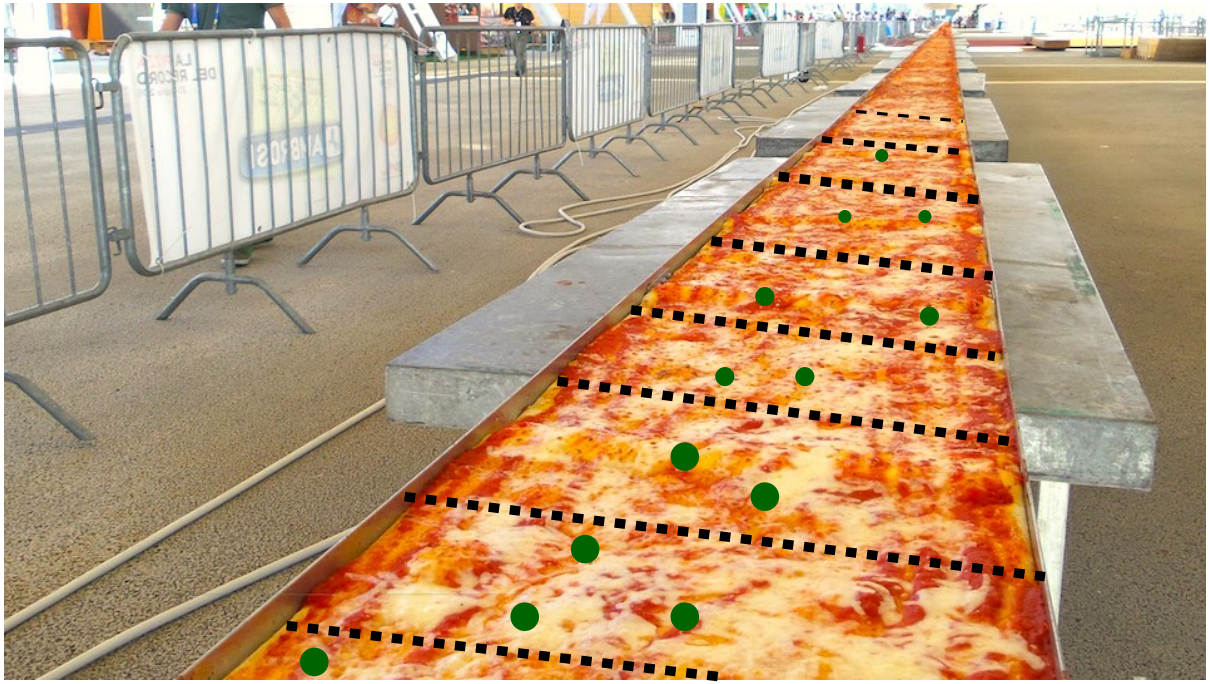
Gustavo's Pizza

Gustavo has very peculiar tastes when it comes to pizza al taglio: he wants his *slice* to have as many olives as possible, but never more than k.

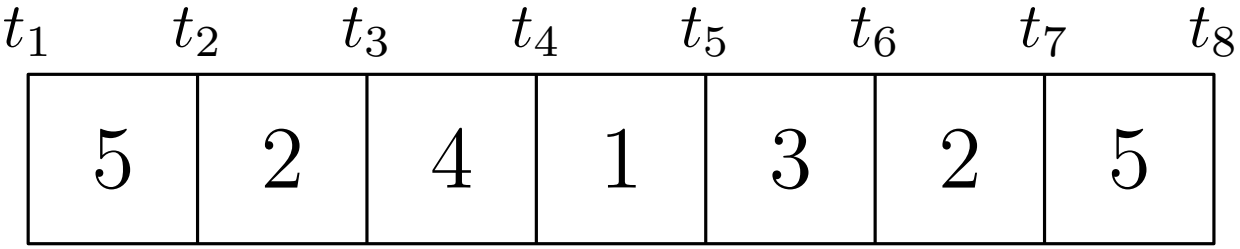
The pizza can be cut into discrete positions $t_1 < \cdots < t_n$. A *slice* (i, j) with $j > i$ represents the interval $[t_i, t_j]$.



The interval $[t_i, t_{i+1}]$ contains η_i olives. Where to cut?

