Divide and Conquer

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- **Divide:** Decompose an instance of a problem into smaller instances of the same problem
- **Conquer:** Solve each subproblem (recursively)
- **Recombine** the subproblems' solutions into a solution to the original problem

Polynomial Multiplication

Problem: Given two polynomials $P(x), Q(x)$ of degree n, compute $R(x) = P(x) \cdot Q(x)$

Instance:

- The coefficients $p_0, p_1, \ldots, p_n \in \mathbb{Z}$ of $P(x) = \sum_{i=0}^{n} p_i x^i$.
- The coefficients $q_0, q_1, \ldots, q_n \in \mathbb{Z}$ of $Q(x) = \sum_{i=0}^{n} q_i x^i$.

Solution:

- The coefficients $r_0, r_1, \ldots, r_{2n} \in \mathbb{Z}$ of
$$R(x) = P(x) \cdot Q(x) = \sum_{i=0}^{2n} r_i x^i.$$

(Assume that arithmetic operations can be performed in $O(1)$ time).

Example

$$P(x) = 1 + 2x + 3x^2$$

$$Q(x) = 3 + 0x + 5x^2$$

$$R(x) = P(x) \cdot Q(x) = 3 + 6x + 14x^2 + 10x^3 + 15x^4$$

How to compute $R(x)$ efficiently?

Intermission: A More General Problem

Given two binary operations $\oplus, \otimes$ and two functions $f, g : \mathbb{Z} \to \mathbb{R}$, the $(\oplus, \otimes)$-discrete convolution of f and g is a function $(f * g) : \mathcal{Z} \to \mathcal{R}$ defined as:

$$(f * g)(n) = \bigoplus_{m=-\infty}^{+\infty} \Big(f(n-m) \otimes g(m) \Big)$$

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Consider the arrays P and Q associated with the polynomials $P(x)$ and $Q(x)$. Define $f(n) = p_n$, $g(n) = q_n$ (and 0 elsewhere) . The $(+, \cdot)$ convolution of P and Q is:

$$r_n = (f * g)(n) = \sum_{m=0}^{n} p_{n-m} q_m$$

Back to Polynomials: A Trivial Solution

$$r_i = \sum_{j=0}^{i} p_{i-j} q_j$$

```
• For $i = 0, \ldots, 2n$ :
    • $r_i \leftarrow 0$
    • For $j = \max\{0, i-n\}, \ldots, \min\{i, n\}$:
        • $r_i \leftarrow r_i + p_{i-j} \cdot q_j$
```

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Time Complexity: $\Theta(n^2)$

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Can we do better?

Divide and Conquer: First Attempt

- Write P as: $P(x) = P'(x) + P''(x) \cdot x^{\lfloor n/2 \rfloor}$, where:

$$P'(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} p_i x^i \qquad \text{and} \qquad P''(x) = \sum_{i=1+\lfloor n/2 \rfloor}^{n} p_i x^{i-\lfloor n/2 \rfloor}$$

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$$P(x) \cdot Q(x) = (P'(x) + P''(x) \cdot x^{\lfloor n/2 \rfloor}) \cdot (Q'(x) + Q''(x) \cdot x^{\lfloor n/2 \rfloor})$$

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$$= P'(x)Q'(x) + (P'(x)Q''(x) + P''(x)Q'(x))x^{\lfloor n/2 \rfloor} + P''(X)Q''(x)x^{2\lfloor n/2 \rfloor}$$

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The problem of computing the product of two polynomials of degree n is reduced to that of computing 4 products of polynomials of degree $\approx n/2$.

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The problem of computing the product of two polynomials of degree n is reduced to that of computing 4 products of polynomials of degree $\approx n/2$.

Recurrence Equation:

$$T(n) = 4T(n/2) + O(n)$$

$O(n)$ time is needed to decompose the polynomials and to recombine the 4 sub-products.

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☹ Still $\Theta(n^2)$

Solution: $\Theta(n^2)$

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We want:

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Define:

$$U = P'(x)Q'(x) \qquad V = P''(x)Q''(x)$$

$$W = (P'(x) + P''(x))(Q'(x) + Q''(x))$$

Divide and Conquer: Second Attempt

We want:

$$\underset{\substack{\Big\downarrow \\ U}}{P'(x)Q'(x)} + (\underset{\substack{\Big\downarrow}}{\underbrace{P'(x)Q''(x) + P''(x)Q'(x)}})x^{\lfloor n/2 \rfloor} + \underset{\substack{\Big\downarrow \\ V}}{P''(X)Q''(x)}x^{2\lfloor n/2 \rfloor}$$

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Only requires 3 multiplications $\Longrightarrow$ 3 subproblems of size $\sim n/2$

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- **Divide:**

  $U = P'(x) \cdot Q'(x)$ (subproblem 1)

  $V = P''(x) \cdot Q''(x)$ (subproblem 2)

  $W = (P'(x) + P''(x)) \cdot (Q'(x) + Q''(x))$ (subproblem 3)

- **Conquer:** Compute U, V, W recursively

- **Recombine:** $U + (W - U - V)x^{\lfloor n/2 \rfloor} + Vx^{2\lfloor n/2 \rfloor}$

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Reurrence Equation: $T(n) = 3T(n/2) + O(n)$

Solution: $O(n^{\log_2 3}) = O(n^{1.585})$

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Trick/Technique: Divide and Conquer

Decompose an instance into smaller instances of the same problem.
Solve recursively and recombine the solutions.

Recursion & Memoization

Fibonacci Numbers

Definition: $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i > 1$

Problem: Given $n \in \mathbb{N}$, compute F_n

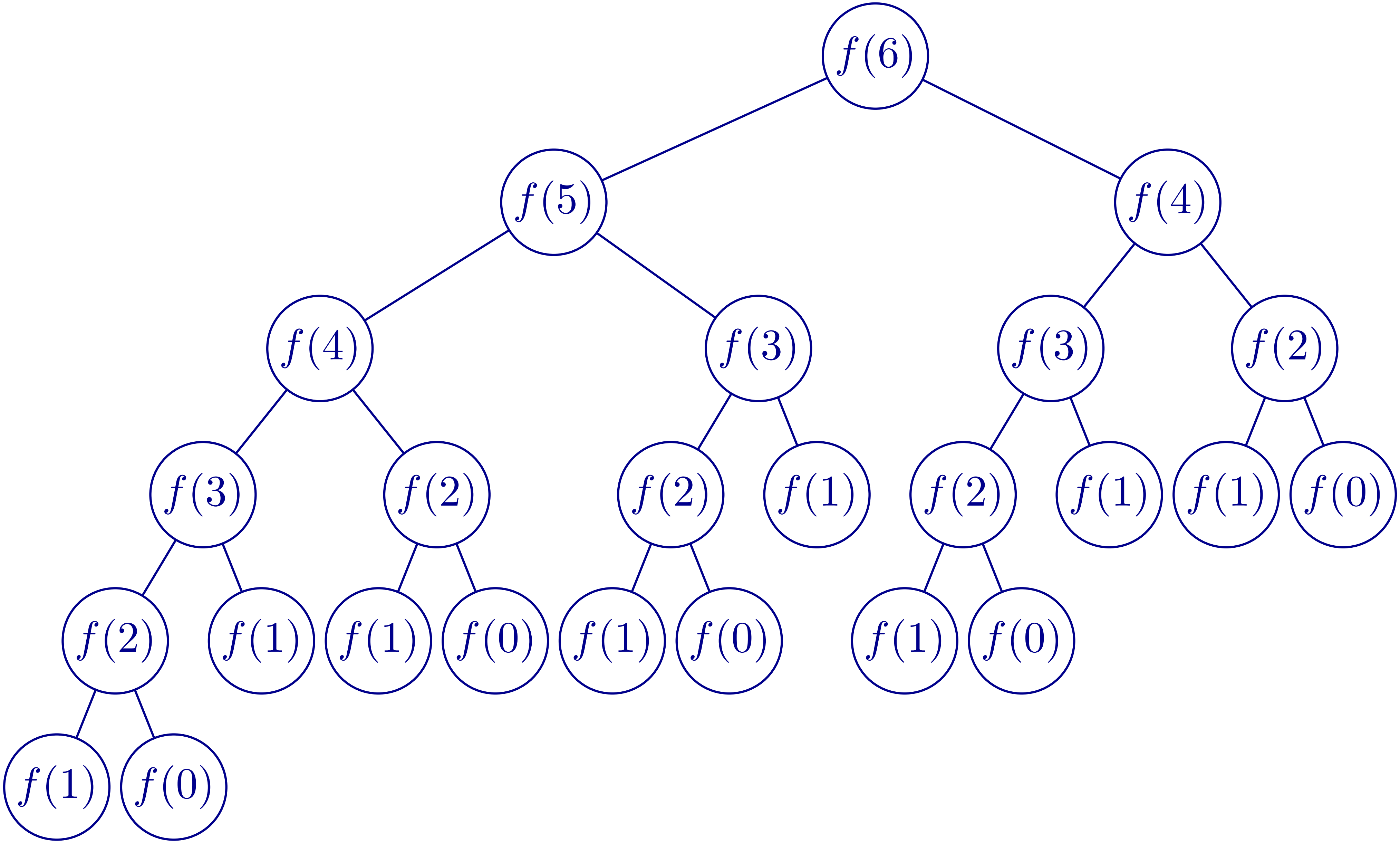
A trivial **recursive** solution:

```
int fibonacci(int n)
{
    if(n<=1)
        return n;

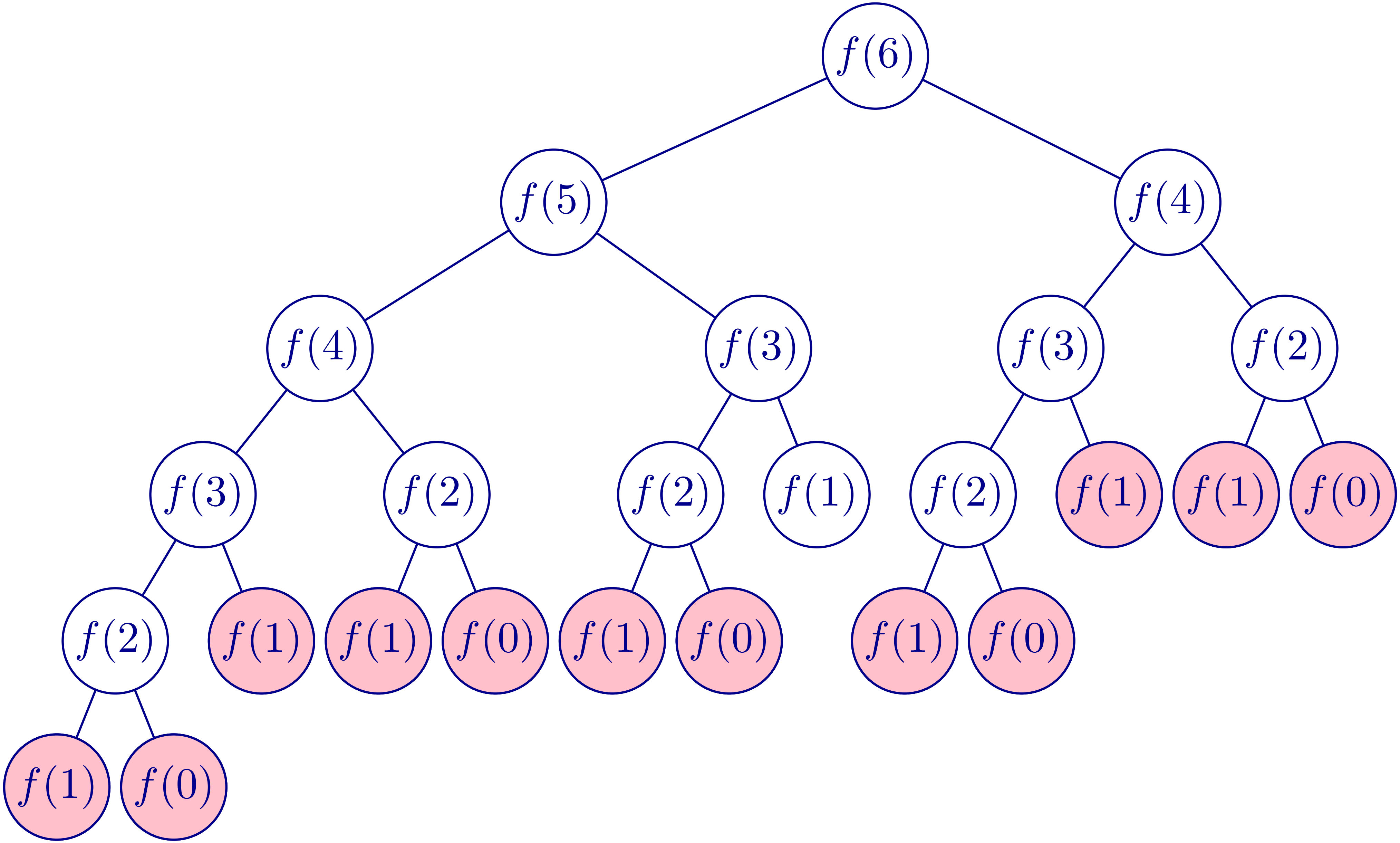
    return fibonacci(n-1) + fibonacci(n-2);
}
```

Computational complexity?

Fibonacci Numbers: Time Complexity



Fibonacci Numbers: Time Complexity



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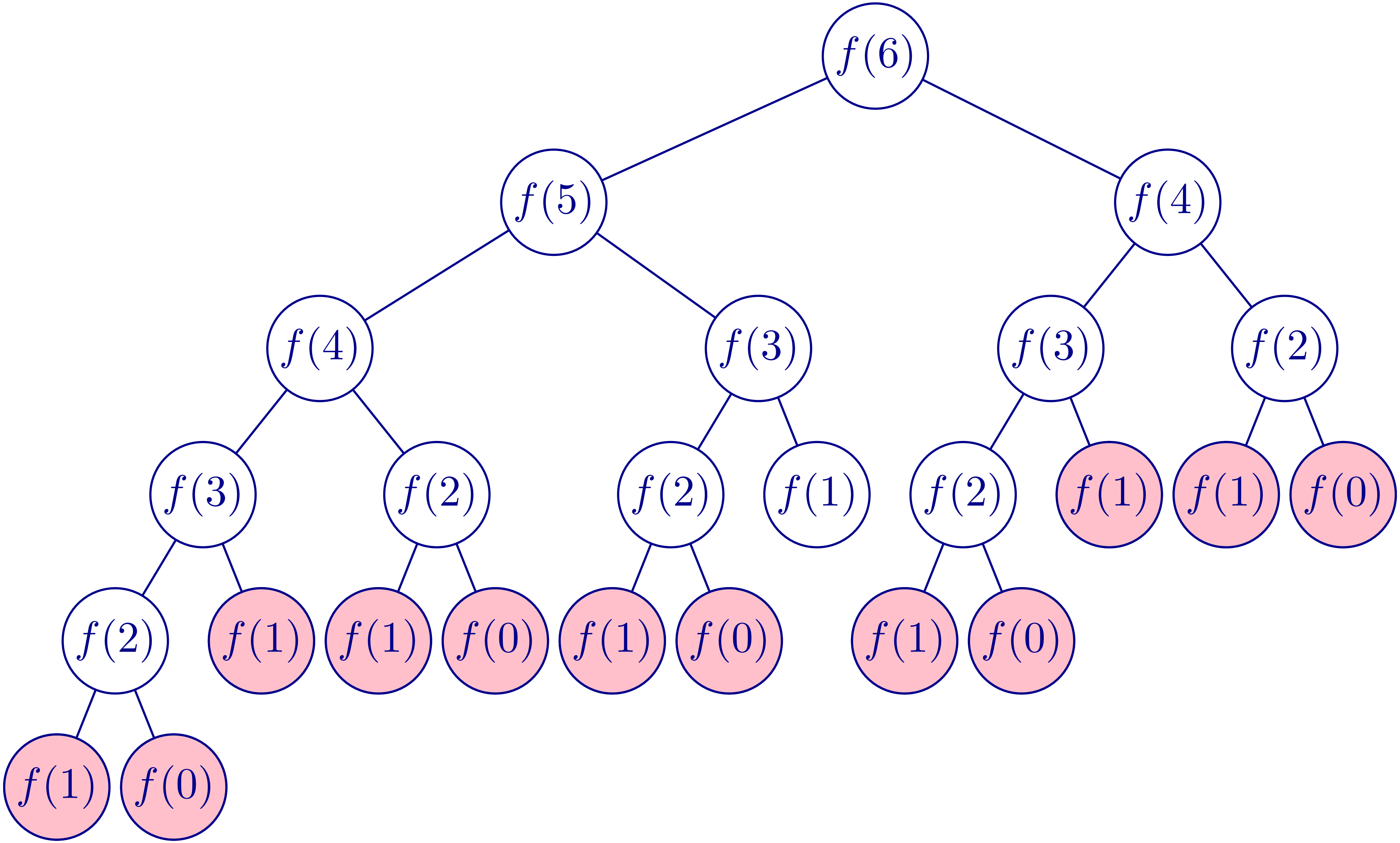
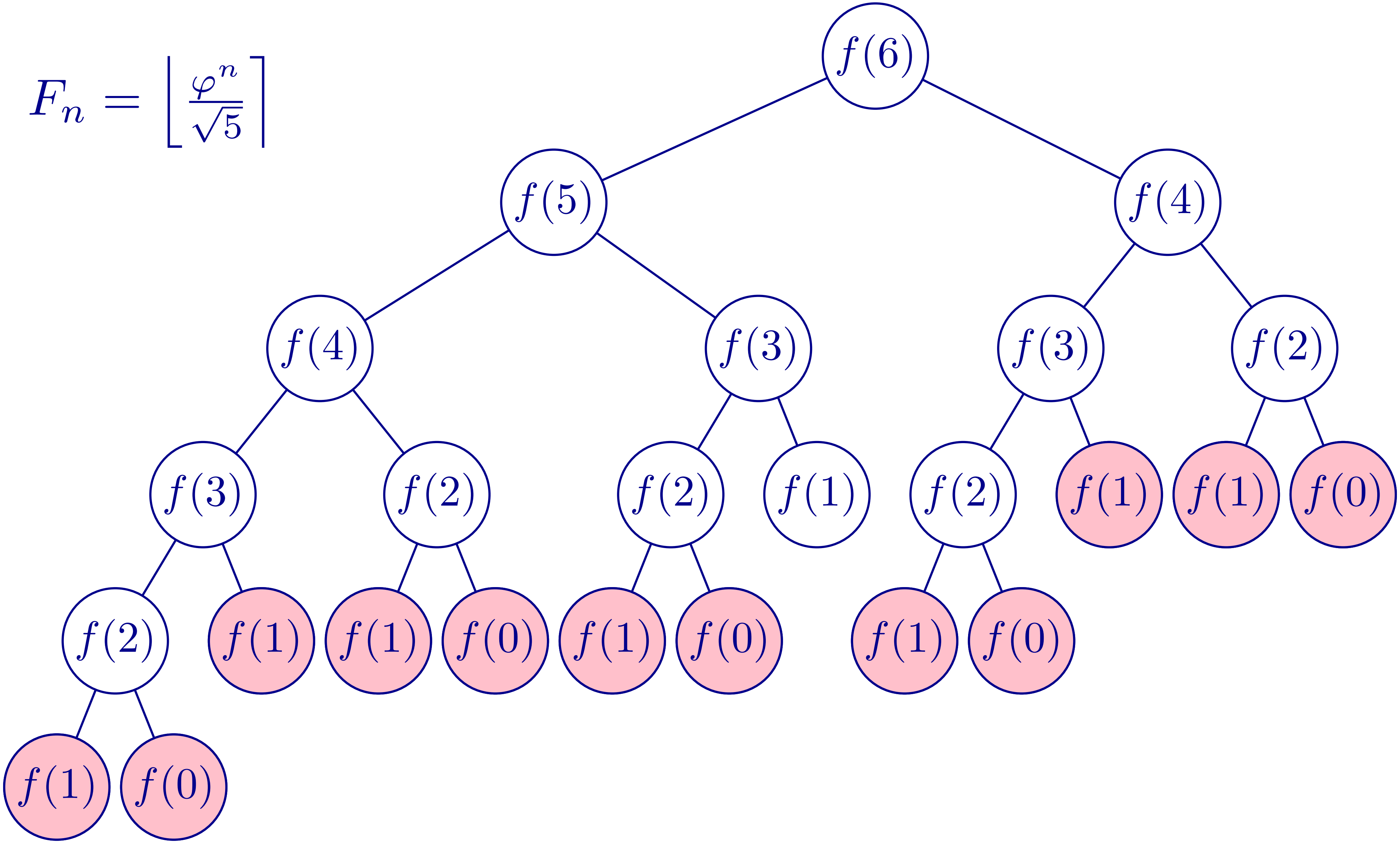


