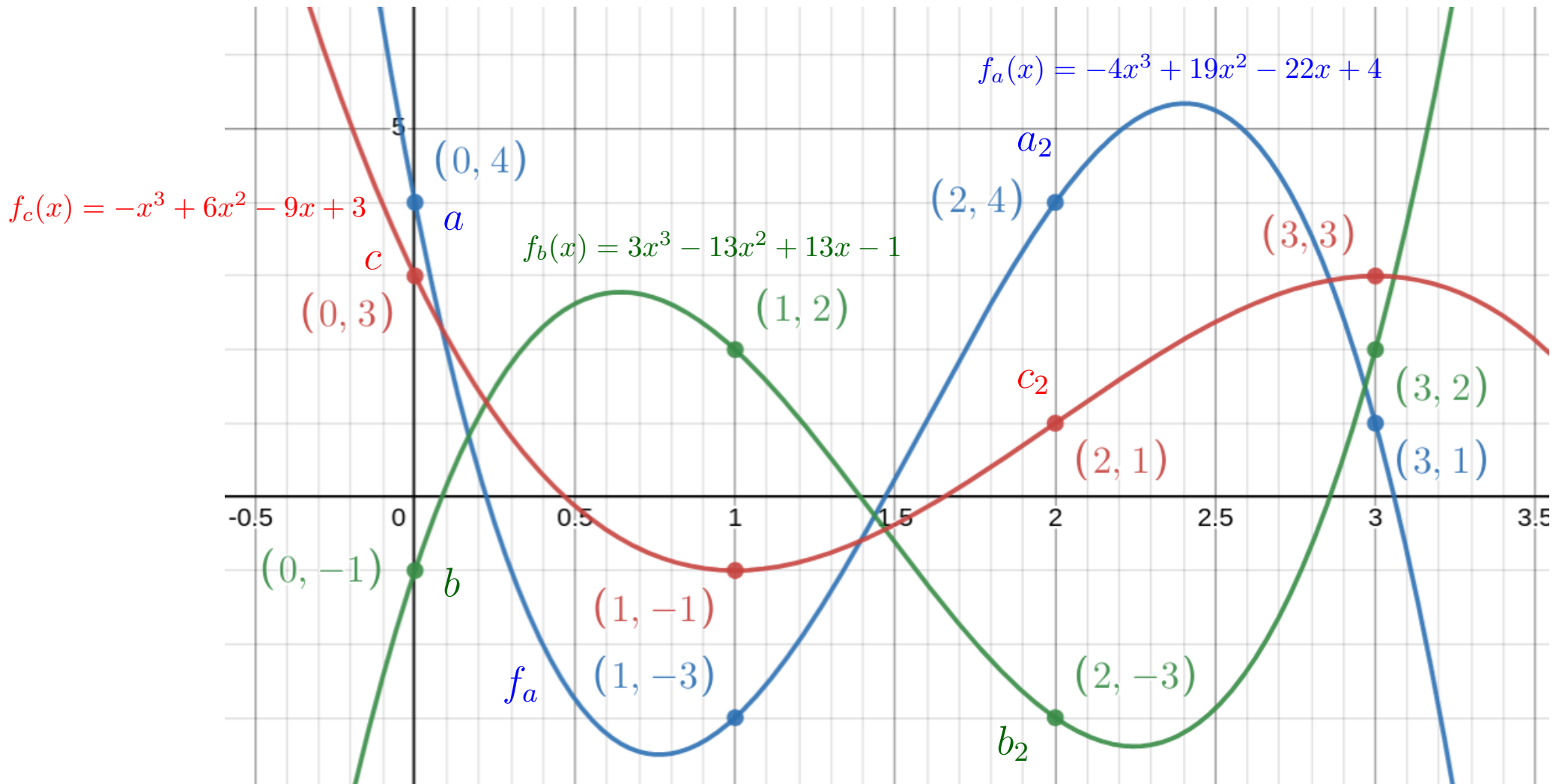
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Addition gates do not require any special care!

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We need to use another property of interpolating polynomials...

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Lemma: Given distinct x_1, \dots, x_n , define $r_j = \prod_{\substack{i=1, \dots, n \\ i \neq j}} x_i \cdot (x_i - x_j)^{-1}$.

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