Example

$k = 9$



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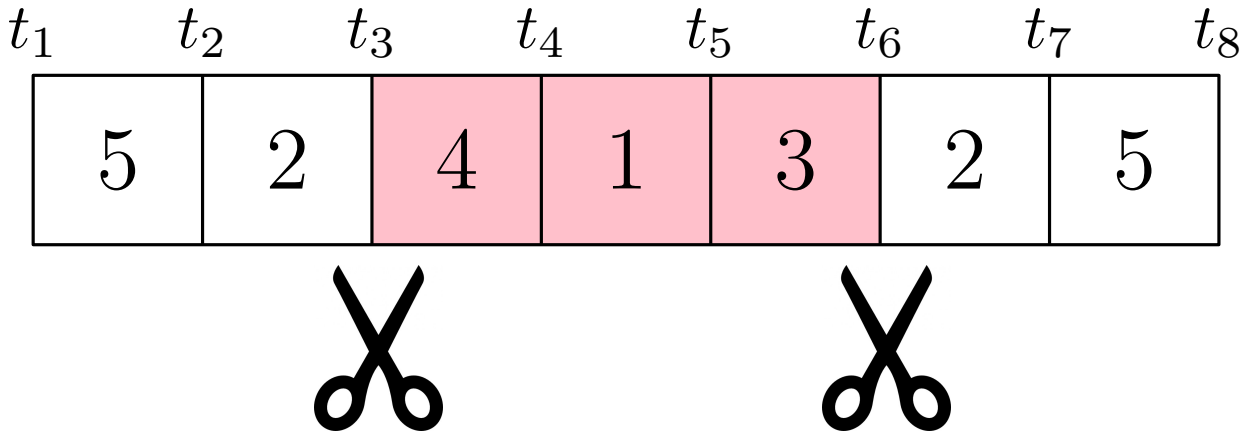
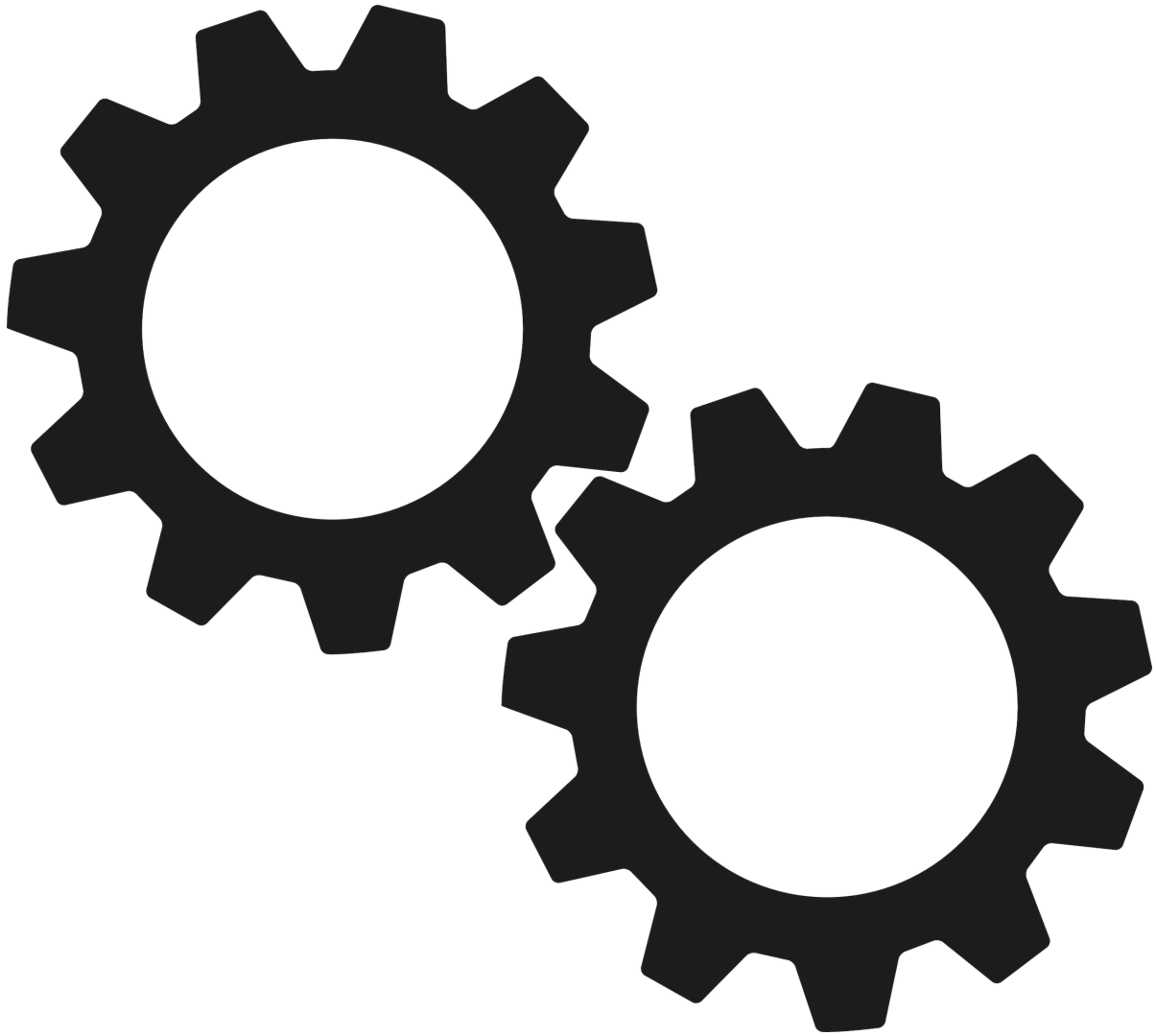


Solution: $(3, 6)$. Number of olives: $\sum_{i=3}^{6-1} \eta_i = 8$.



A Naive Solution

- For $i = 1, \ldots, n-1$:
  - For $j = i+1, \ldots, n$:
    - olives $\leftarrow 0$
    - For $k = i, \ldots, j-1$:
      - olives $\leftarrow$ olives $+ \eta_k$

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A Naive Solution

- For $i = 1, \ldots, n-1$: $O(n)$
  - For $j = i+1, \ldots, n$: $O(n)$
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Total time: $O(n^3)$

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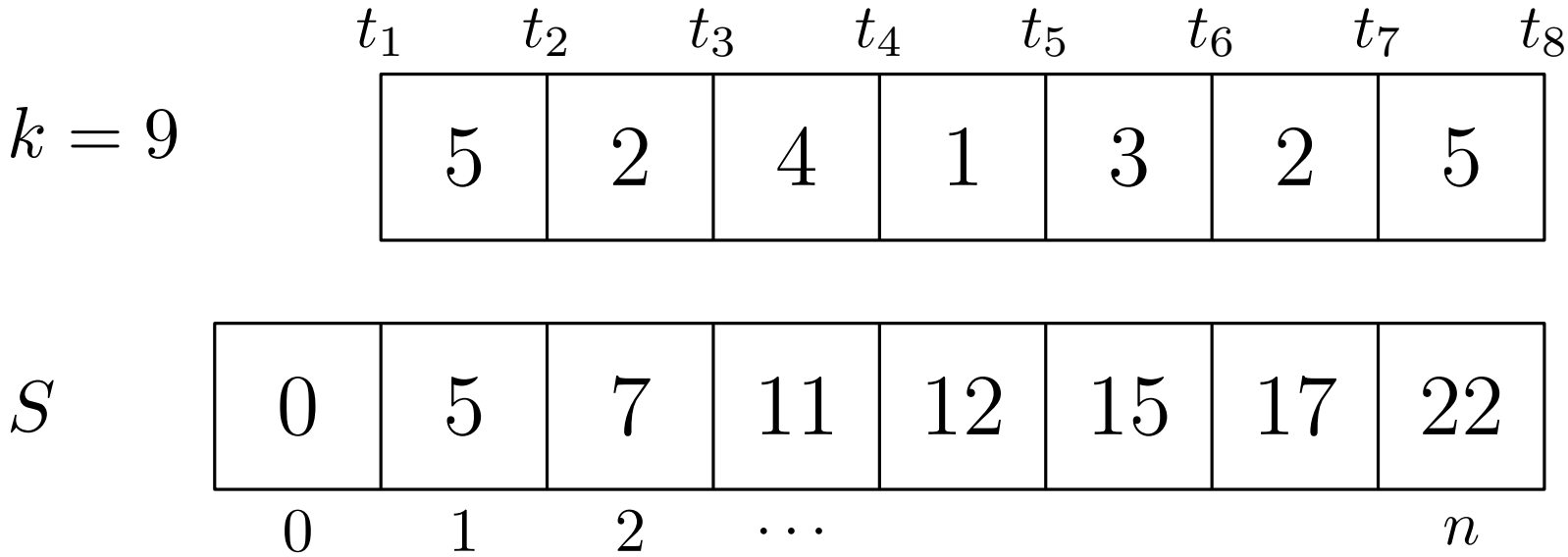
Can we do better?