Time $= \Theta(1) \cdot$ #Nodes $= \Theta($#Leaves$) = \Theta(F_n)$

Fibonacci Numbers: Time Complexity

$$F_n = \left\lfloor \frac{\varphi^n}{\sqrt{5}} \right\rceil$$



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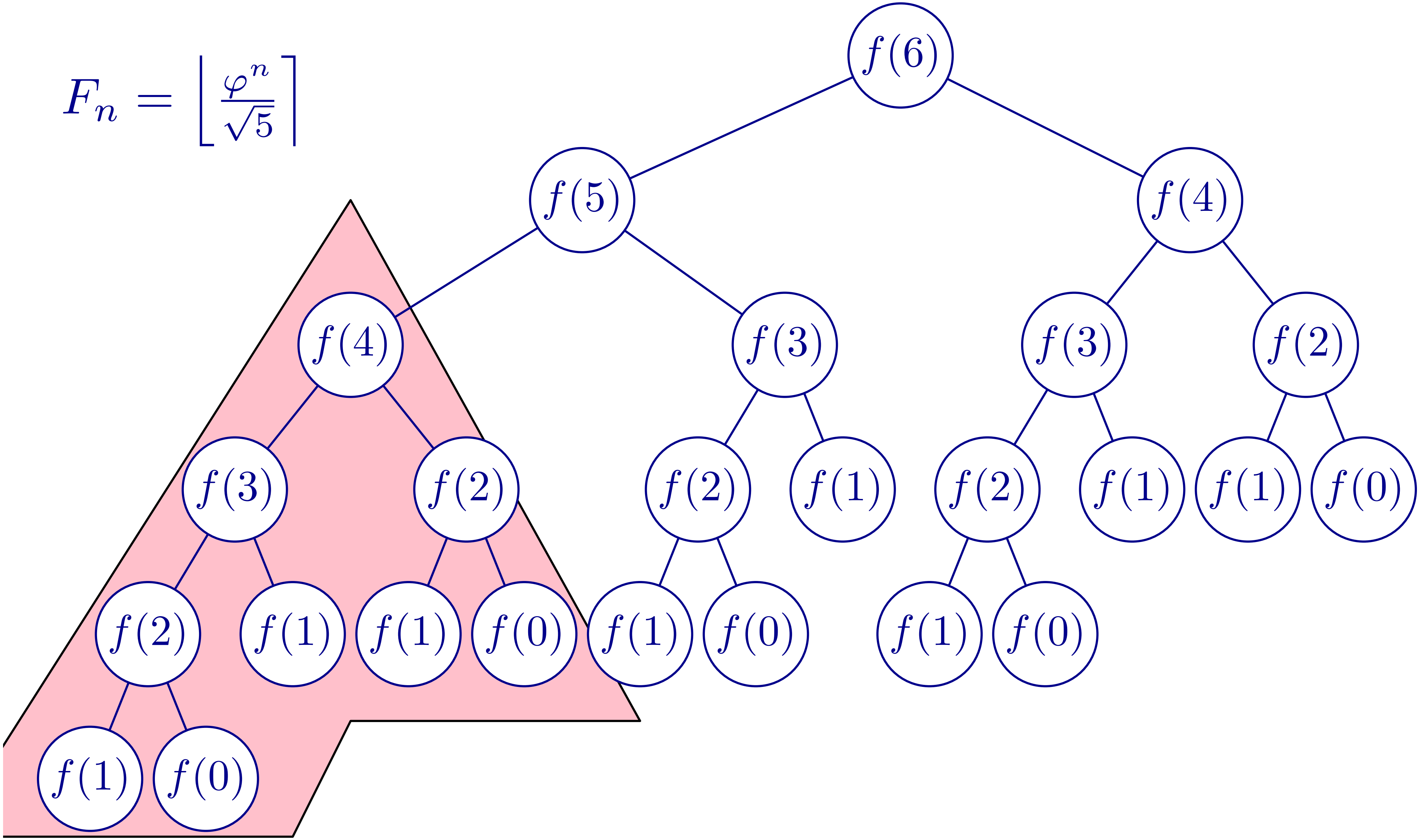

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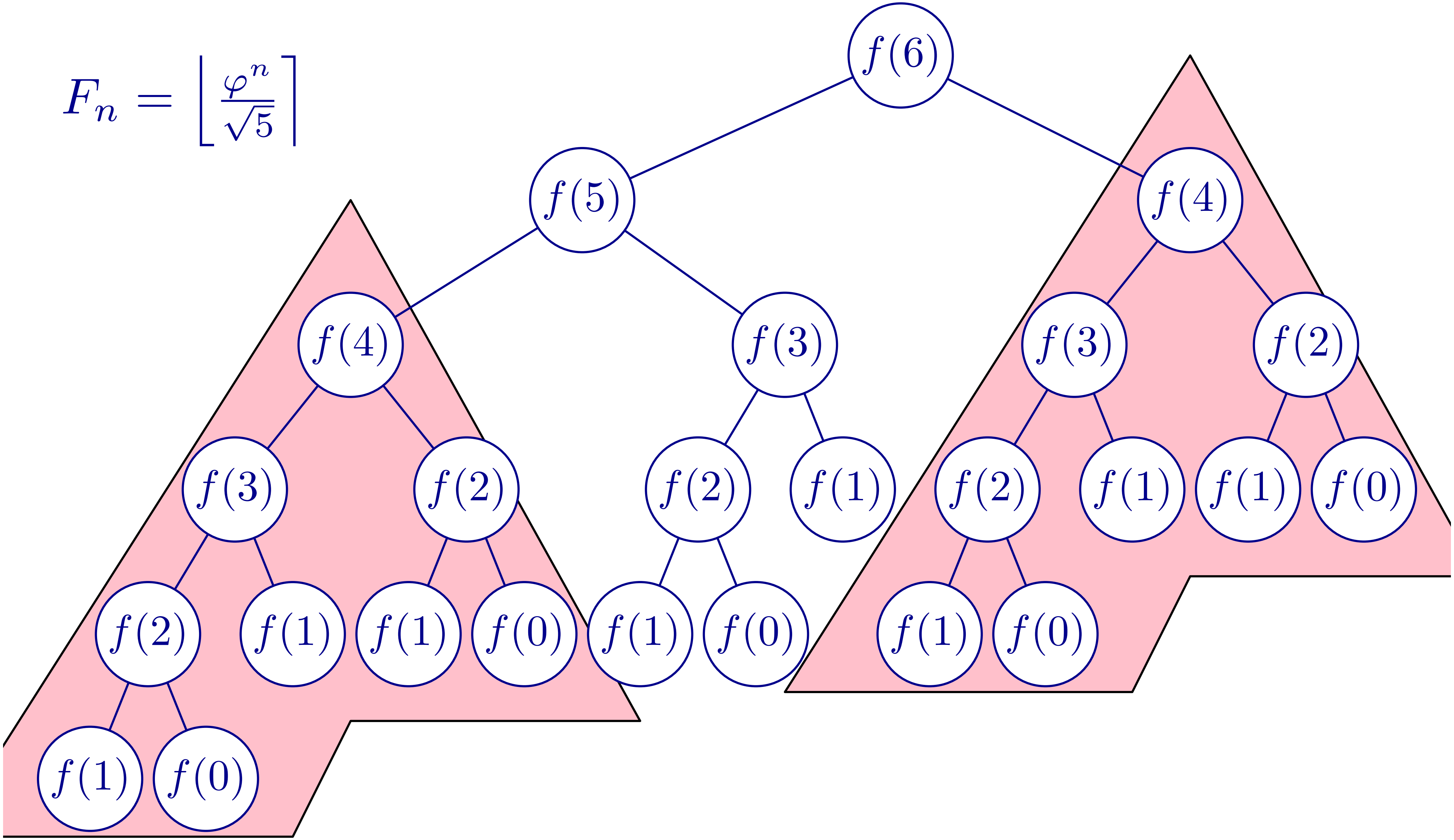

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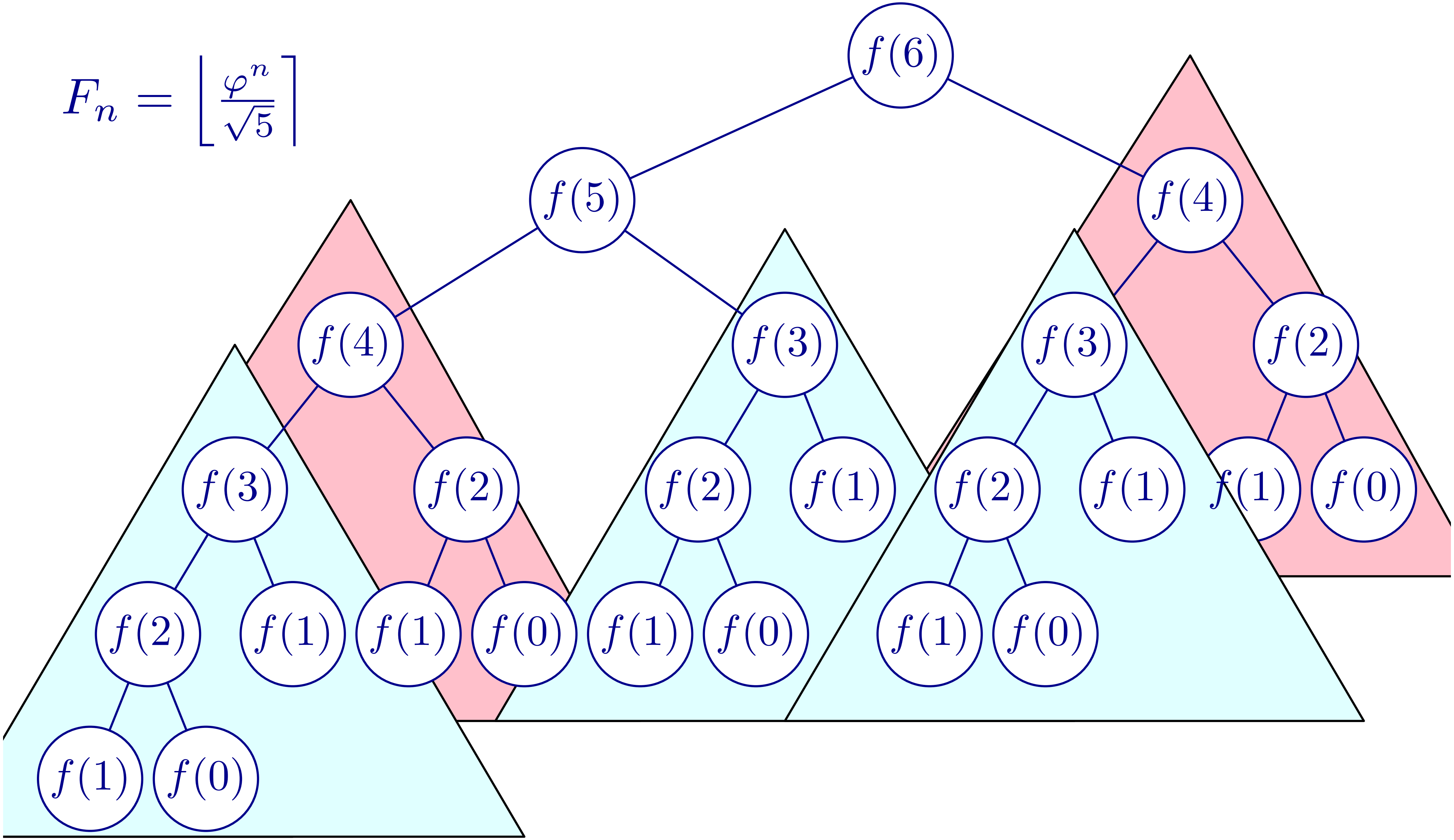

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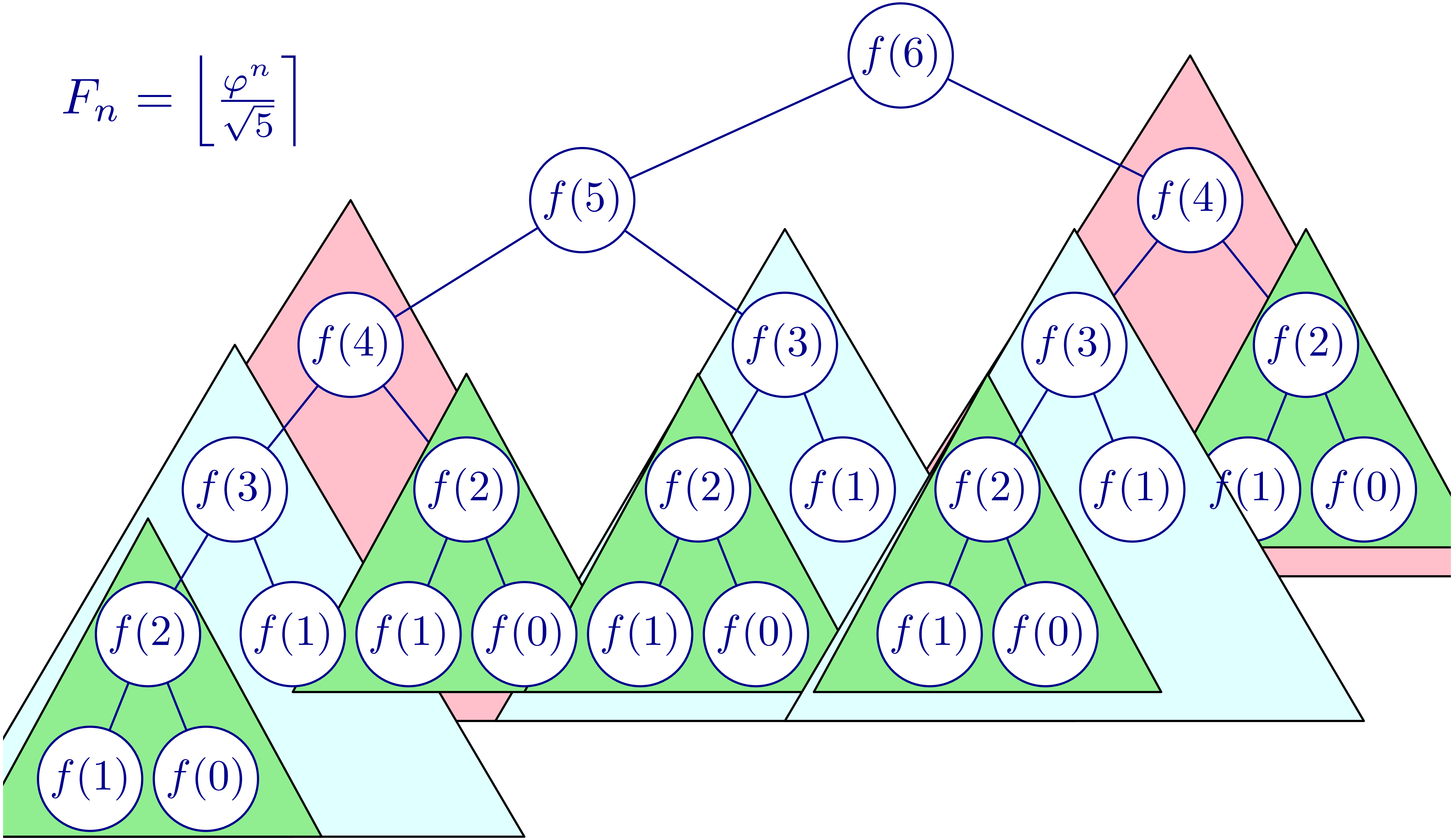

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Fibonacci Numbers: Memoization

Idea: Do not recompute duplicate values:

- Store values in memory
- If value is in memory, recall it
- Otherwise, compute and store it

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std::vector<int> memo(n+1, 0);

int fibonacci(int n)
{
    if(n<=1) return n;

    if(memo[n]) return memo[n];