Partial Sums



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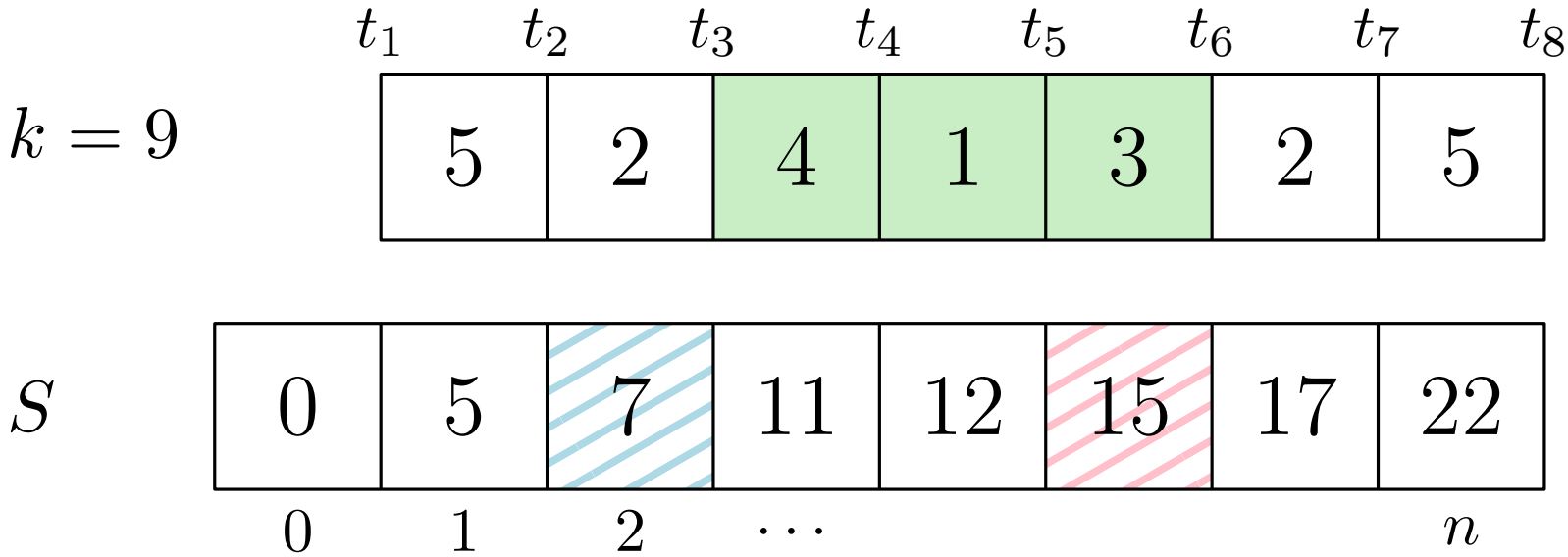
Partial Sums



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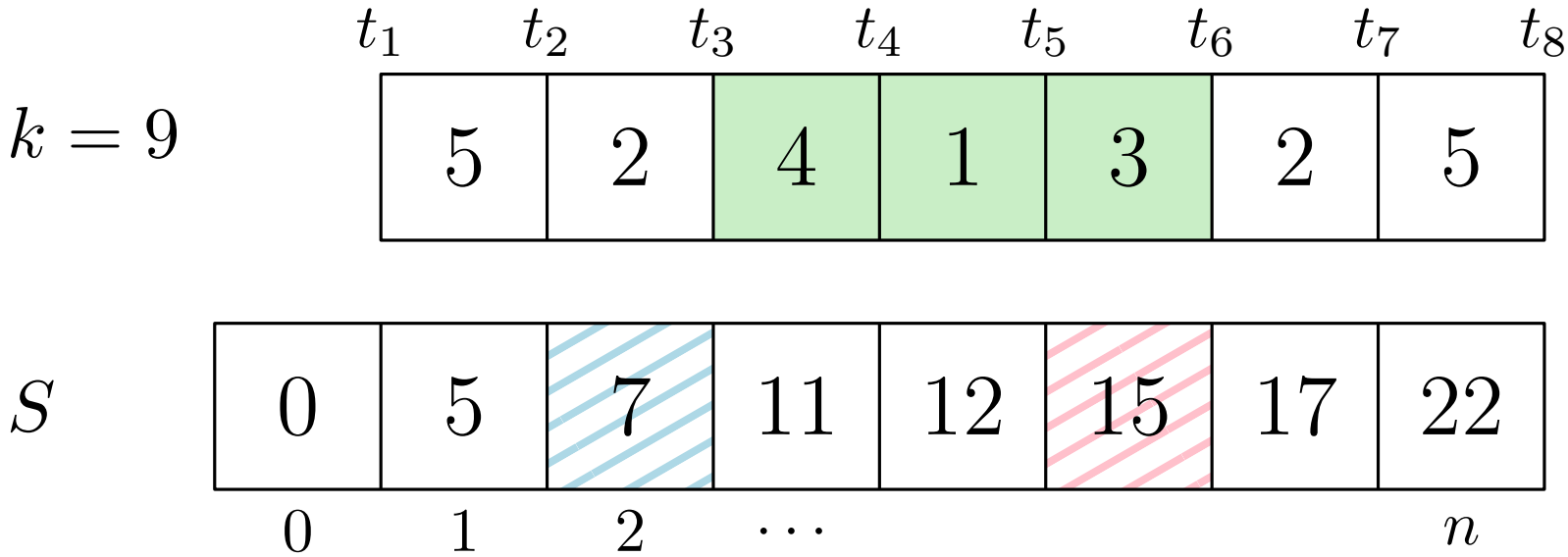

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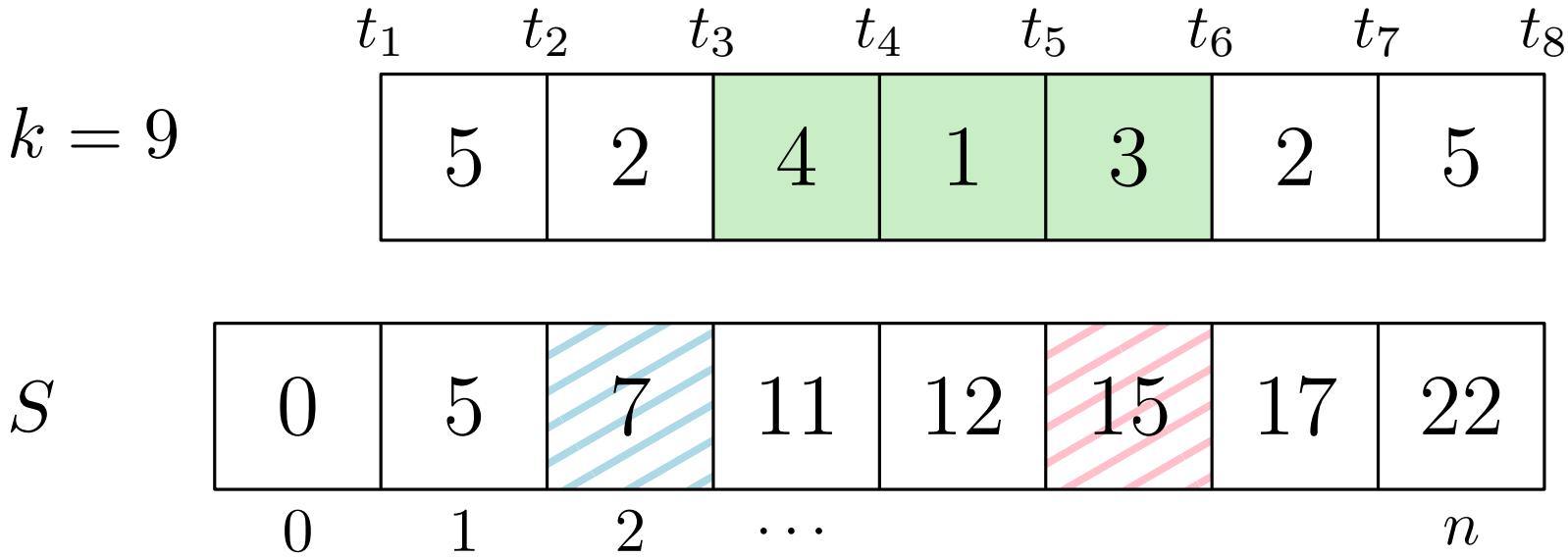

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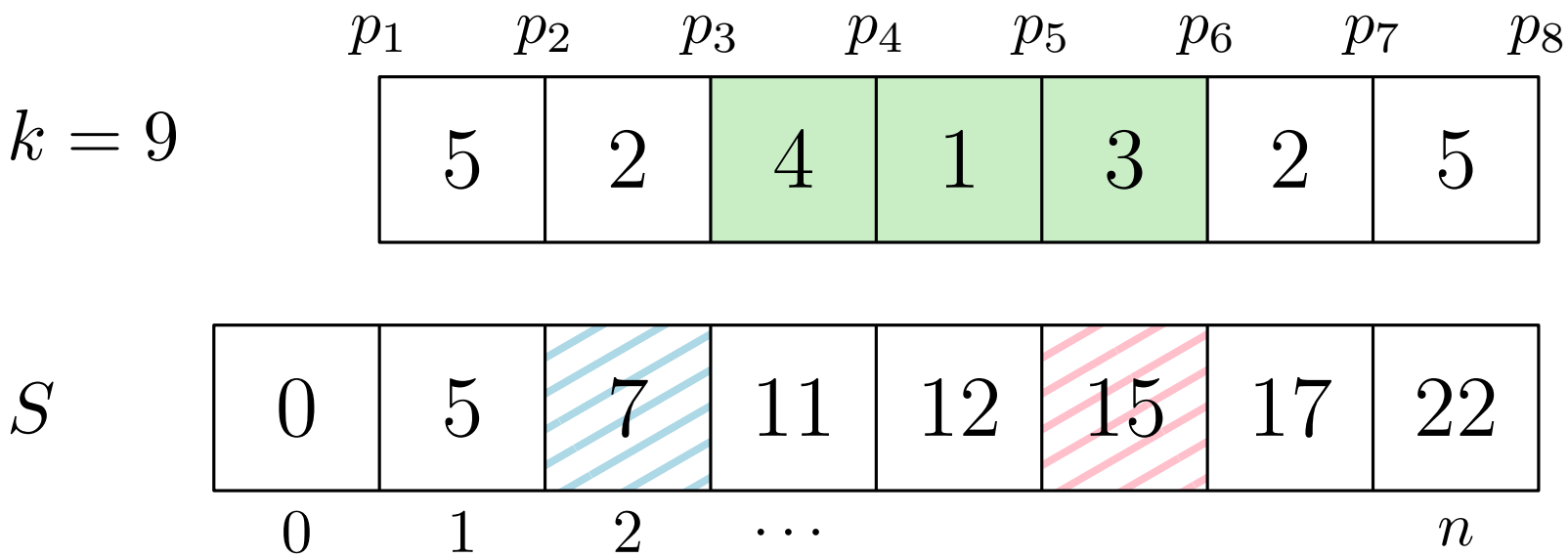
Partial Sums + Binary Search