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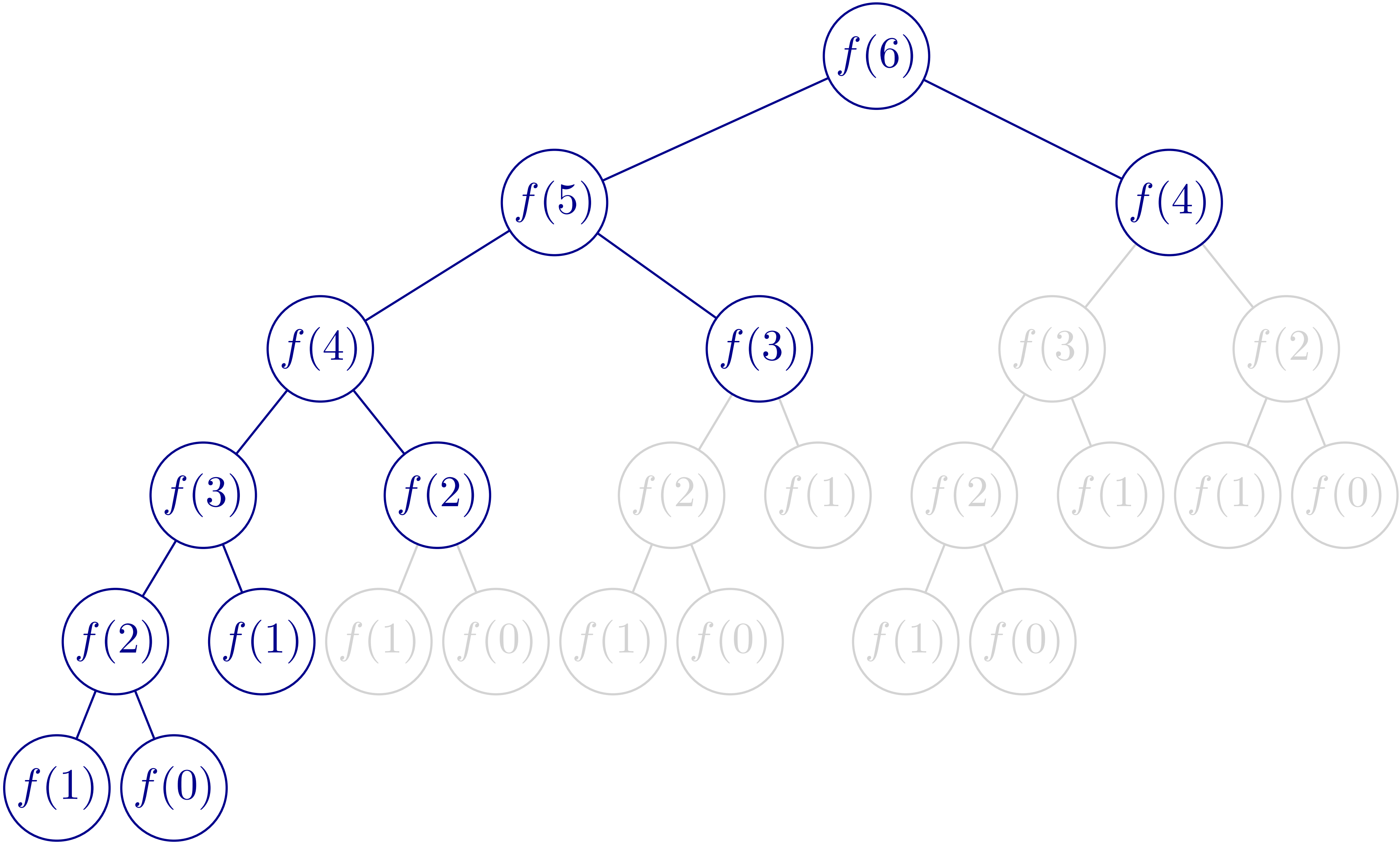
Time Complexity with Memoization



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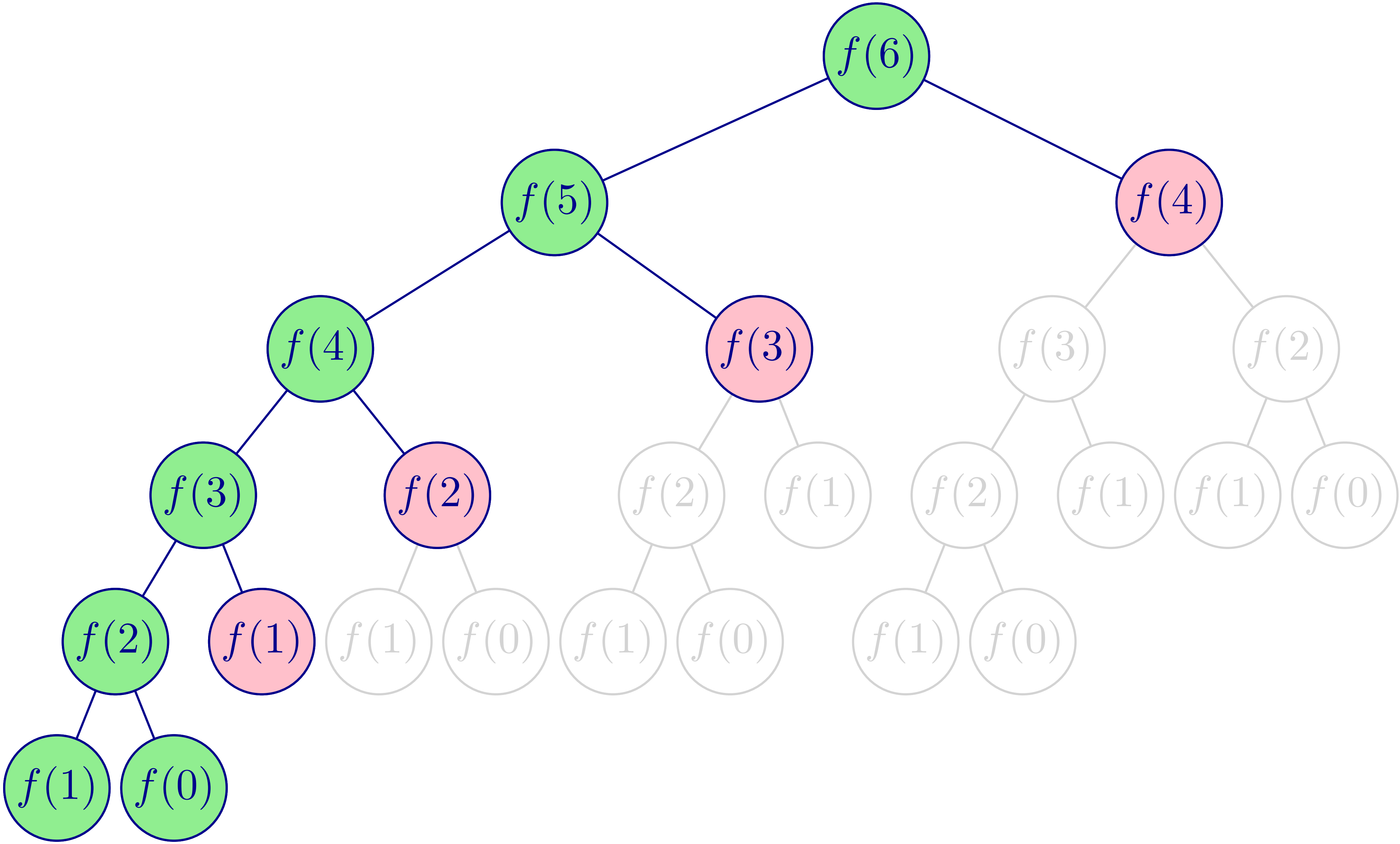

Time $= \Theta(1) \cdot$ #Green Nodes $= \Theta(n)$

The Memoization Recipe

- Design a recursive algorithm for the problem (hard)
- Add memoization (easy)

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Trick/Technique: Memoization

Avoid recomputing solutions to duplicate subproblems by storing results in memory.

Memoization: Pitfalls

Let $G_{-1} = G_0 = 1$, and $G_i = \begin{cases} 2G_{i-1} & \text{if } i \text{ is even} \\ G_{i-2} + 3 & \text{if } i \text{ is odd} \end{cases}$, for $i \geq 1$.

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std::vector<int> memo(n+1, 0);

int g(int n)
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    if(n<=0) return 1;

    memo[n] = (i%2)?(g(n-2)+3):(2*g(n-1));
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}
```

Does this code work?

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Does this code work?   No! `n` can be -1!

Solution: check base cases before the memo table.

Memoization: Pitfalls

$G_0 = 0$, $G_1 = 1$, and $G_i = (G_{i-1} + G_{i-2} + 1) \bmod 2$, for $i \geq 1$.

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}
```

Too slow! Why?

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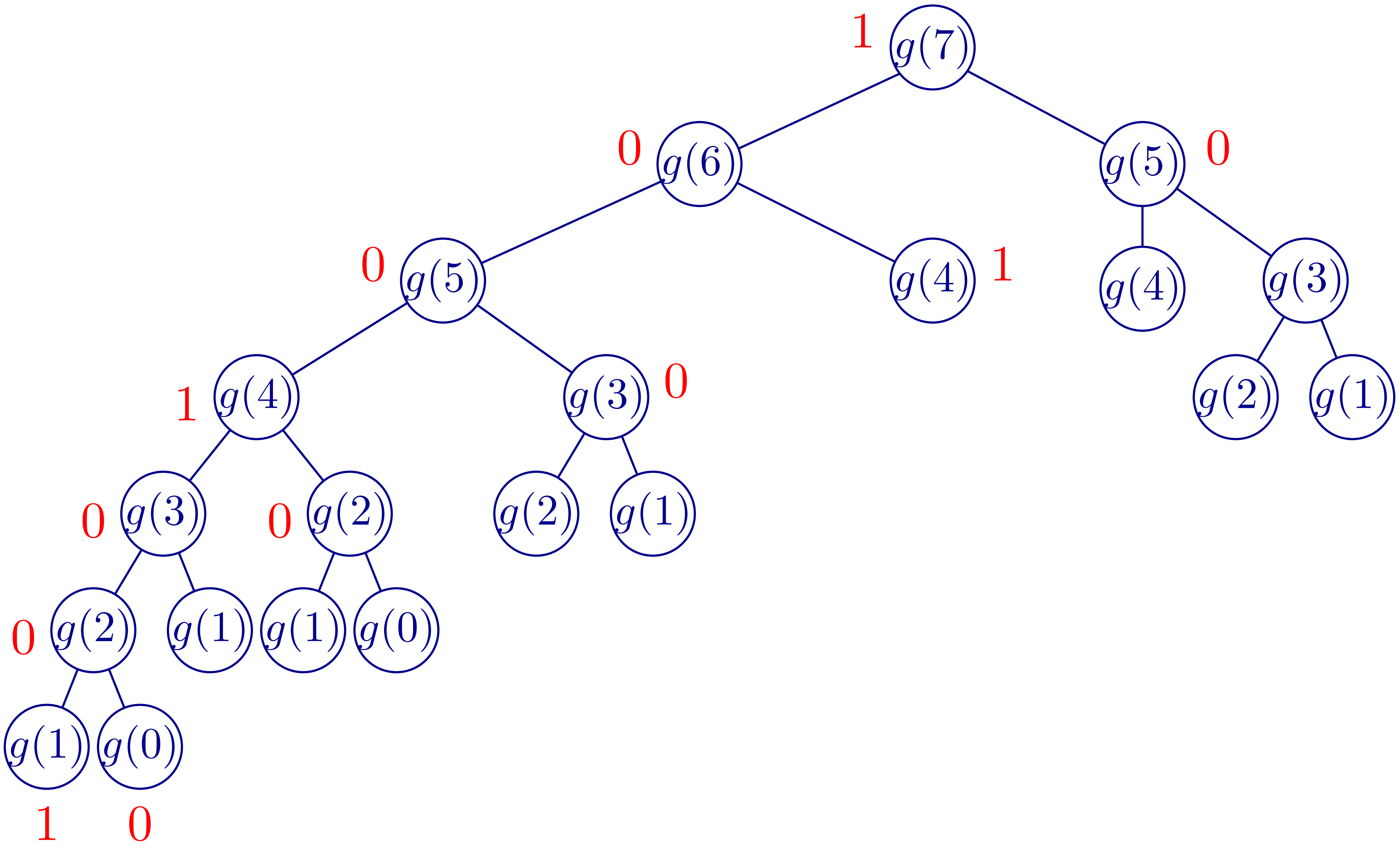
Too slow! Why?

0 is a possible value of G_i !

Memoization: Pitfalls

$n = 7$



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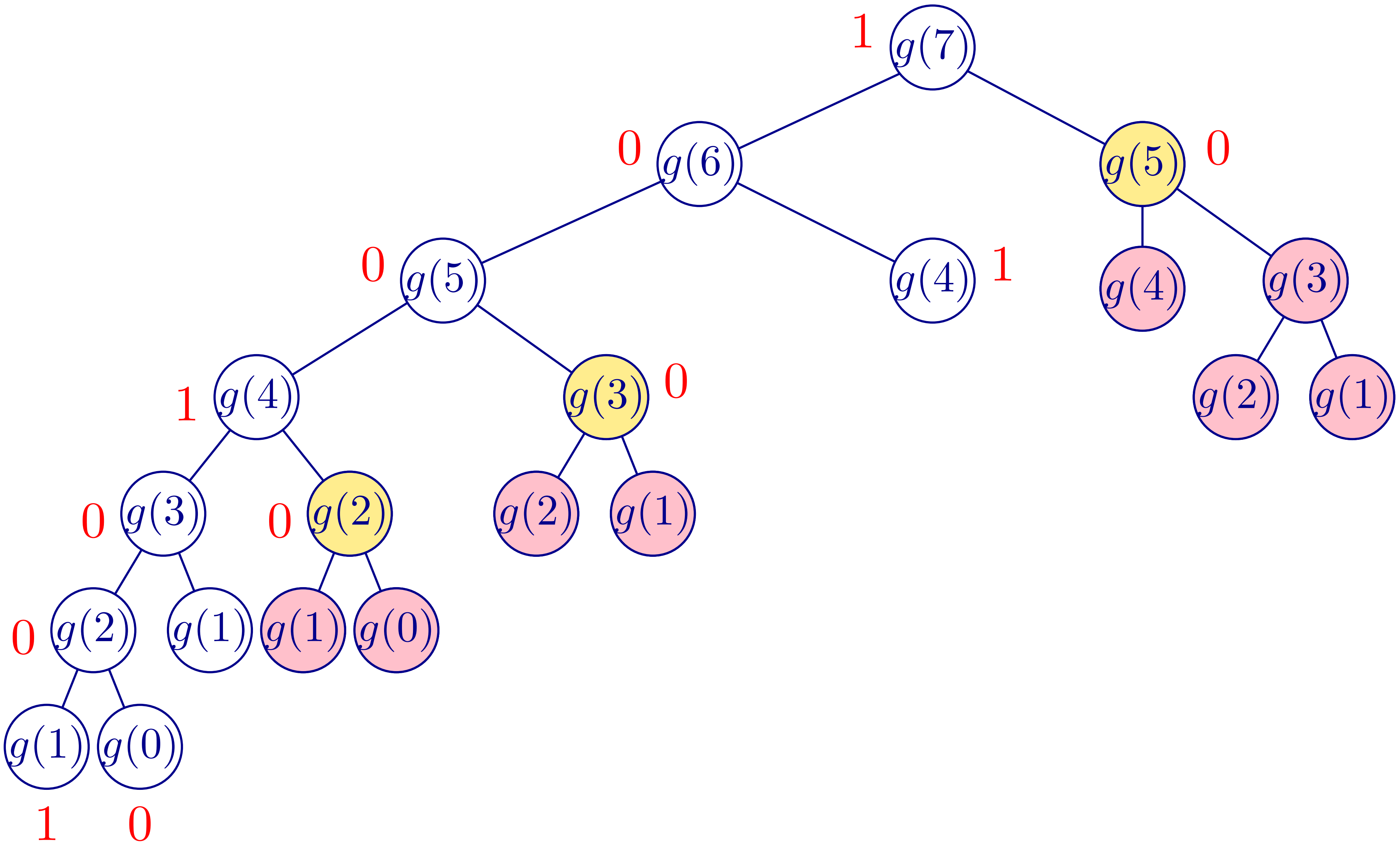

Dynamic Programming

Dynamic Programming

I spent the Fall quarter (of 1950) at RAND. [...] We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word research. [...] he would get violent if people used the term research in his presence. [...] The RAND Corporation was employed by the Air Force, and the Air Force had Wilson as its boss, essentially. Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. [...] I decided therefore to use the word "programming". I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying. [...] Let's take a word that has an absolutely precise meaning, namely dynamic, in the classical physical sense. It also has a very interesting property as an adjective, and that is it's impossible to use the word dynamic in a pejorative sense. [...] Thus, I thought **dynamic programming** *was a good name. It was something not even a Congressman could object to.*

Richard E. Bellman,
Eye of the Hurricane: An Autobiography

Dynamic Programming: Idea

- Decompose a problem into a series of “overlapping” subproblems
- The solutions to the “smallest” subproblems are trivially known
- The optimal solution to a subproblem can be reconstructed from the optimal solutions of “smaller” subproblems
- Systematically solve subproblems in a suitable order (from the “smaller” to the “larger” ones)
- Eventually, either the solution to the original problem is explicitly computed or it can be reconstructed from the subproblem’s solutions

Dynamic Programming: Idea

- Decompose a problem into a series of “overlapping” subproblems (hard)
- The solutions to the “smallest” subproblems are trivially known (easy)
- The optimal solution to a subproblem can be reconstructed from the optimal solutions of “smaller” subproblems (hard)
- Systematically solve subproblems in a suitable order (from the “smaller” to the “larger” ones) (easy)
- Eventually, either the solution to the original problem is explicitly computed or it can be reconstructed from the subproblem’s solutions (easy)

Fibonacci, Revisited

- i-th subproblem: Compute the value of F_i
- Base cases: $i = 0, i = 1.$
- Compute F_i in increasing order of i: $\quad F_i = F_{i-1} + F_{i-2}$
- Both F_{i-1} and F_{i-2} are already known when F_i is considered.
- Solution: F_n

```
std::vector<int> F(n+1);
F[0]=0; F[1]=1;

for(int i=2; i<=n; i++)
   F[i] = F[i-1] + F[i-2];

return F[n];
```

Fibonacci, Revisited

Trick to reduce space:

- Once we compute F_i, the values $F_0, \ldots, F_{i-2}$ will not be used anymore.
- Keep track of just two values x_0, x_1.
- At the end of iteration i, $F_i = x_{i \bmod 2}$ and $F_{i-1} = x_{(i-1) \bmod 2}$.

```
int x[2] = {0, 1};

for(int i=2; i<=n; i++)
    x[i%2] = x[(i-1)%2] + x[(i-2)%2];

return x[n%2];
```