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- For $i = 1, \ldots, n-1$: $O(n)$
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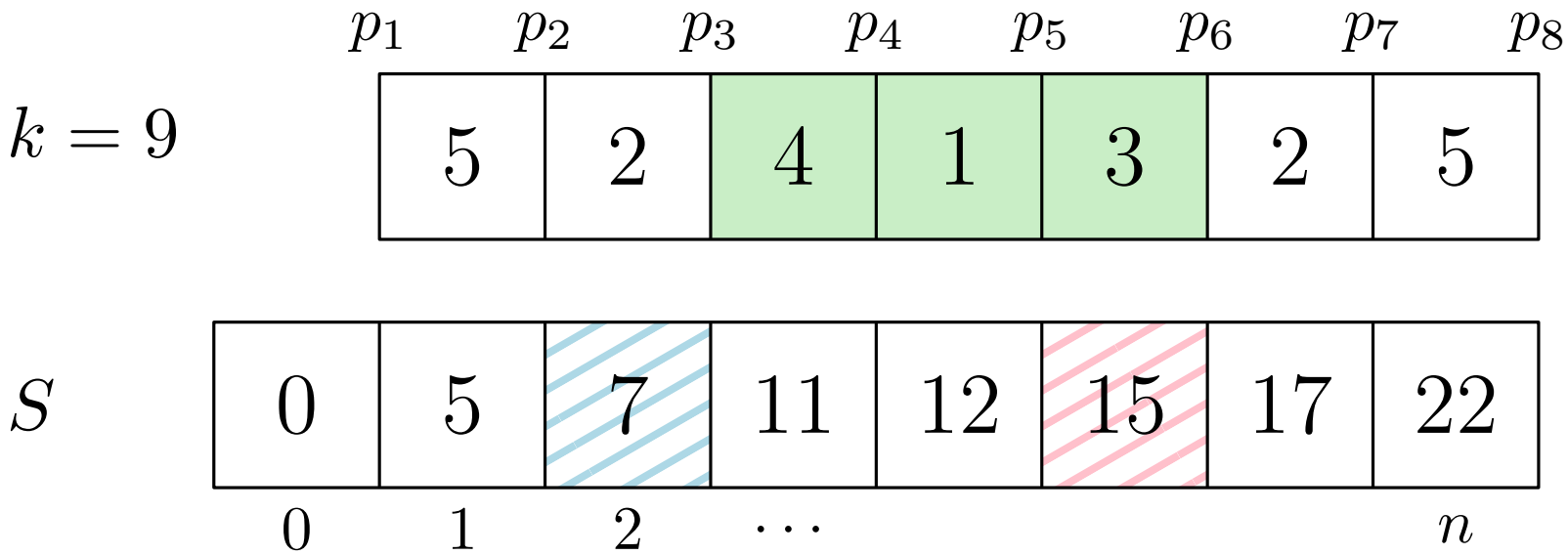
Partial Sums + Binary Search



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Total time: $O(n \log n)$

Recap

- Naive

  $O(n^3)$

- Partial Sums

  $O(n^2)$

- Partial Sums + Binary Search

  $O(n \log n)$

Time

Recap

- Naive

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Time

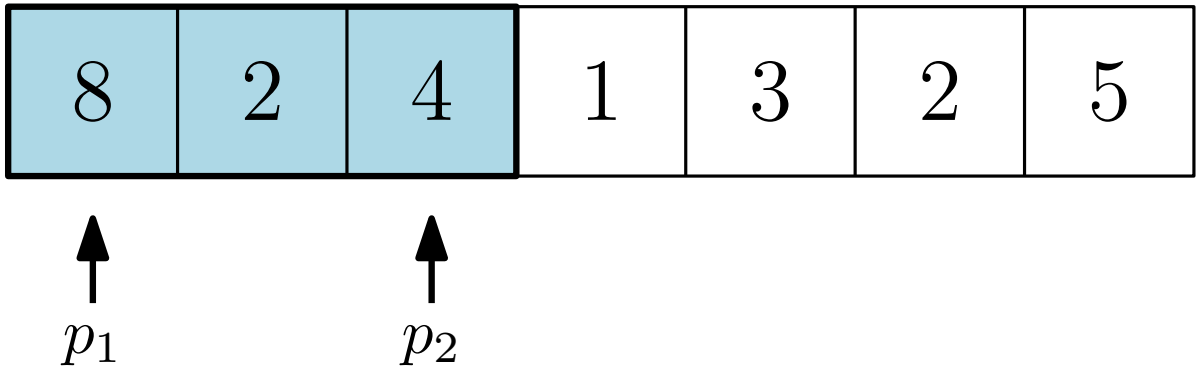
Can we do better?

Sliding Window

Sliding Window: Idea

- Keep two pointers p_1, p_2 to keep track of the current *window* W, i.e., the subsequence between p_1 and p_2.