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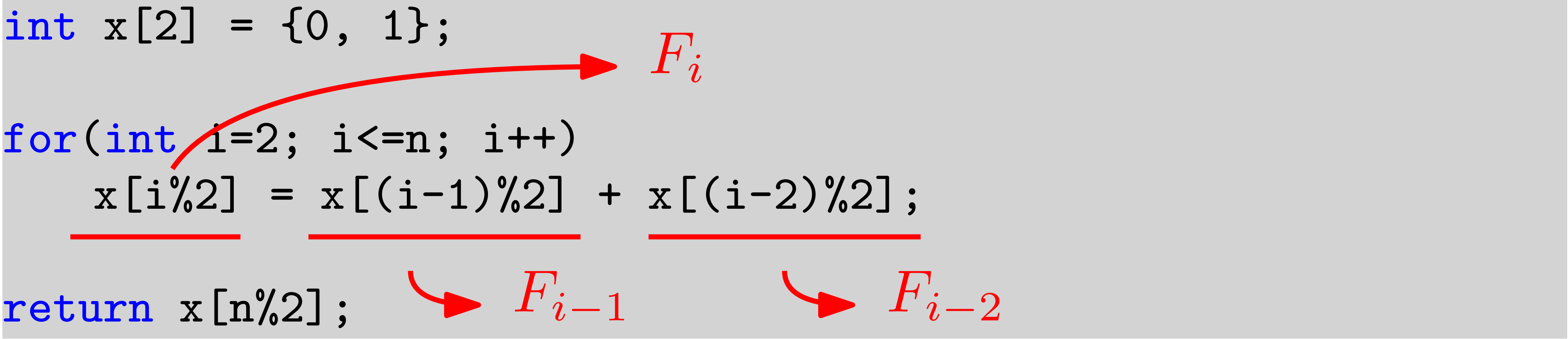

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return x[n%2];
```

Drink as much as possible

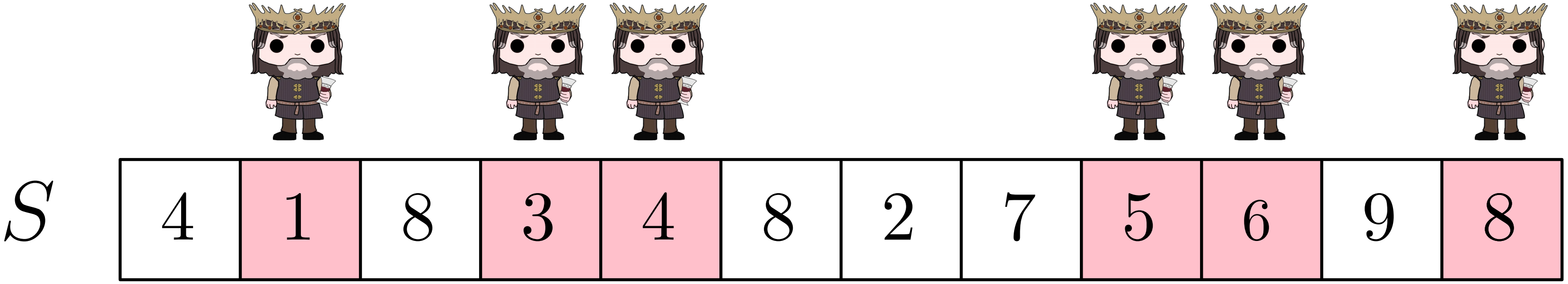
Robert wants to drink as much a possible.

- Robert walks through the streets of King's Landing and encounters n taverns $t_1, t_2, \dots, t_n$, in order
- When Robert encounters a tavern t_i, he can either stop for a drink or continue walking
- The wine served in tavern t_i has strength $s_i \in \mathbb{N}$ (the higher, the stronger)
- The strength of robert's drinks must increase over time
- **Goal:** Compute the maximum number of drinking stops of Robert

Example

S	4	1	8	3	4	8	2	7	5	6	9	8

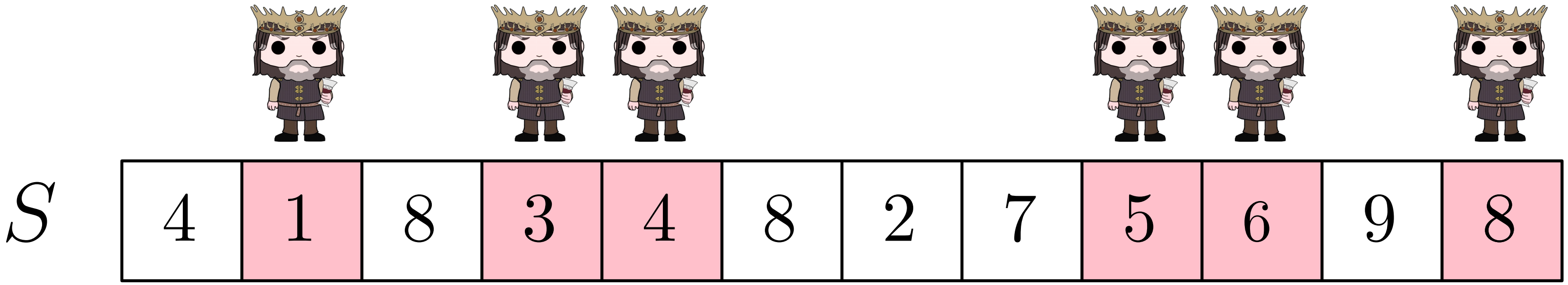
Example



S	4	1	8	3	4	8	2	7	5	6	9	8

Solution: 6

Example



S	4	1	8	3	4	8	2	7	5	6	9	8

Solution: 6

This is a classic problem known as:
Longest Increasing Subsequence (LIS)

A DP Algorithm: First Attempt

- Subproblem definition

$$OPT[i] = \text{Length of the LIS in } S[1], \ldots, S[i]$$

A DP Algorithm: First Attempt

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  $OPT[i] =$ Length of the LIS in $S[1], \ldots, S[i]$

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  $OPT[1] = 1$

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- Recursive formula

A DP Algorithm: Second Attempt

Tip: Sometimes adding constraints to subproblems helps!

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S	4	1	8	3	4	8	2	7	5	6	9	8
	1	2	3	4	5	6	7	8	9	10	11	12
OPT	1	1	2	2	3	4	2	4				

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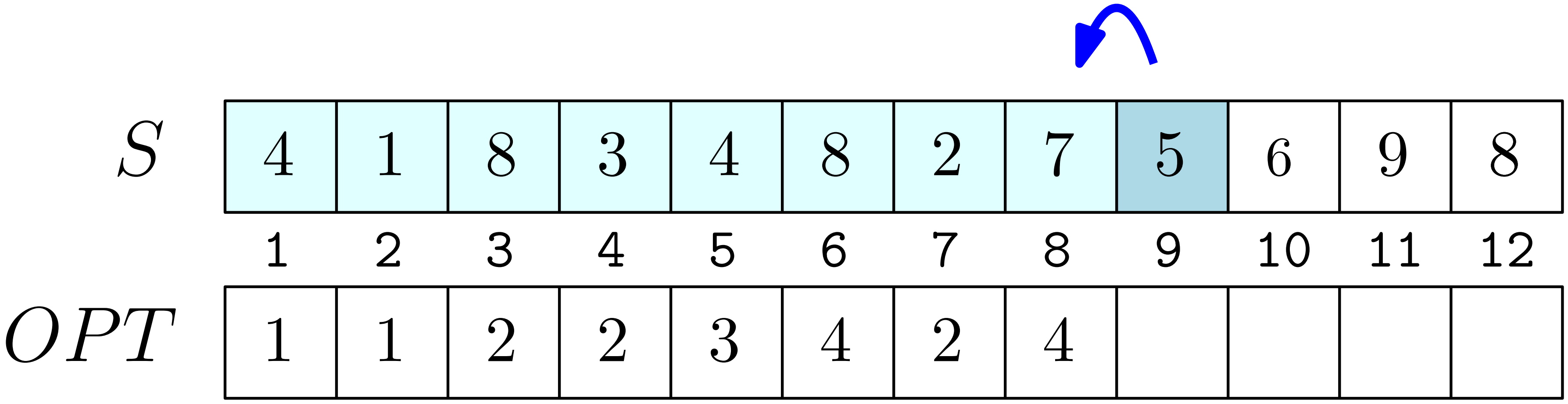

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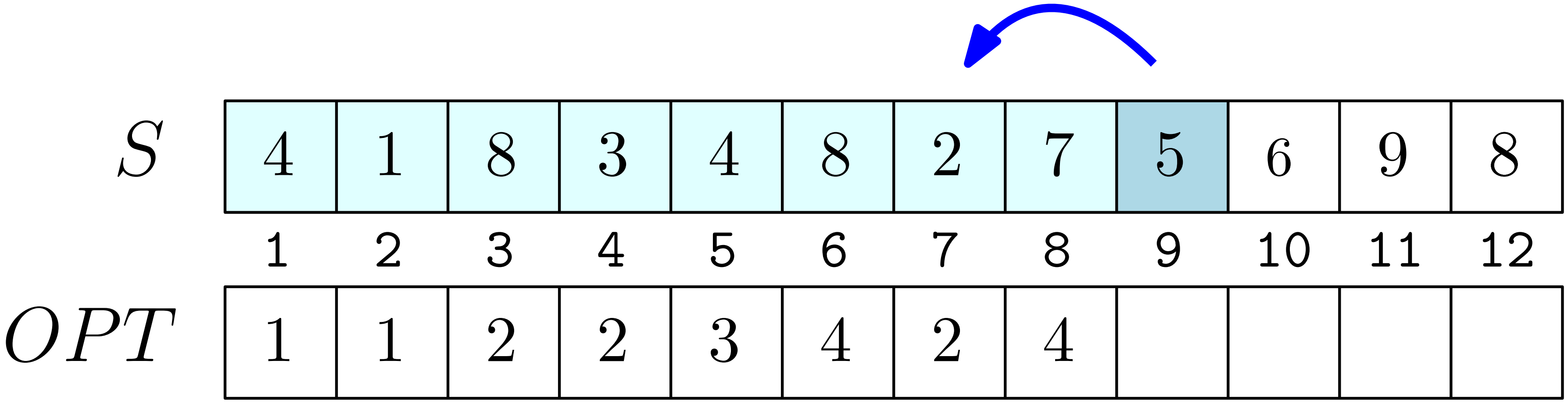

	1	2	3	4	5	6	7	8	9	10	11	12
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OPT	1	1	2	2	3	4	2	4				

Possible lengths: 3

A DP Algorithm: Second Attempt

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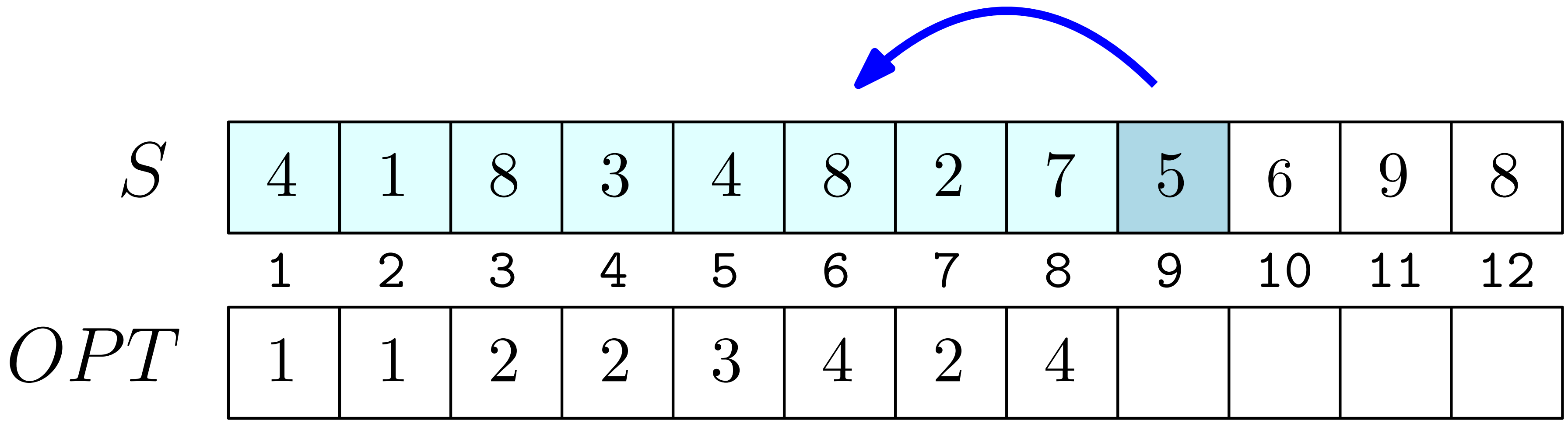

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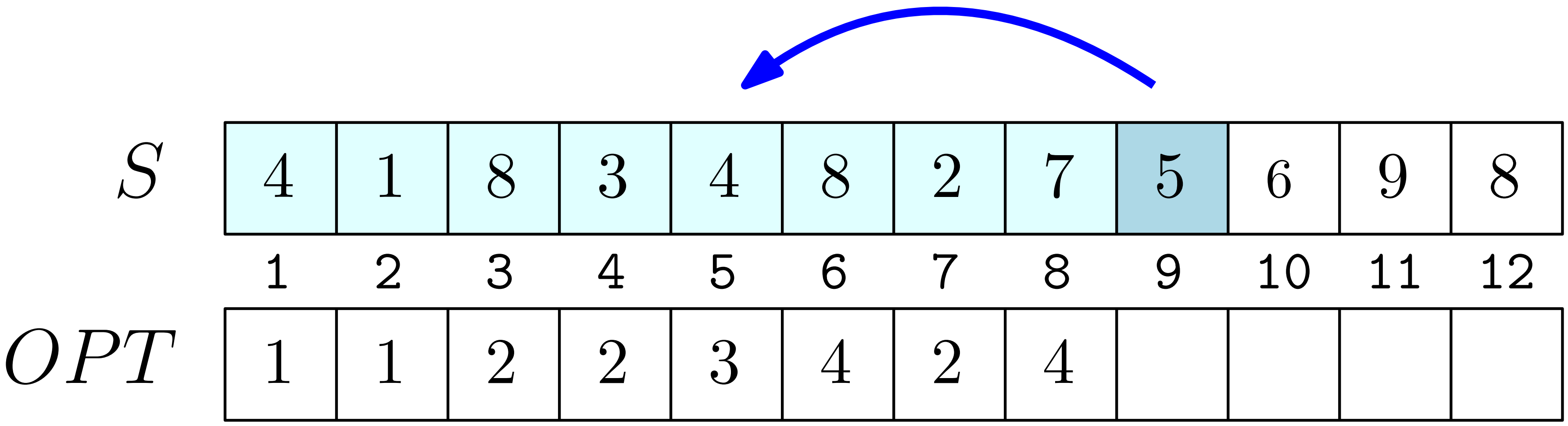

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8
OPT	1	1	2	2	3	4	2	4				

Possible lengths: $3 \quad 4$

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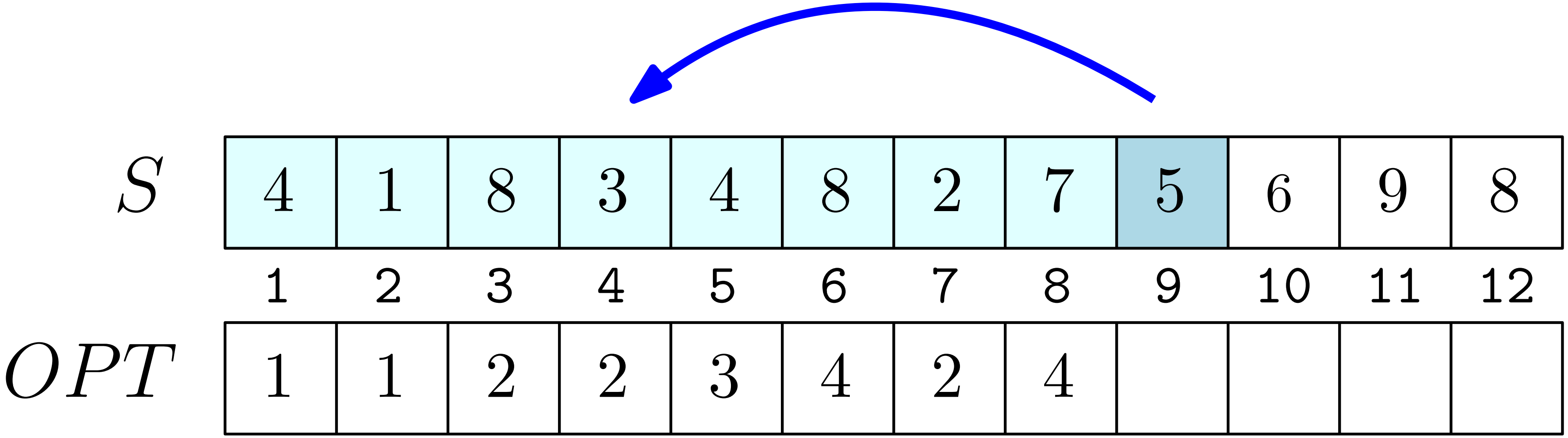

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8
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Possible lengths: $3 \quad 4 \quad 3$

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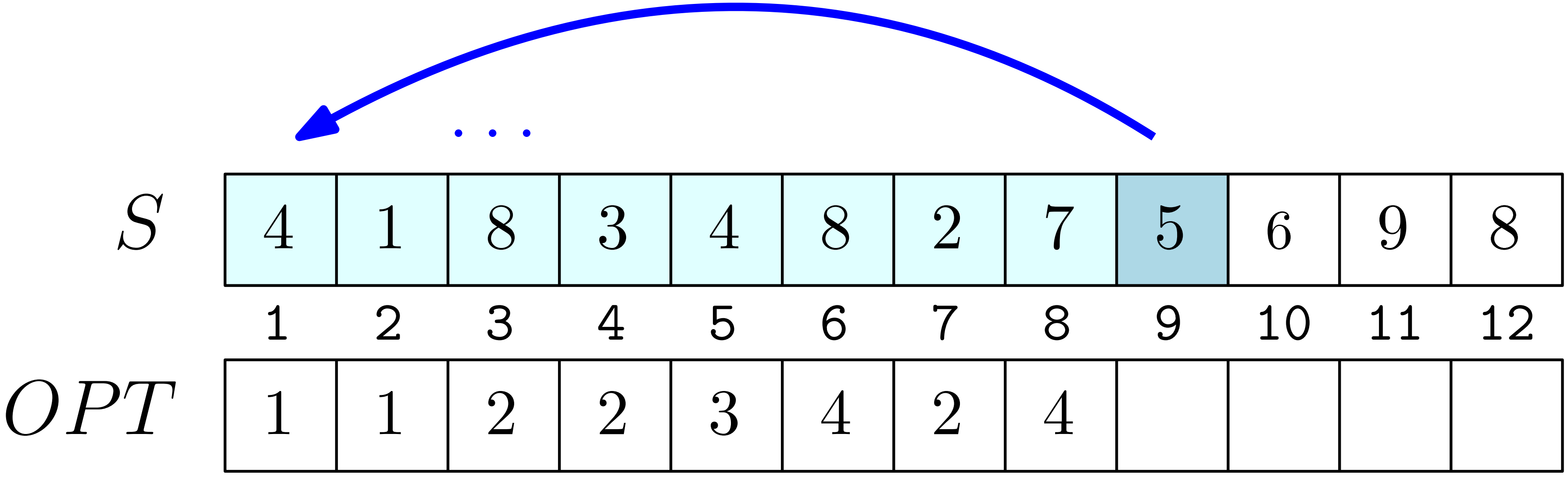

S	4	1	8	3	4	8	2	7	5	6	9	8
	1	2	3	4	5	6	7	8	9	10	11	12
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Possible lengths: 3 4 3 2 2

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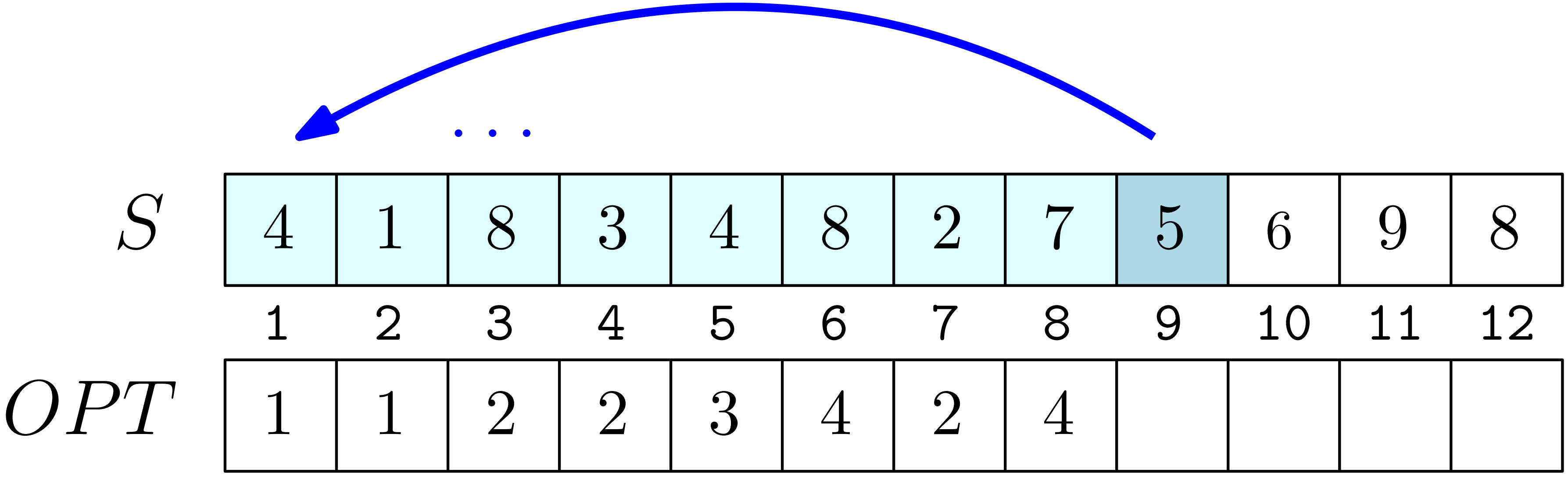

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8
OPT	1	1	2	2	3	4	2	4				

Possible lengths: 3 4 3 2 2 1

Sequence containing only $S[i]$

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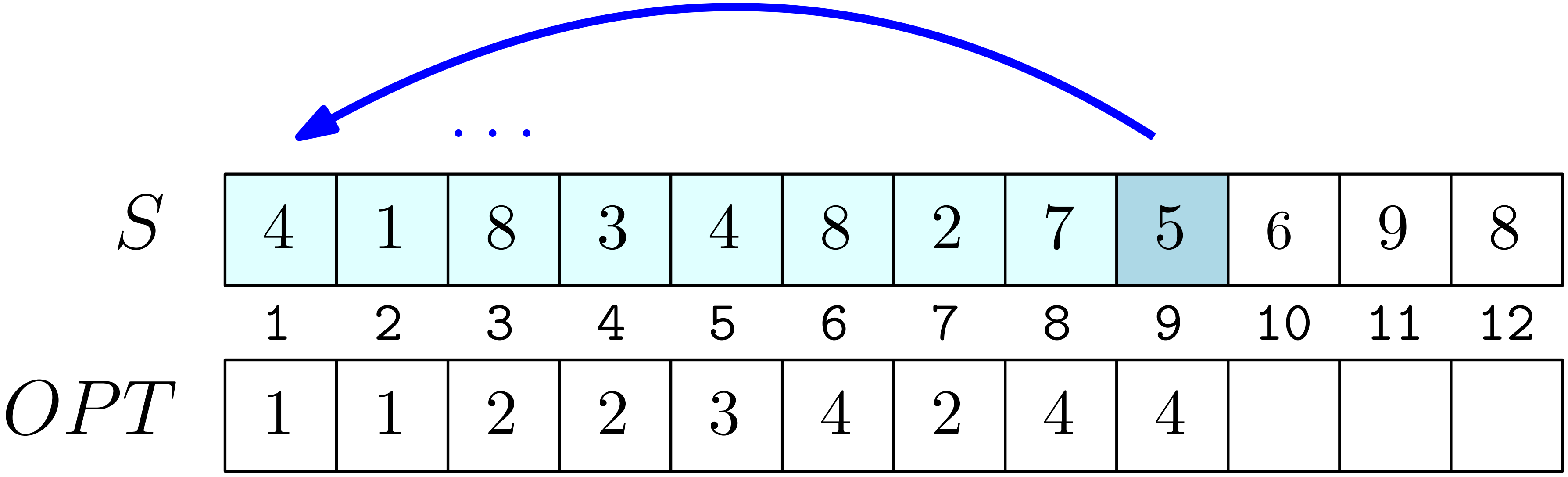

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8
OPT	1	1	2	2	3	4	2	4	4			

Possible lengths: $3 \quad 4 \quad 3 \quad 2 \quad 2 \quad 1$ $\qquad OPT[9] = 4$

Sequence containing only $S[i]$

The Dynamic Proramming Algorithm

- Subproblem definition

  $OPT[i] =$ Length of the LIS that ends with $S[i]$

- Base cases

  $$OPT[1] = 1$$

- Recursive formula

  $$OPT[i] = \max\left\{1, 1 + \max_{\substack{j=1,\ldots,i-1\\ S[j]<S[i]}} OPT[j]\right\}$$

- Subproblems' order

  $$OPT[1], OPT[2], \ldots, OPT[n]$$

- Solution:

  $\max_{i=1,\ldots,n} OPT[i]$

Time Complexity

- $O(n)$ subproblems

- Base cases are handled in constant time

- $OPT[i]$ is computed in time $\Theta(i)$

$$OPT[i] = \max\left\{1, 1 + \max_{\substack{j=1,\ldots,i-1\\ S[j]<S[i]}} OPT[j]\right\}$$

Time Complexity

- $O(n)$ subproblems
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- $OPT[i]$ is computed in time $\Theta(i)$

$$OPT[i] = \max\left\{1, 1 + \max_{\substack{j=1,\ldots,i-1\\ S[j]<S[i]}} OPT[j]\right\}$$

Overall time: $O\left(\sum_{i=1}^{n} i\right) = O(n^2)$.

A possible implementation (DP)

```
std::vector<int> OPT(n+1);
OPT[1]=1;

for(int i=2; i<=n; i++)
{
    OPT[i]=1;
    for(int j=1; j<i; j++)
        if(S[j] < S[i])
            OPT[i] = std::max(OPT[i], 1+OPT[j]);
}

return std::max_element(OPT.begin()+1, OPT.end());
```

A possible implementation (Memo)

```
std::vector<int> memo(n+1, 0);

int LIS(std::vector &S, int i)
{
    if(i==1) return 1;

    if(memo[i]) return memo[i];

    int r=1;
    for(int j=1; j<i; j++)
        if(S[j]<S[i])
            r=std::max(r, 1+LIS(S, j));

    return memo[i]=r;
}
```

Memoization vs. DP

✓ Top-Down approach (more intuitive)

✓ Easier to index subproblems by other objects (e.g., sets).

✓ Only computes necessary subproblems

✗ Function calls overhead

✗ Call stack (recusion depth) is bounded

✗ Time complexity is harder to analyze

✗ Bottom-Up approach (harder to grasp)

✗ Need to index subproblems with integers

✗ Always computes all subproblems

✓ No recursion. Less overhead. More cache efficient.

✓ Short and clean code

✓ Time complexity analysis is easy(/ier)

Another DP Algorithm for LIS

- Subproblem definition

$OPT[i, \ell]$ = Index j of the smallest element $S[j]$ with $j \leq i$ that ends an increasing subsequence of length ℓ, or $\perp$ if no such subsequence exists

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8

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$OPT[8, 2] =$

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$OPT[8, 2] = 7$

Another DP Algorithm for LIS

- Subproblem definition

$OPT[i, \ell]$ = Index j of the smallest element $S[j]$ with $j \leq i$ that ends an increasing subsequence of length ℓ, or $\perp$ if no such subsequence exists

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8

$OPT[8, 2] = 7$ $\qquad$ $OPT[8, 3] =$

Another DP Algorithm for LIS

- Subproblem definition

  $OPT[i,\ell]$ = Index j of the smallest element $S[j]$ with $j \le i$ that ends an increasing subsequence of length ℓ, or $\perp$ if no such subsequence exists

	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8

$OPT[8,2] = 7$ $\qquad$ $OPT[8,3] = 5$

Another DP Algorithm for LIS

- Subproblem definition

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	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8

$OPT[8, 2] = 7$ $\qquad$ $OPT[8, 3] = 5$

$OPT[8, 5] =$

Another DP Algorithm for LIS

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	1	2	3	4	5	6	7	8	9	10	11	12
S	4	1	8	3	4	8	2	7	5	6	9	8

$OPT[8, 2] = 7$ $\quad\quad$ $OPT[8, 3] = 5$

$OPT[8, 5] = \perp$

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \bot$:

$$OPT[i, \ell] = \bot \qquad\qquad (= OPT[i-1, \ell])$$