$k = 9$



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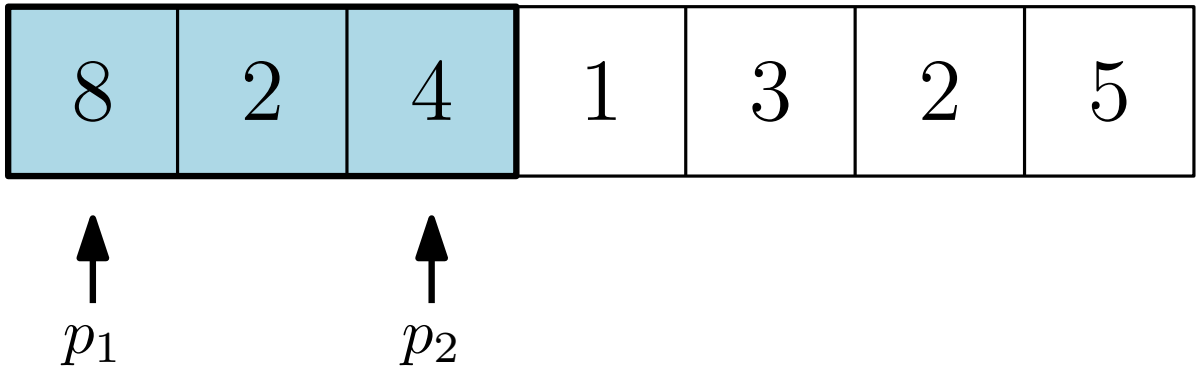

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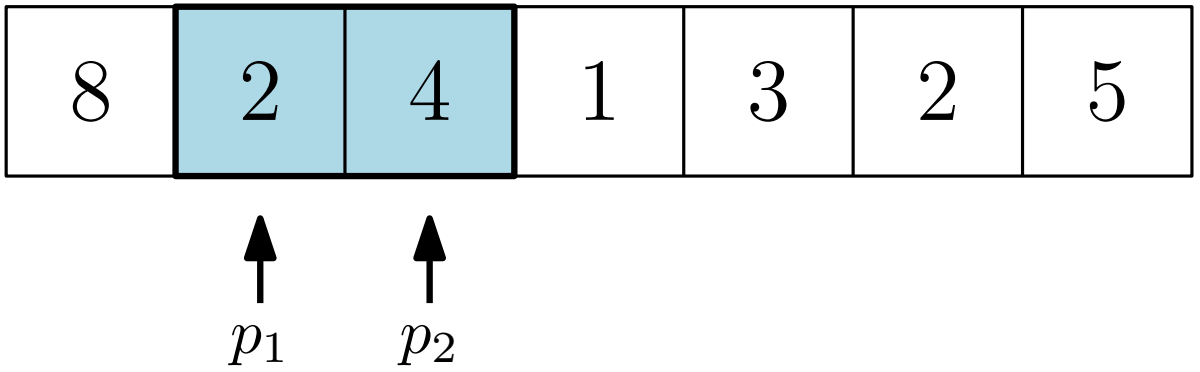

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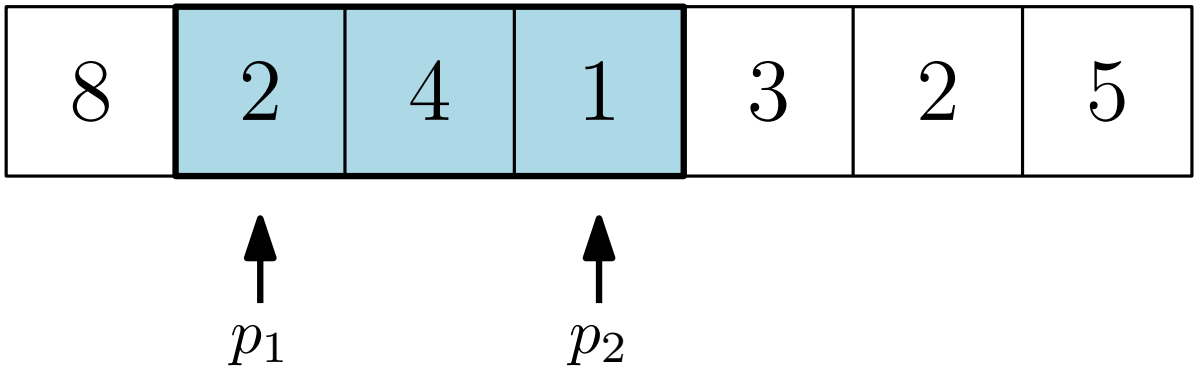

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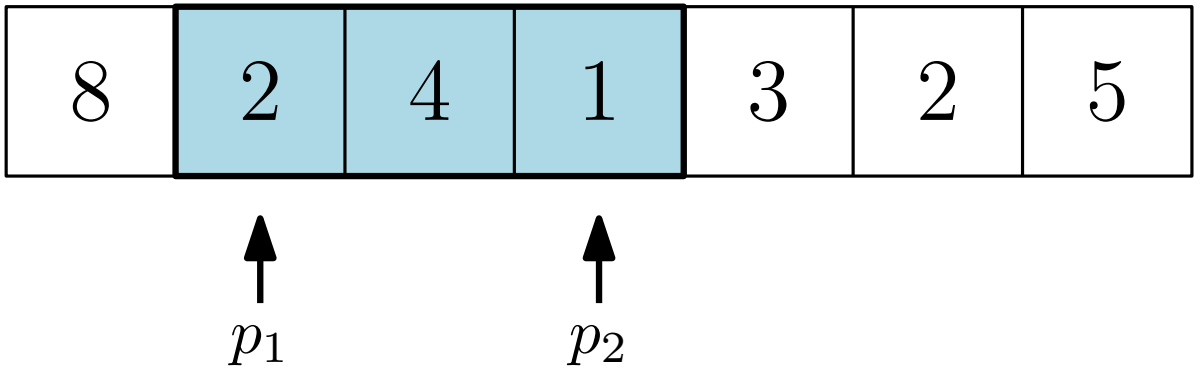

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- If $\sigma(p_1, p_2)$ is too large: increase p_1.
- If $\sigma(p_1, p_2)$ is too small: increase p_2.
- Return "best" feasible window among those considered.