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \perp$:

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- If $S[i] \leq S[OPT[i-1, \ell-1]]$

$$OPT[i, \ell] = OPT[i-1, \ell]$$

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \perp$:

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- If $S[i] \leq S[OPT[i-1, \ell-1]]$

$$OPT[i, \ell] = OPT[i-1, \ell]$$

The above two cases can be merged into a single case.

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \perp$ or $S[i] \le S[OPT[i-1, \ell-1]]$

$$OPT[i, \ell] = OPT[i-1, \ell]$$

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \perp$ or $S[i] \leq S[OPT[i-1, \ell-1]]$

$$OPT[i, \ell] = OPT[i-1, \ell]$$

- If $OPT[i-1, \ell-1] \neq \perp$ and $S[i] > S[OPT[i-1, \ell-1]]$

$$OPT[i, \ell] = \begin{cases} i & \text{if } OPT[i-1, \ell] = \perp \text{ or} \\ & \quad S[i] \leq S[OPT[i-1, \ell]] \\ OPT[i-1, \ell] & \text{otherwise} \end{cases}$$

Computing $OPT[i, \ell]$

- If $OPT[i-1, \ell-1] = \perp$ or $S[i] \leq S[OPT[i-1, \ell-1]]$

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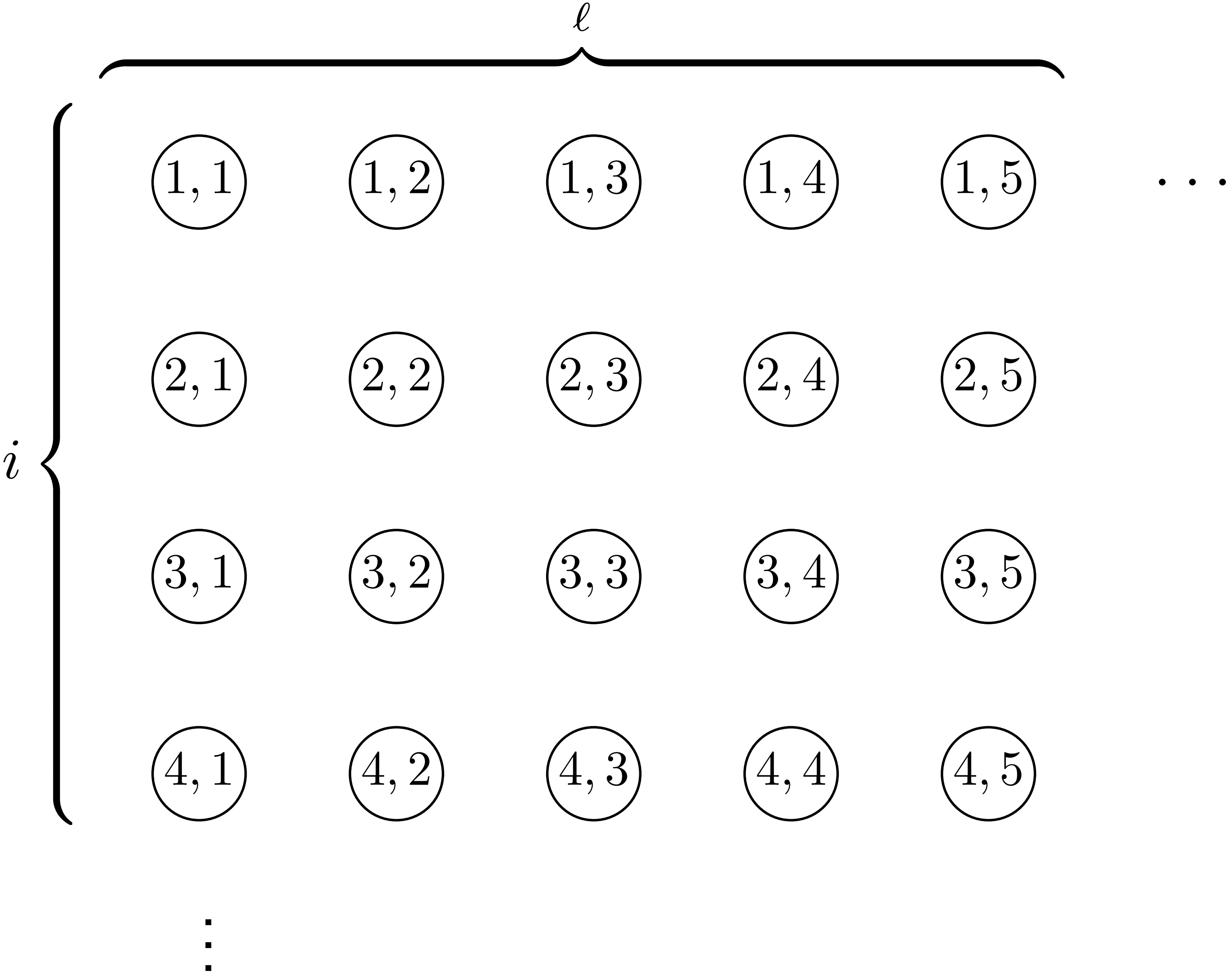
- Solution:

$$\max\{\ell = 1, \ldots, n \mid OPT[n, \ell] \neq \perp\}$$

Base Cases / Order of Subproblems



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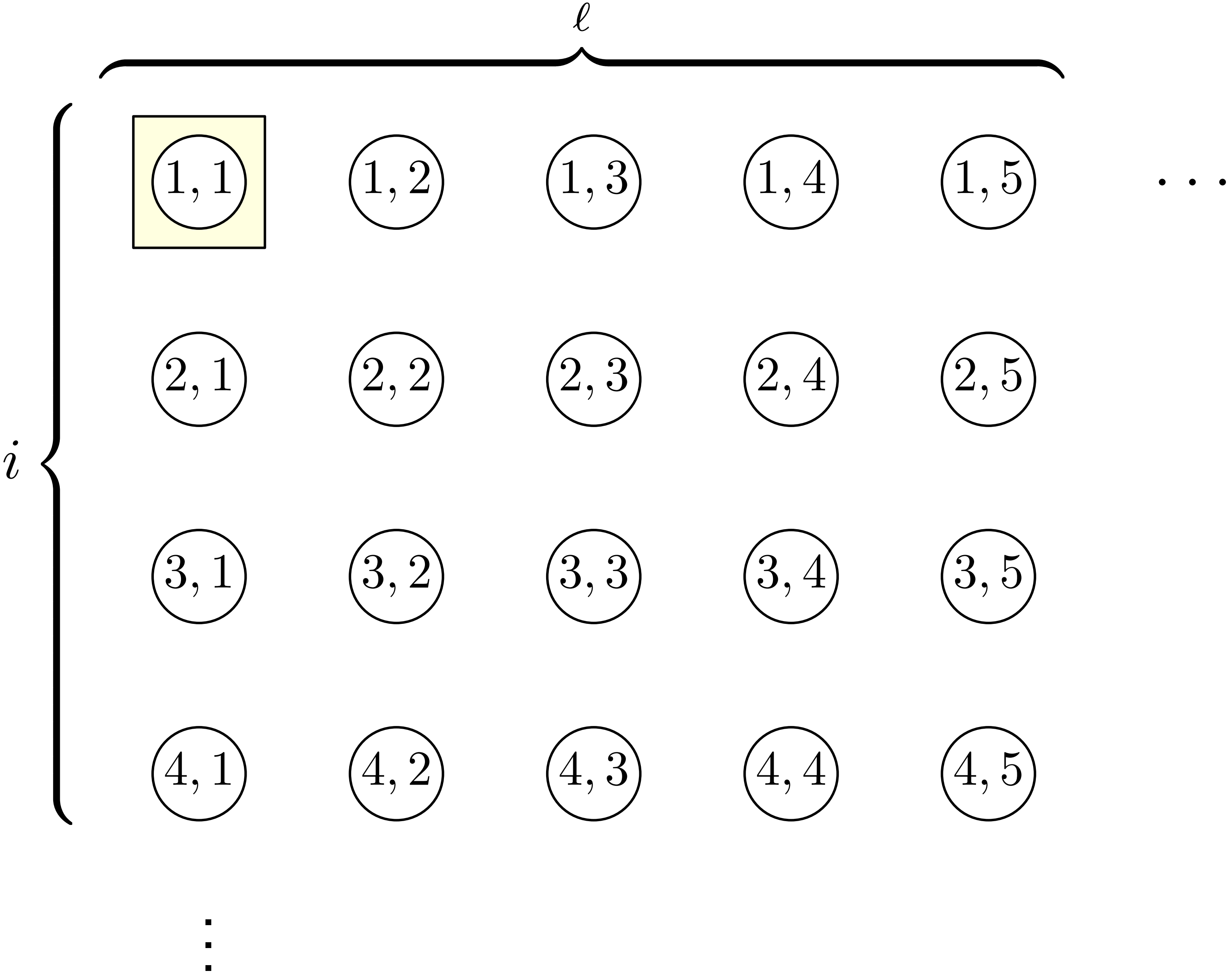
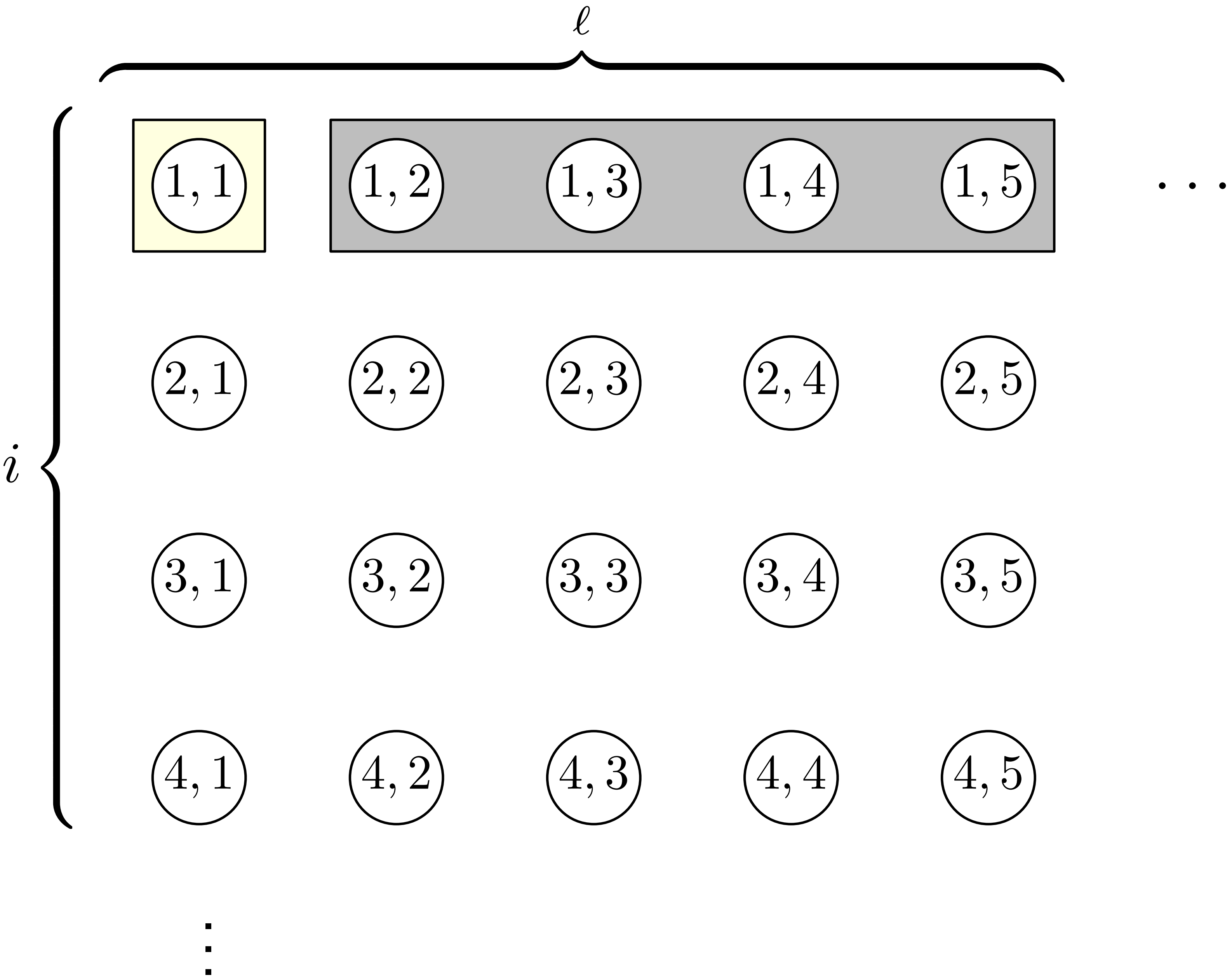


$1, 1$	$1, 2$	$1, 3$	$1, 4$	$1, 5$	$\cdots$
$2, 1$	$2, 2$	$2, 3$	$2, 4$	$2, 5$	
$3, 1$	$3, 2$	$3, 3$	$3, 4$	$3, 5$	
$4, 1$	$4, 2$	$4, 3$	$4, 4$	$4, 5$	
$\vdots$					

Base Cases / Order of Subproblems



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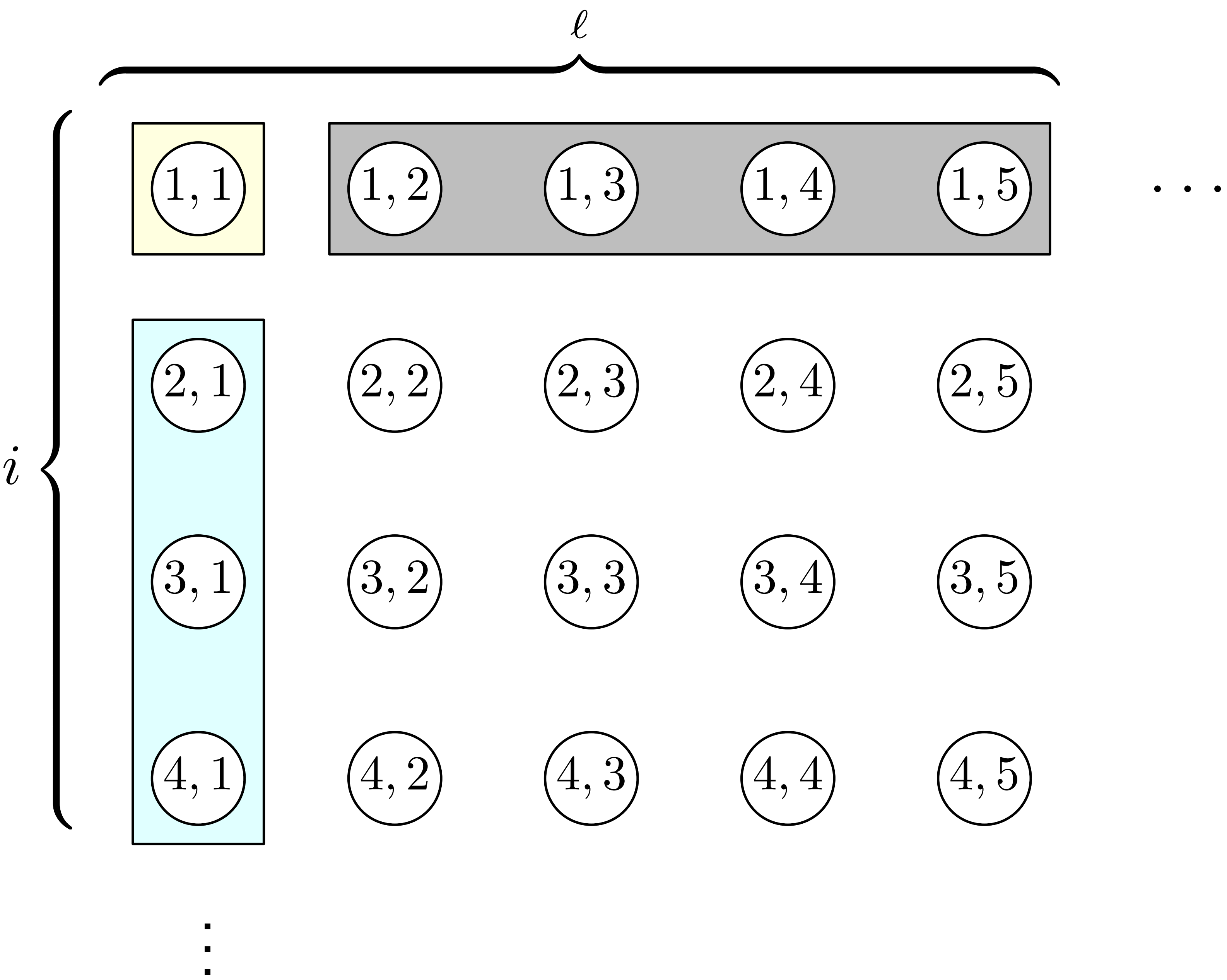

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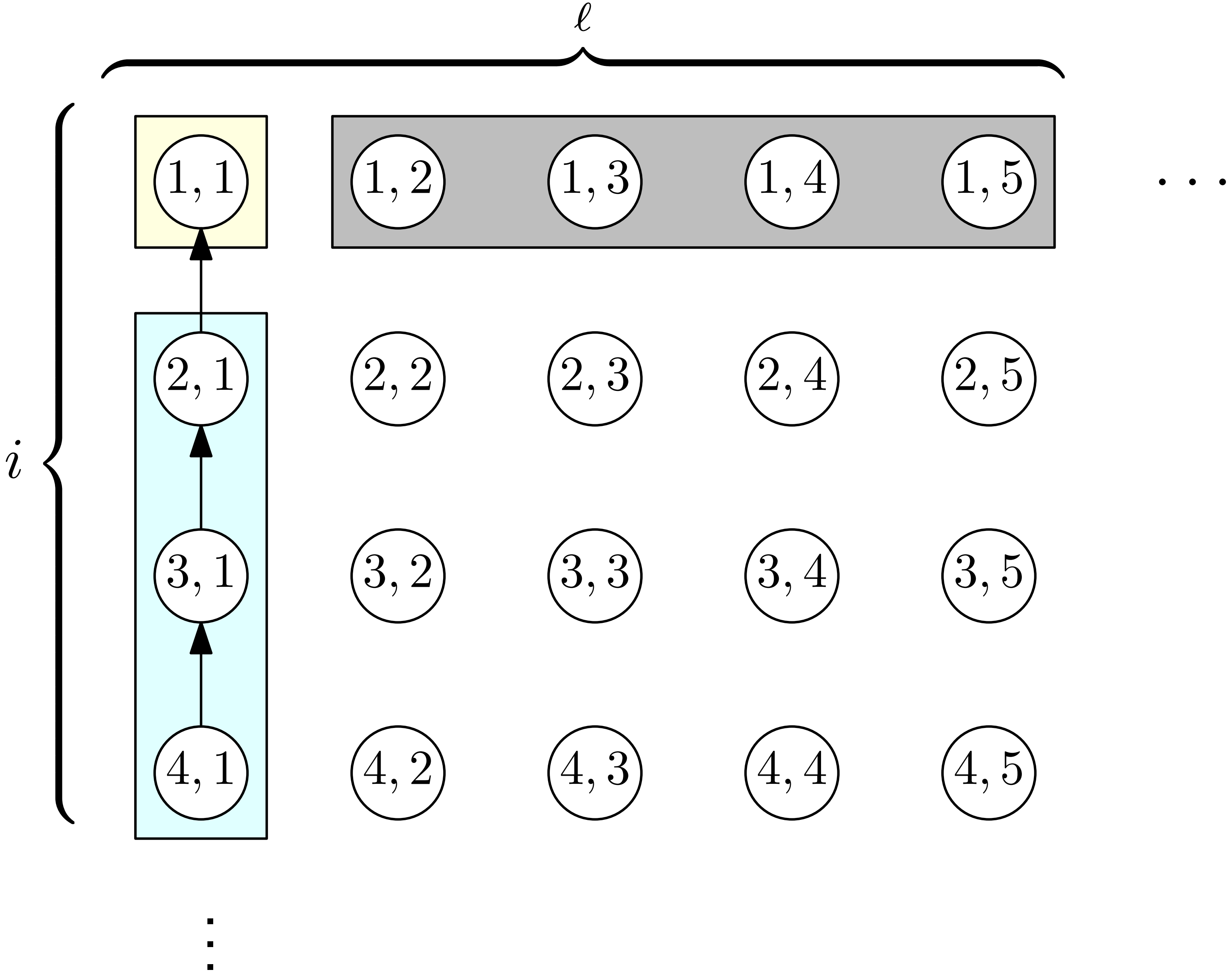

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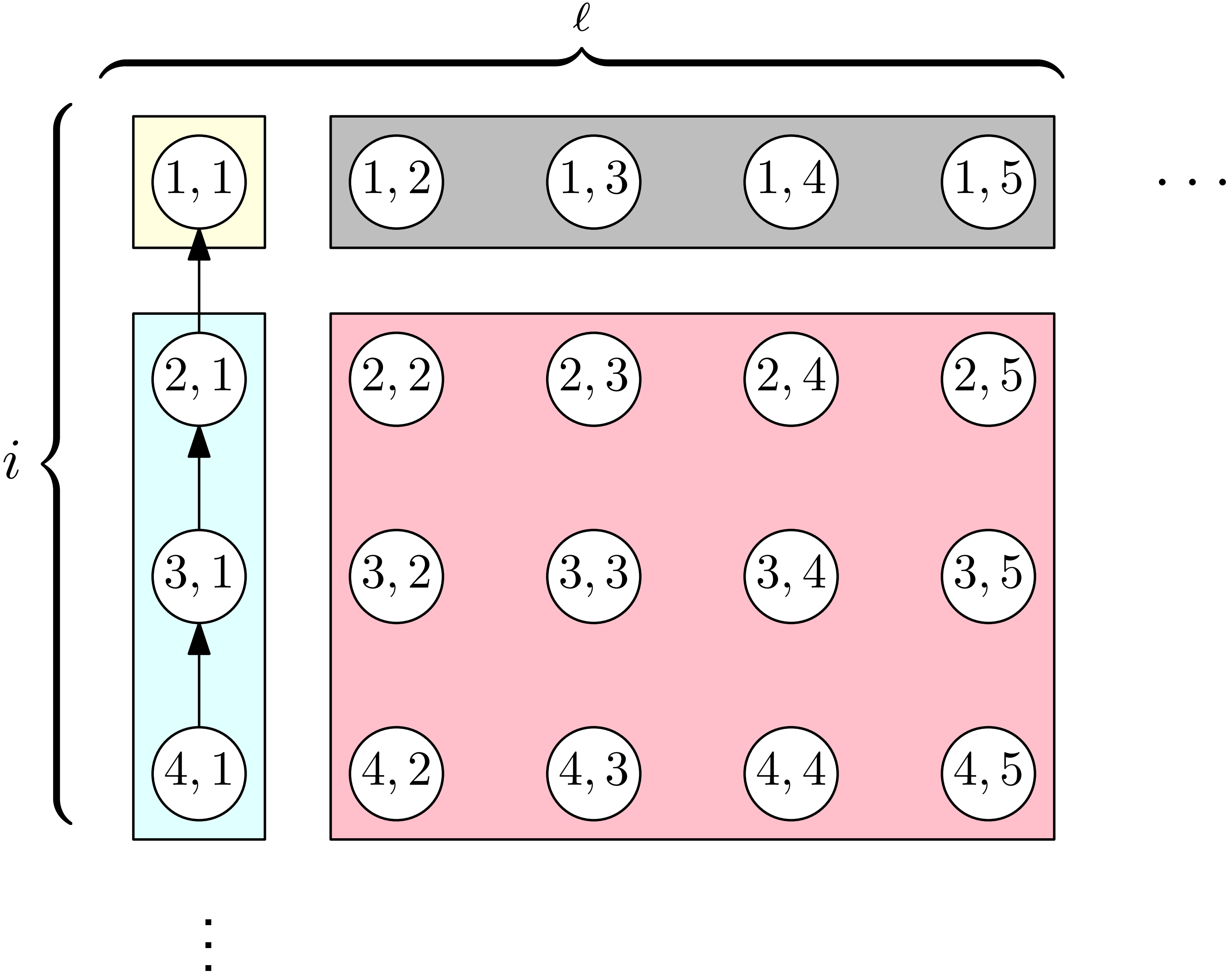

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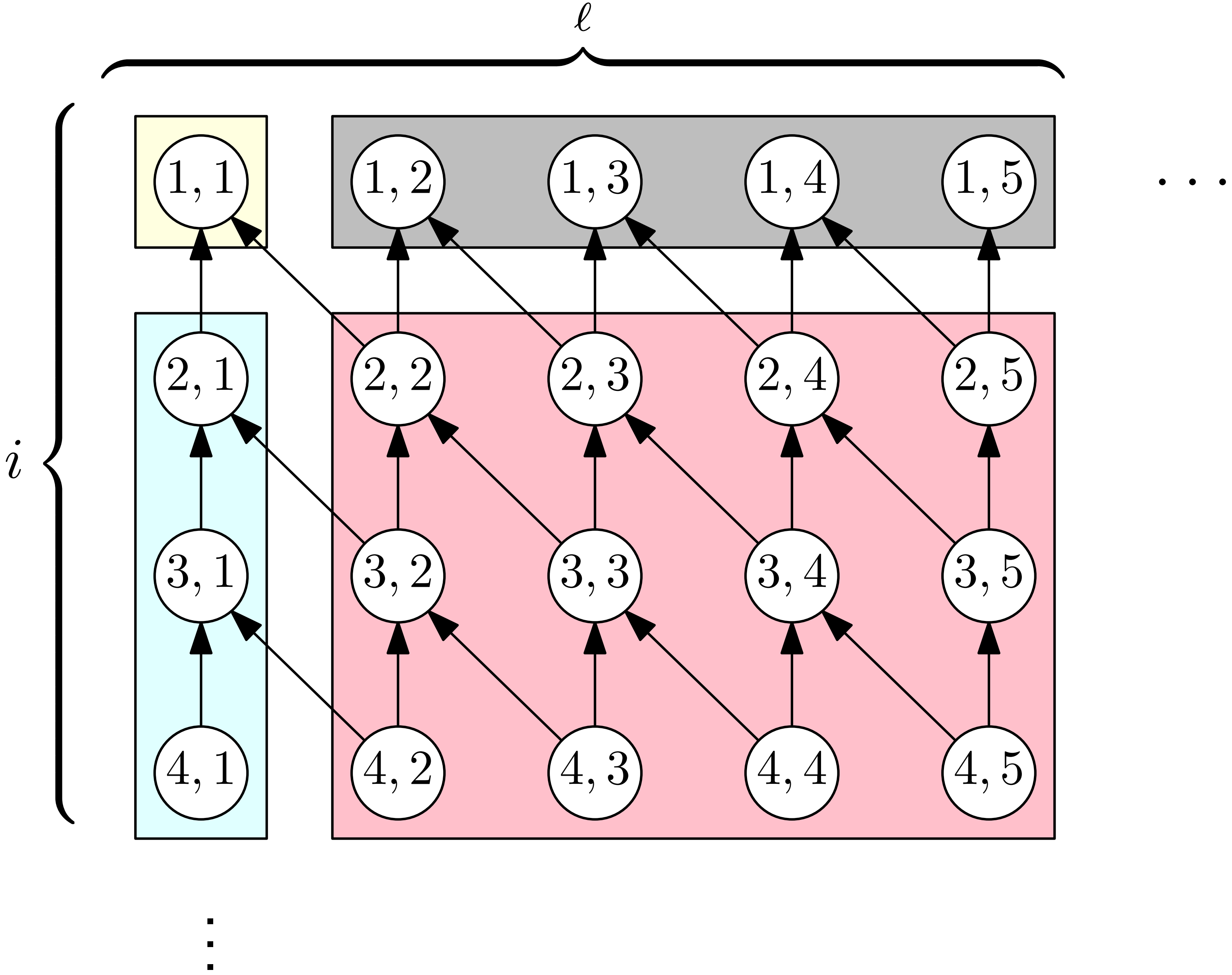

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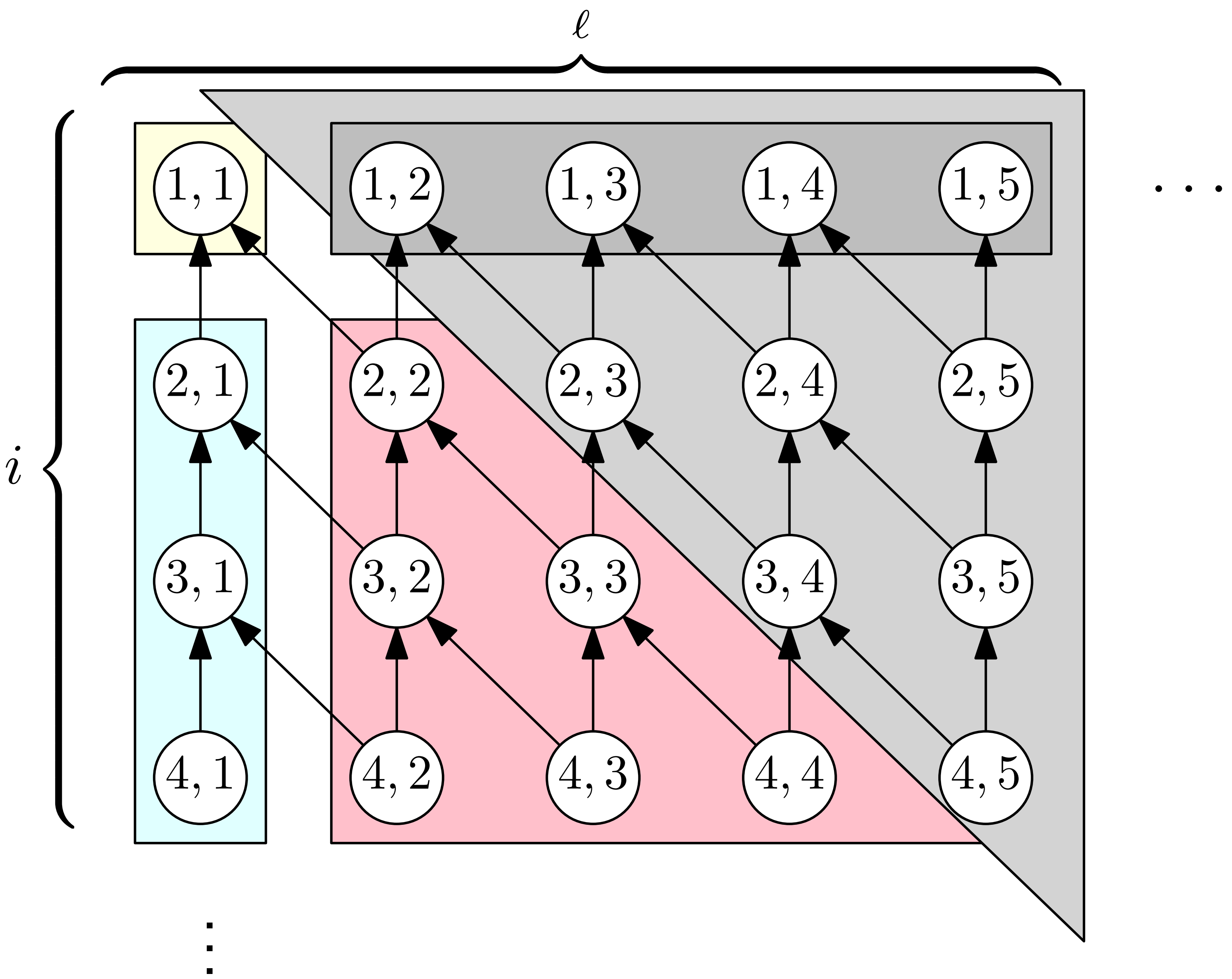

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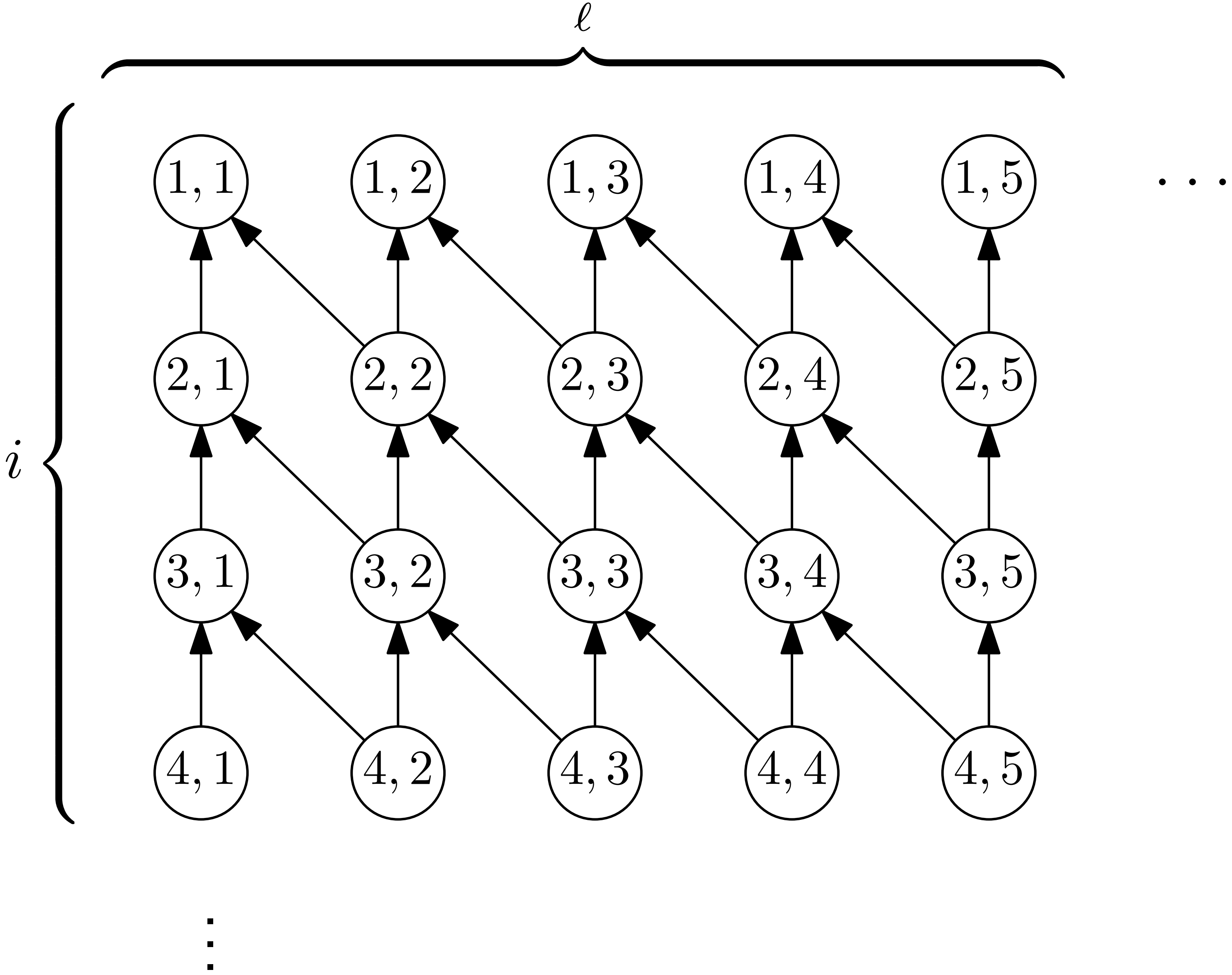

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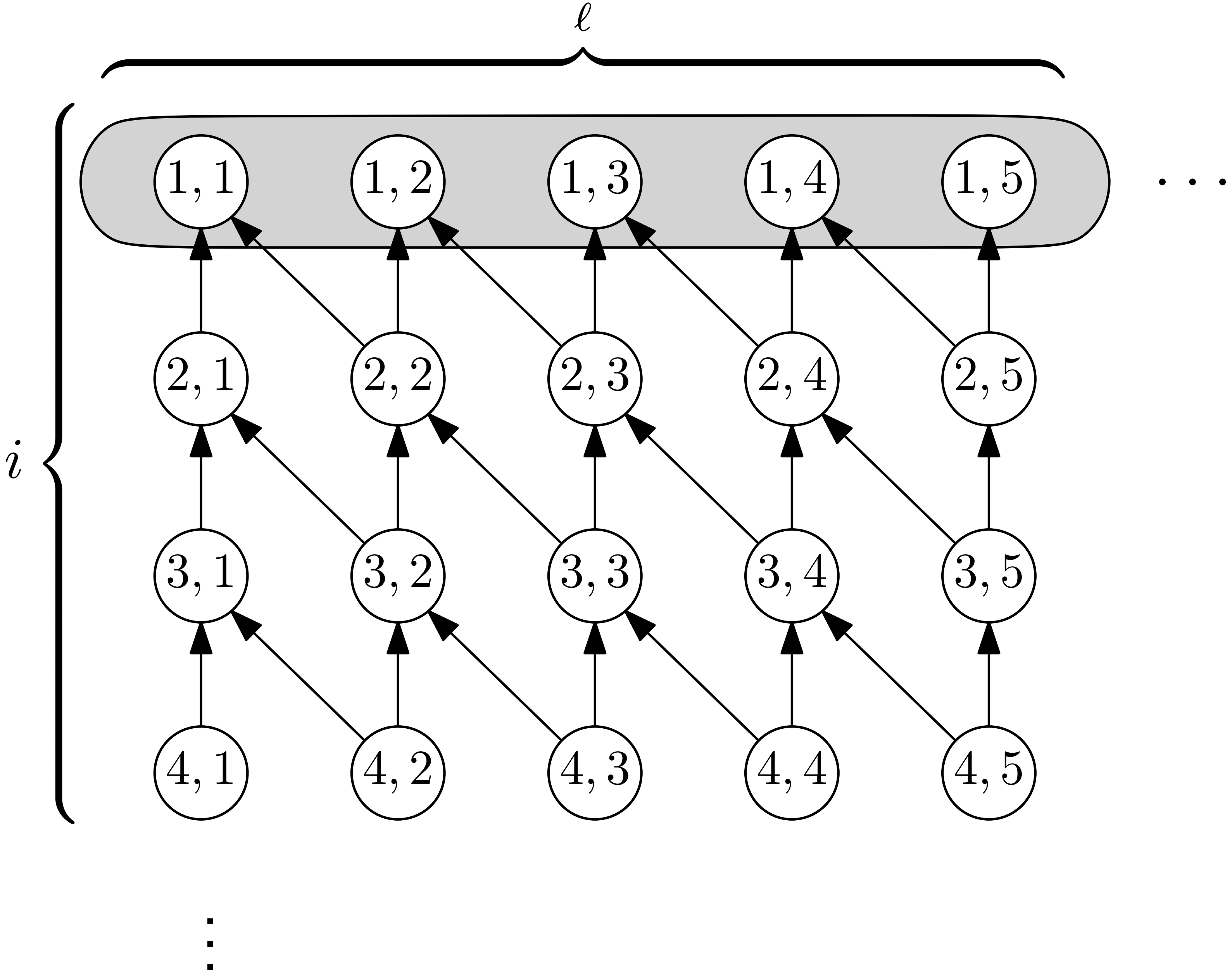

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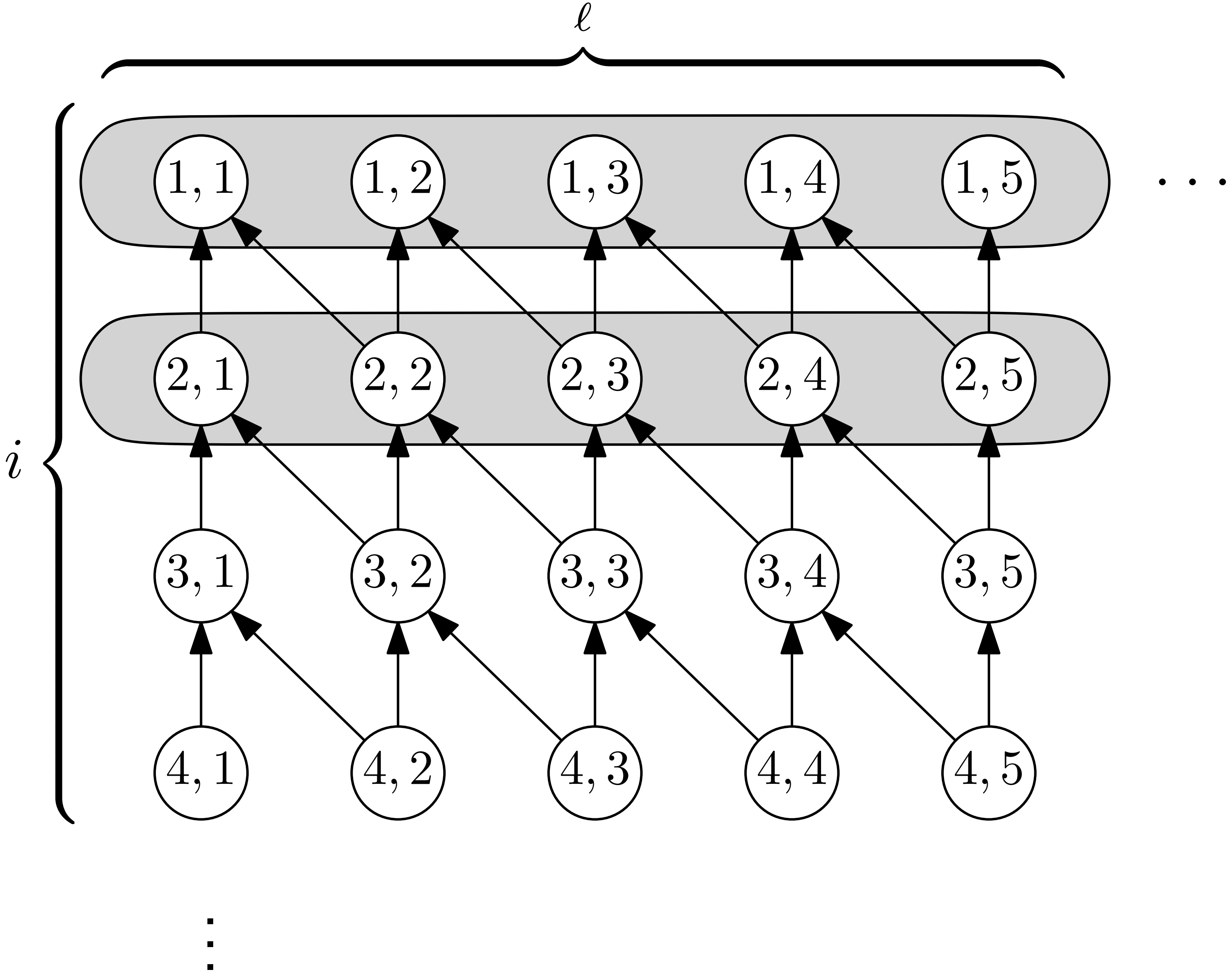

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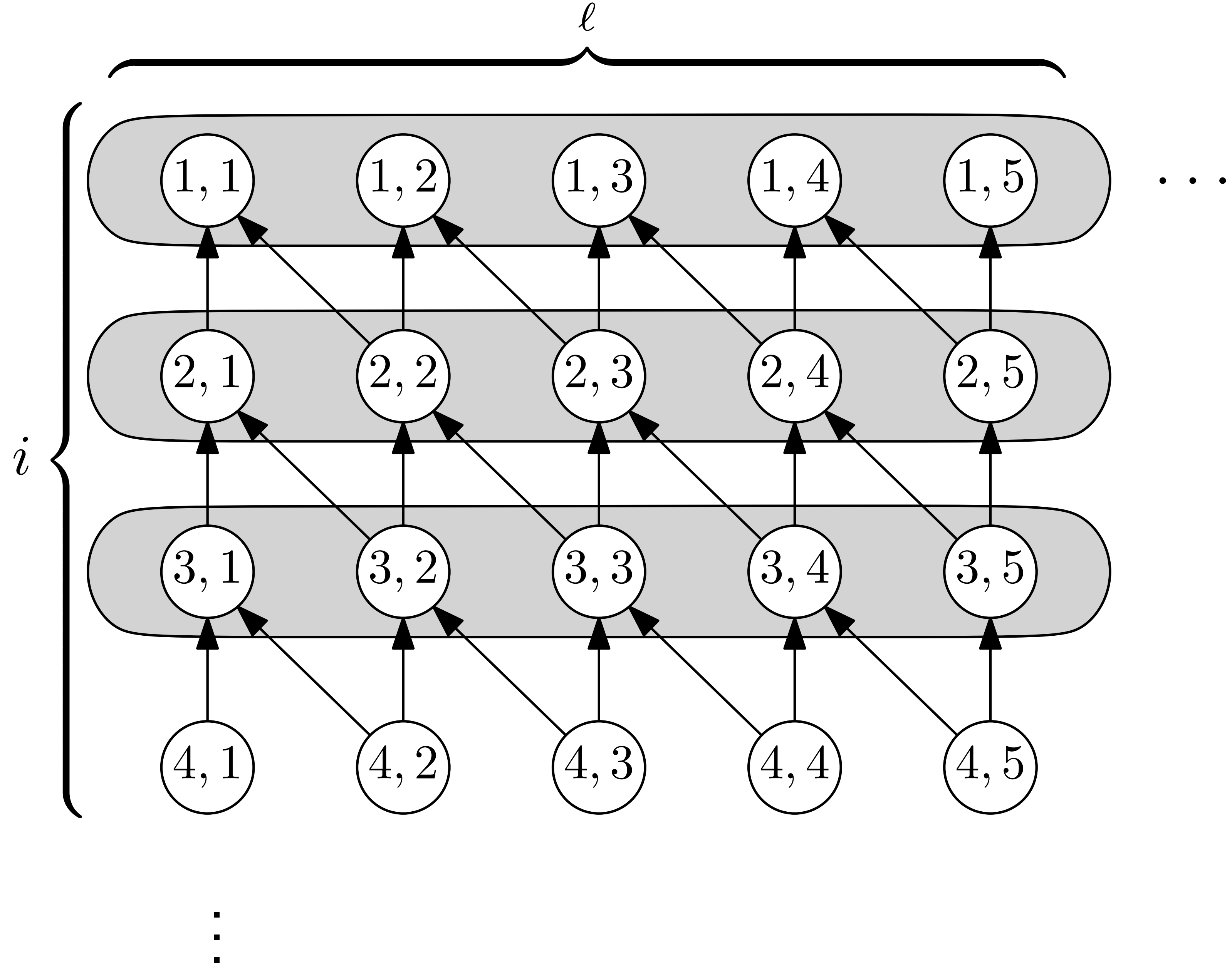

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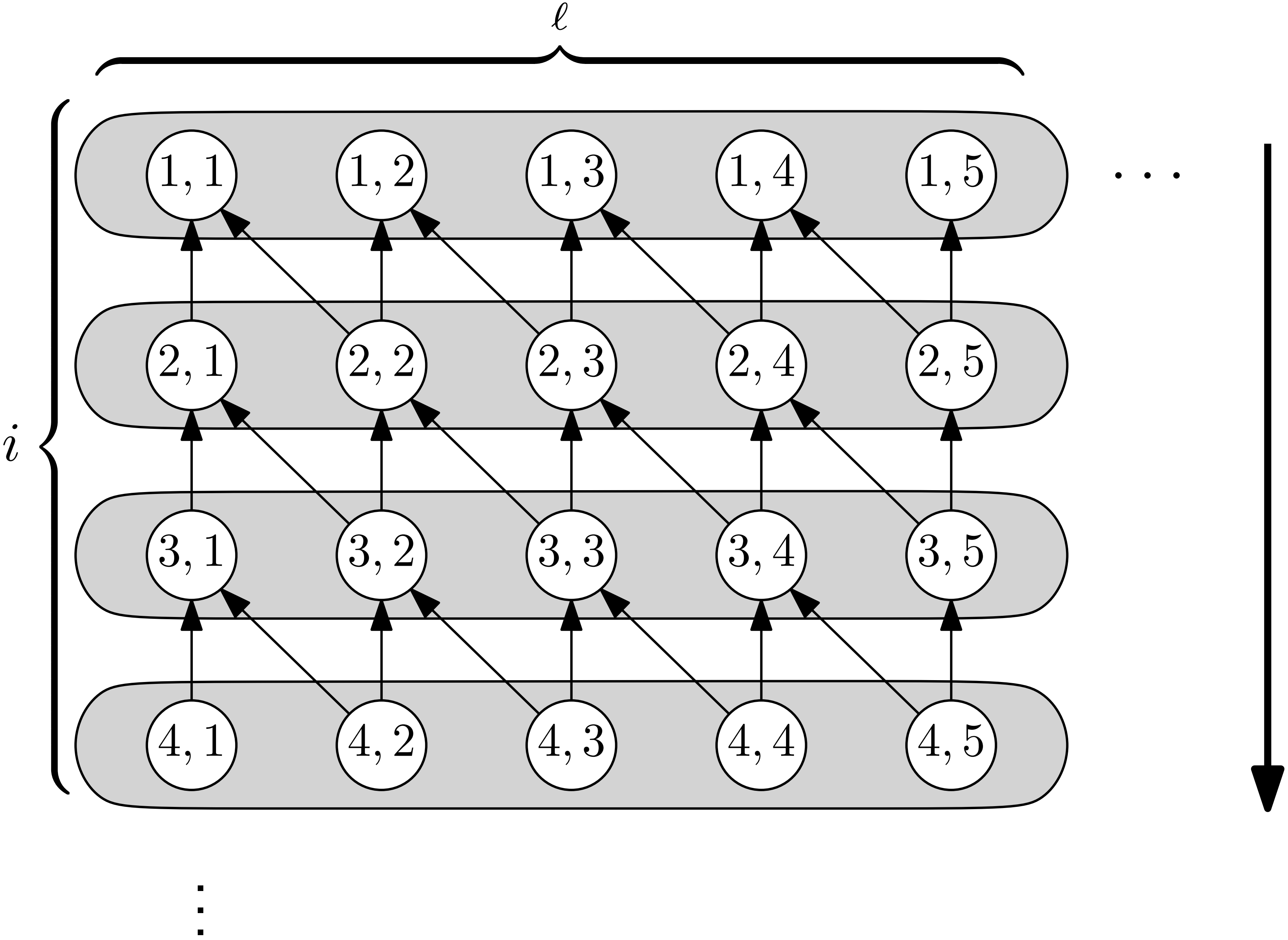

Time Complexity

$$\left.\begin{array}{l}\bullet\ O(n^2)\ \text{subproblems}\\[1em]\bullet\ O(1)\ \text{time per subproblem}\end{array}\right\}O(n^2)$$

Time Complexity

$$\left.\begin{array}{l}\bullet\ O(n^2)\text{ subproblems}\\[2em] \bullet\ O(1)\text{ time per subproblem}\end{array}\right\}O(n^2)$$

Can we do better?

Some Properties

Lemma: Given $i > 1$, let ℓ^* be the length of a LIS L of $S[1], \ldots, S[i]$ that ends with $S[i]$.

1) $OPT[i, \ell^*] = i$.

2) For $\ell \neq \ell^*$: $OPT[i, \ell] = OPT[i-1, \ell]$.

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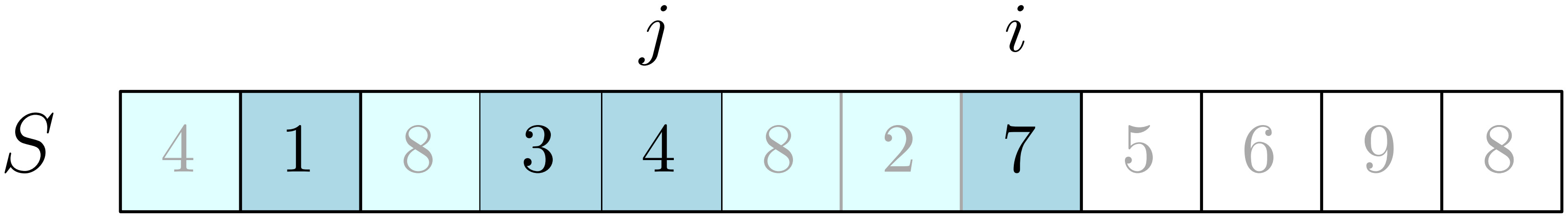
Proof of 1 (sketch): (Case $\ell^* \geq 2$)



					j			i				
S	4	1	8	3	4	8	2	7	5	6	9	8

- $j =$ index of the one-to-last element of L
- $S[i] > S[j] \geq S[\underbrace{OPT[i-1, \ell^* - 1]}_{\neq \perp}]$

Some Properties

Lemma: Given $i > 1$, let ℓ^* be the length of a LIS L of $S[1], \ldots, S[i]$ that ends with $S[i]$.

1) $OPT[i, \ell^*] = i$.

2) For $\ell \neq \ell^*$: $OPT[i, \ell] = OPT[i-1, \ell]$.

Proof of 1 (sketch): (Case $\ell^* \geq 2$)

$$OPT[i, \ell^*] = \begin{cases} i & \text{if } OPT[i-1, \ell^*] = \bot \text{ or} \\ & \quad S[i] \leq S[OPT[i-1, \ell^*]] \\ OPT[i-1, \ell^*] & \text{otherwise} \end{cases}$$

- If $OPT[i, \ell^*] \neq i$ then:

  $OPT[i-1, \ell^*] \neq \bot$ and $S[i] > S[OPT[i-1, \ell^*]]$

- Contradiction: wrong choice of ℓ^*!

Some Properties

Lemma: Given $i > 1$, let ℓ^* be the length of a LIS L of $S[1], \ldots, S[i]$ that ends with $S[i]$.

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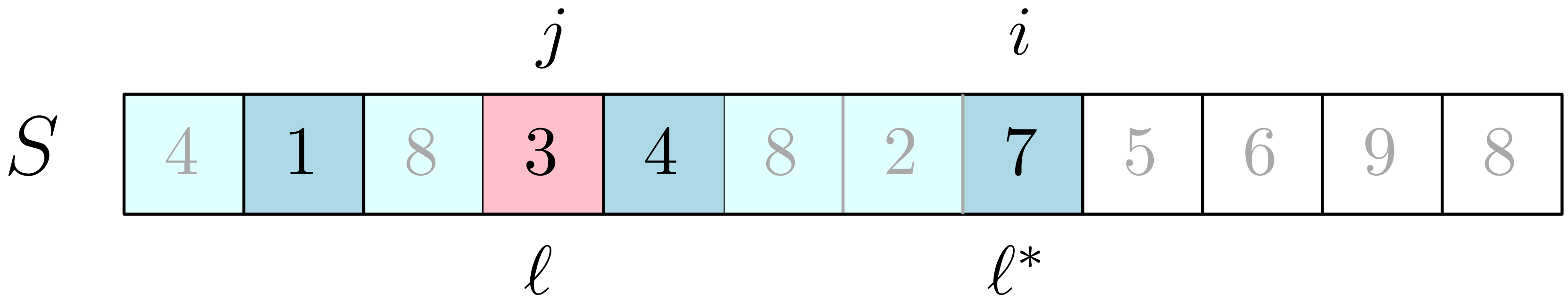
2) For $\ell \neq \ell^*$: $OPT[i, \ell] = OPT[i-1, \ell]$.

Proof of 2: Trivially true if $\ell > \ell^*$. Consider $\ell < \ell^*$:

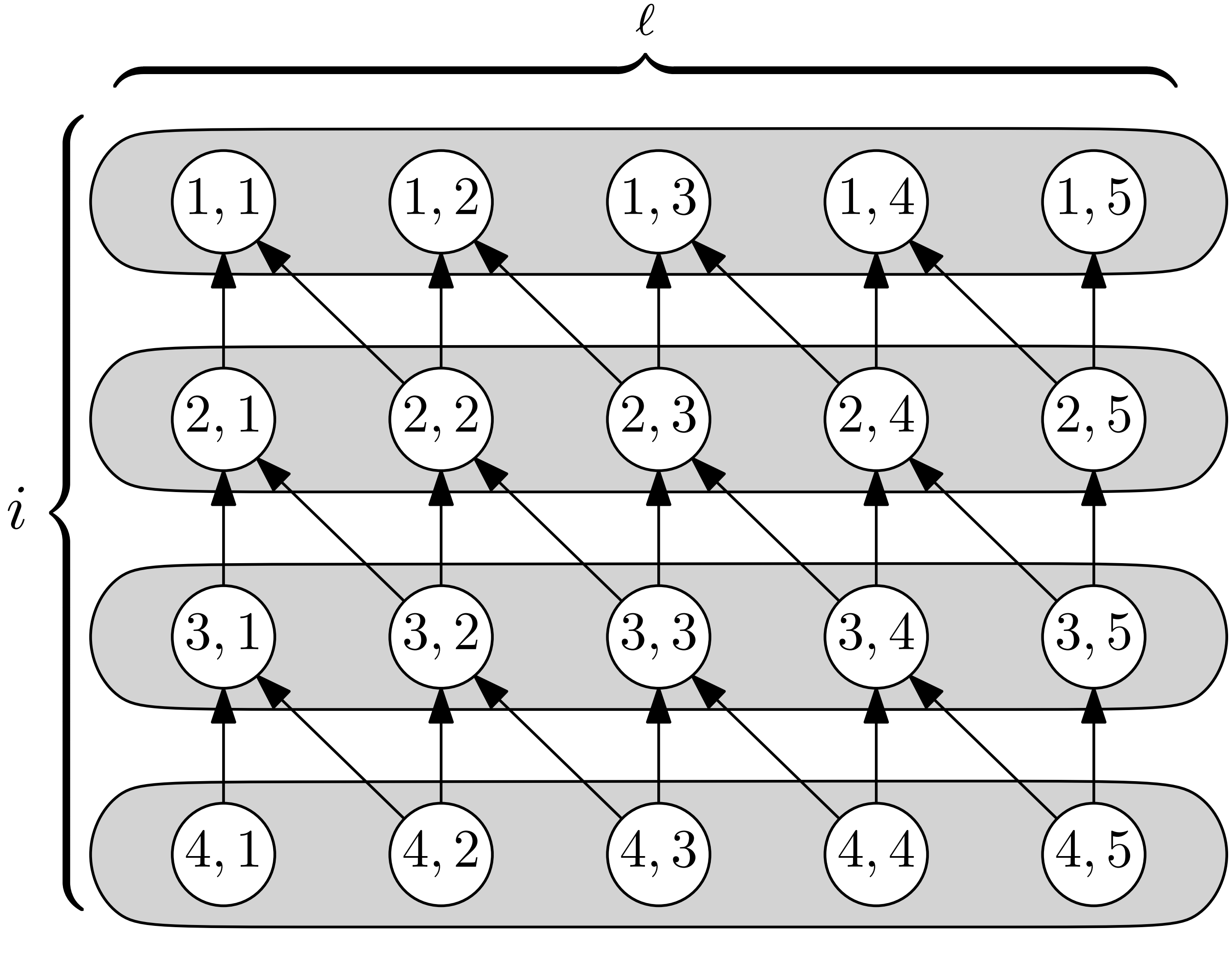


4	1	8	3	4	8	2	7	5	6	9	8

- The ℓ-th term in the IS of length ℓ^* ending in $OPT[i, \ell^*] = i$ appears in some position $j < i$.
- $S[j] < S[i] \implies OPT[i, \ell] \neq i$

A possible implementation

Observation 2: After the i-th iteration, all values $OPT[1, \ell], \ldots, OPT[i-1, \ell]$ will never be used anymore!



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Idea: only keep track of $OPT[\ell] := OPT[i,\ell]$, where i is the current iteration.

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```
• $OPT[1] = 1$, $OPT[2] = \cdots = OPT[n] = \perp$
• For $i = 2,\ldots,n$:
    • $\ell^* \leftarrow 1$                                   // Find $\ell^*$
    • For $\ell = 1,\ldots,i-1$:
        • If $OPT[\ell] \neq \perp$ and $S[OPT[\ell]] < S[i]$:
            • $\ell^* = \ell + 1$
    • $OPT[\ell^*] = i$
• Return $\max\{\ell = 1,\ldots,n \mid OPT[\ell] \neq \perp\}$
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```