(plus suitable handling of edge cases)

A possible implementation

```
int left=0, right=-1, sum=0;
int best_left=-1, best_right=-1, best_sum=-1;

do
{
    if(sum<=k && right<n-1)
        sum += A[++right];
    else
        sum -= A[left++];

    if(sum<=k && sum>best_sum)
    {
        best_sum = sum; best_left = left; best_right = right;
    }
} while(left<n-1 || right<n-1);

std::cout << "Cut from position " << best_left+1
          << " to position " << best_right+2 << "\n";
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$O(n)$

Why does it work?

Observation: p_1 (and p_2) will get all values from 1 to n.

- Let $W^* = [p_1^*, p_2^*]$ be an optimal interval minimizing p_1^*.
- Consider the first instant where $p_1 = p_1^*$ or $p_2 = p_2^*$.

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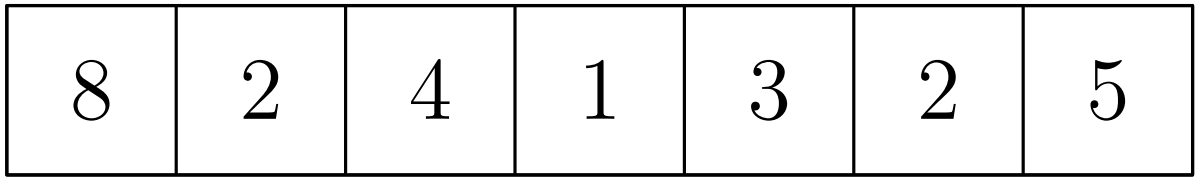

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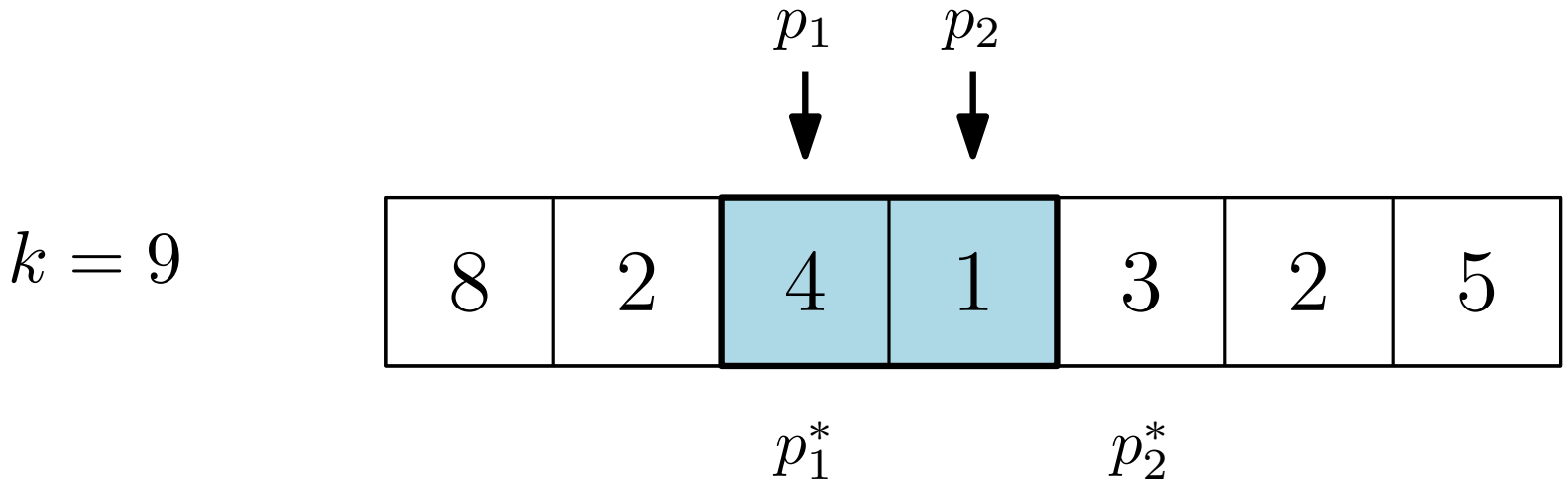

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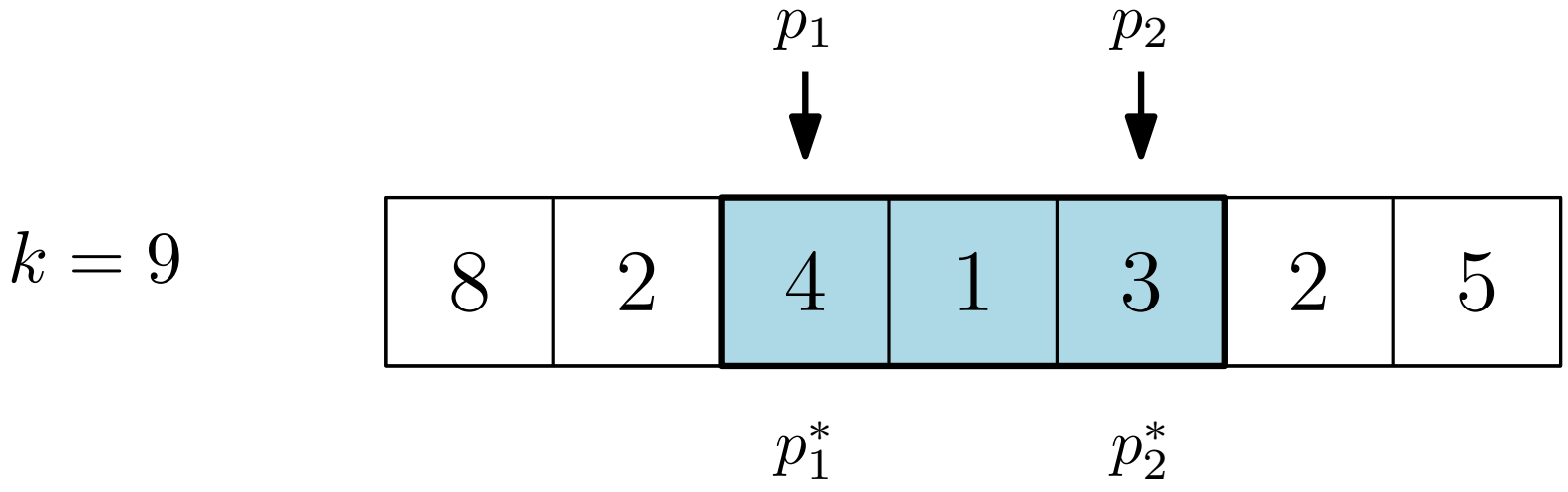
- Therefore, p_2 will be incremented until it reaches p_2^* while $p_1 = p_1^*$ remains constant $\Longrightarrow$ the algorithm considers W^*.



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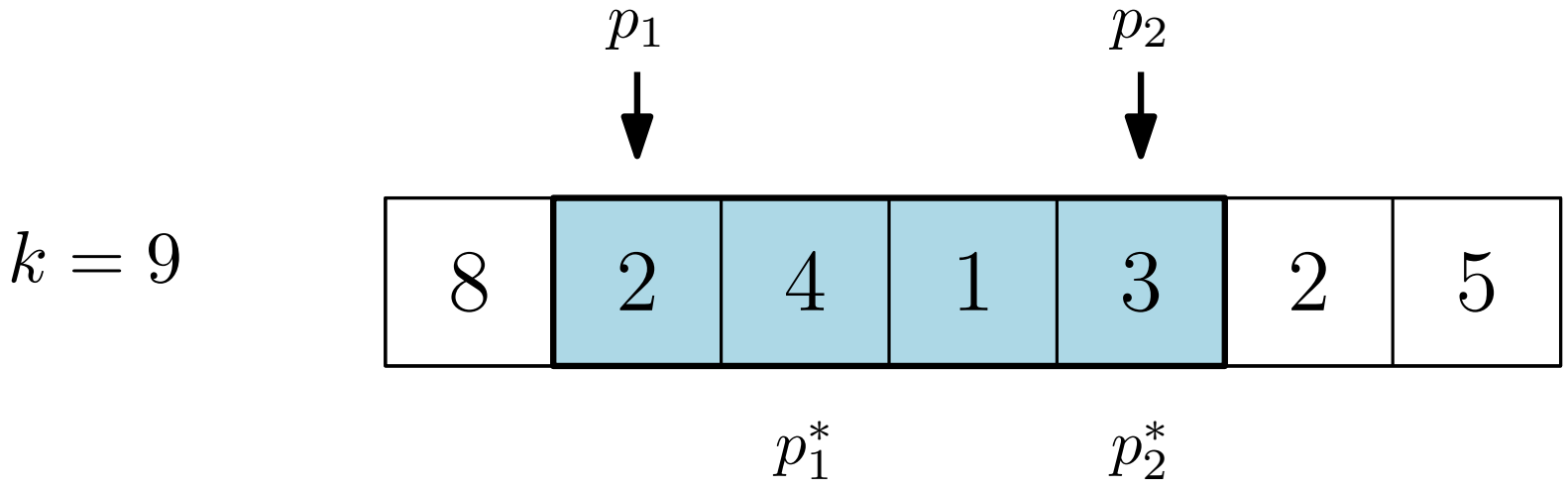

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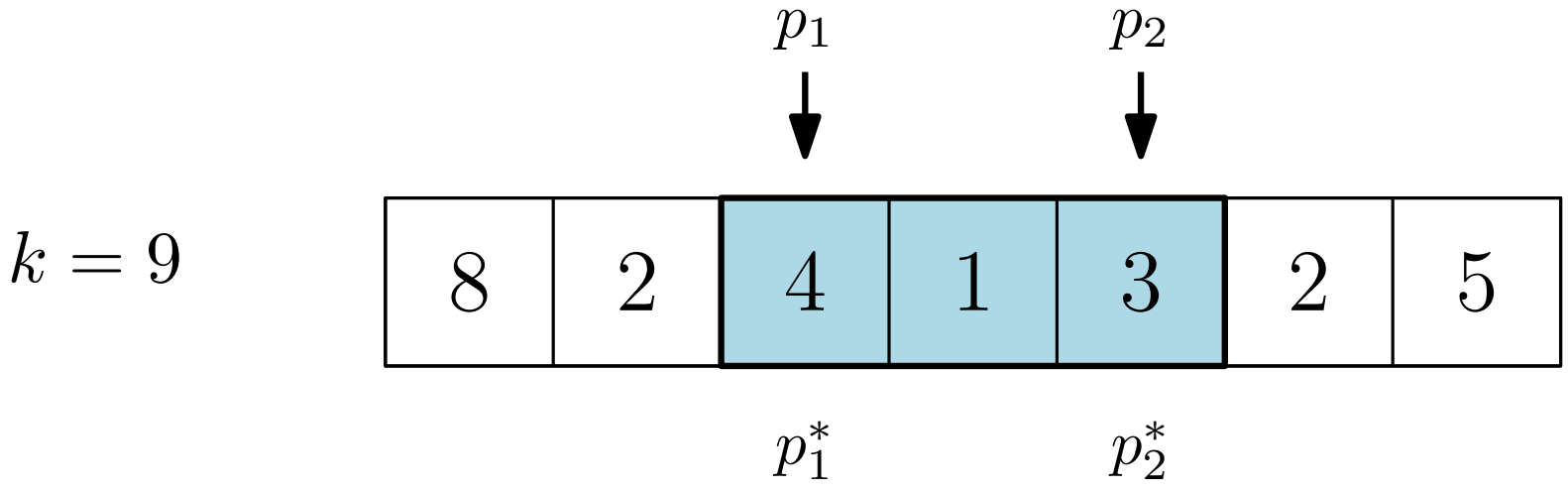

Sliding Window

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Trick/Technique: Sliding Window

Some problems in which you need to find an interval can be solved in linear time using a sliding window approach, if you can ensure that an optimal interval will be considered.