• $OPT[1] = 1$, $OPT[2] = \cdots = OPT[n] = \bot$
• For $i = 2, \dots, n$:                                              $O(n)$
    • $\ell^* \leftarrow 1$                                  // Find $\ell^*$
    • For $\ell = 1, \dots, i-1$:
        • If $OPT[\ell] \neq \bot$ and $S[OPT[\ell]] < S[i]$:        $O(n)$
            • $\ell^* = \ell + 1$
    • $OPT[\ell^*] = i$
• Return $\max\{\ell = 1, \dots, n \mid OPT[\ell] \neq \bot\}$
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Observation 2: After the i-th iteration, all values $OPT[1,\ell],\dots,OPT[i-1,\ell]$ will never be used anymore!

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- $OPT[1] = 1, OPT[2] = \cdots = OPT[n] = \bot$
-

☹ Still $O(n^2)$ time

$n)$

  - $OPT[\ell^*] = i$
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A possible implementation

Observation 3: $S[OPT[\ell]]$ is monotonically increasing w.r.t. ℓ (until $OPT[\ell] = \perp$).

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• $OPT[1] = 1$, $OPT[2] = \cdots = OPT[n] = \bot$
• For $i = 2, \ldots, n$:                                              $O(n)$
    • $\ell^* \leftarrow 1$
    • For $\ell = 1, \ldots, i - 1$:
            • If $OPT[\ell] \neq \bot$ and $S[OPT[\ell]] < S[i]$:      $\}\ O(n)$
            • $\ell^* = \ell + 1$
    • $OPT[\ell^*] = i$
• Return $\max\{\ell = 1, \ldots, n \mid OPT[\ell] \neq \bot\}$
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A possible implementation

Observation 3: $S[OPT[\ell]]$ is monotonically increasing w.r.t. ℓ (until $OPT[\ell] = \bot$).

Idea: use binary search to find ℓ^*!

- $OPT[1] = 1$, $OPT[2] = \cdots = OPT[n] = \bot$
- For $i = 2, \ldots, n$: $O(n)$
  - Binary search for largest value ℓ such that $OPT[\ell] \neq \bot$ and $S[OPT[\ell]] < S[i]$, if any.
  - $\ell^* \leftarrow \ell + 1$, if ℓ exists, otherwise 1

    $\Big\}\ O(\log n)$
  - $OPT[\ell^*] = i$
- Return $\max\{\ell = 1, \ldots, n \mid OPT[\ell] \neq \bot\}$

Total time: $O(n \log n)$

Recap

Trick/Technique: Divide and Conquer

Decompose an instance into smaller instances of the same problem.
Solve recursively and recombine the solutions.

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Avoid recomputing solutions to duplicate subproblems by storing results in memory.

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Trick/Technique: Dynamic Programming

Define overlapping subproblems (possibly w/additional constraints). Systematically solve subproblems using an order that allows previous solutions to be recombined. Compute solution to the original problem from the subproblems' solutions.