More on Dynamic Programming

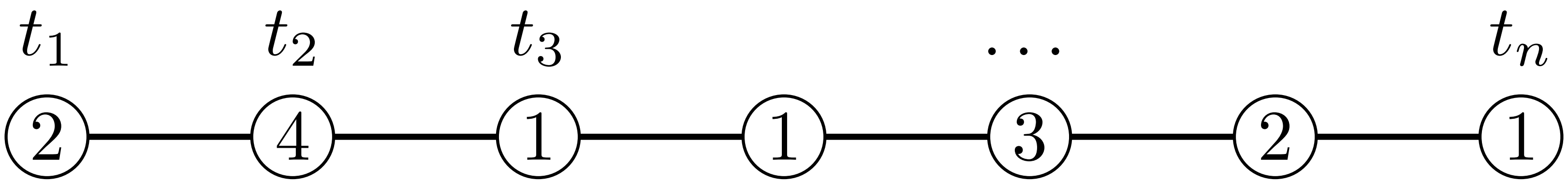
Drink as Much as Possible: A Variant

Robert still wants to drink as much a possible.

- Robert walks through the streets of King's Landing and encounters n taverns $t_1, t_2, \ldots, t_n$, in order
- When Robert encounters a tavern t_i, he can either stop for a drink or continue walking.
- Tavern t_i has $w_i \in \mathbb{N}$ liters of wine.
- If Robert drinks in tavern t_i then he will be too drunk to drink in tavern t_{i+1}. He will be able to drink again by the time he reaches t_{i+2}
- **Goal:** Compute the maximum amount of wine (in liters) Roberts can drink

Max-Weight Independent Set on Paths

Definition: An *independent set* (IS) of a graph $G = (V, E)$ is a set $\mathcal{I} \subseteq V$ such that $\forall (u, v) \in E,\ u \notin \mathcal{I}$ or $v \notin \mathcal{I}$.



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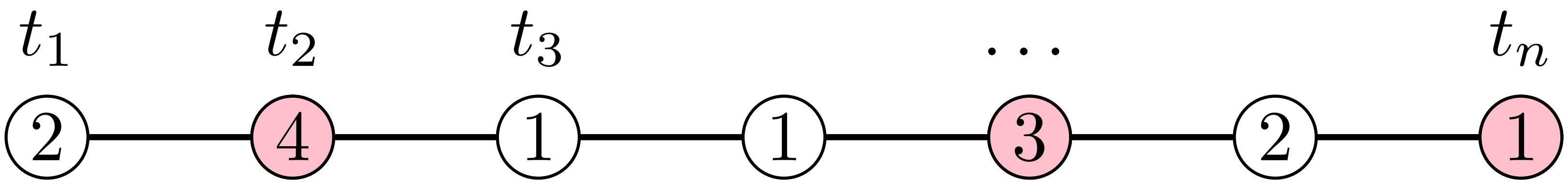

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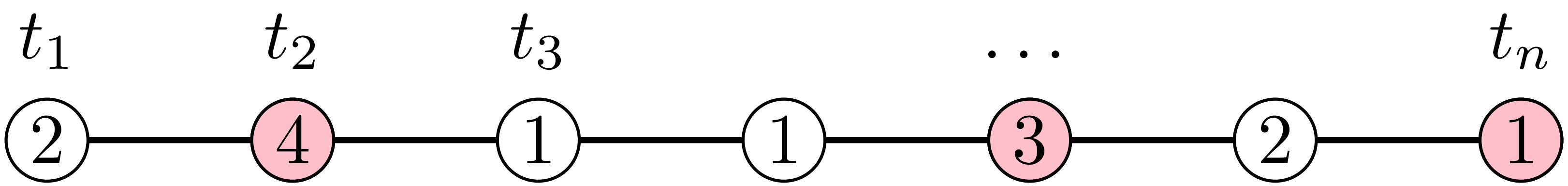

Linear-Time Dynamic Programming Algorithm

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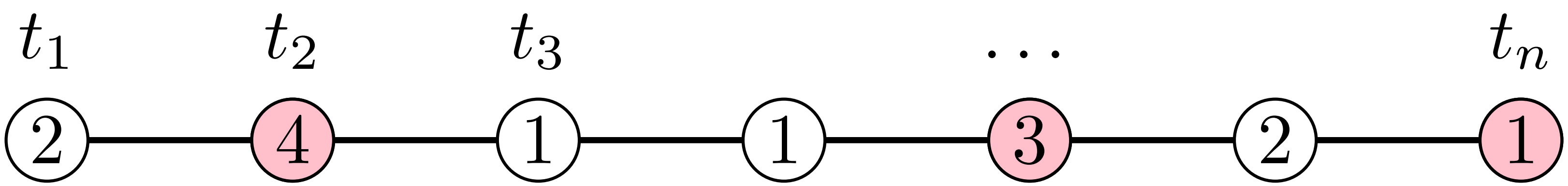

Linear-Time Dynamic Programming Algorithm

Sketch of the algorithm:

$OPT[i] =$ Maximum-weight IS w.r.t. the subpath $t_1, \dots, t_i$

$$OPT[0] = 0 \qquad OPT[1] = w_1$$

$$OPT[i] = \max\{w_i + OPT[i-2], OPT[i-1]\}$$

Max-Weight Independent Set on Trees

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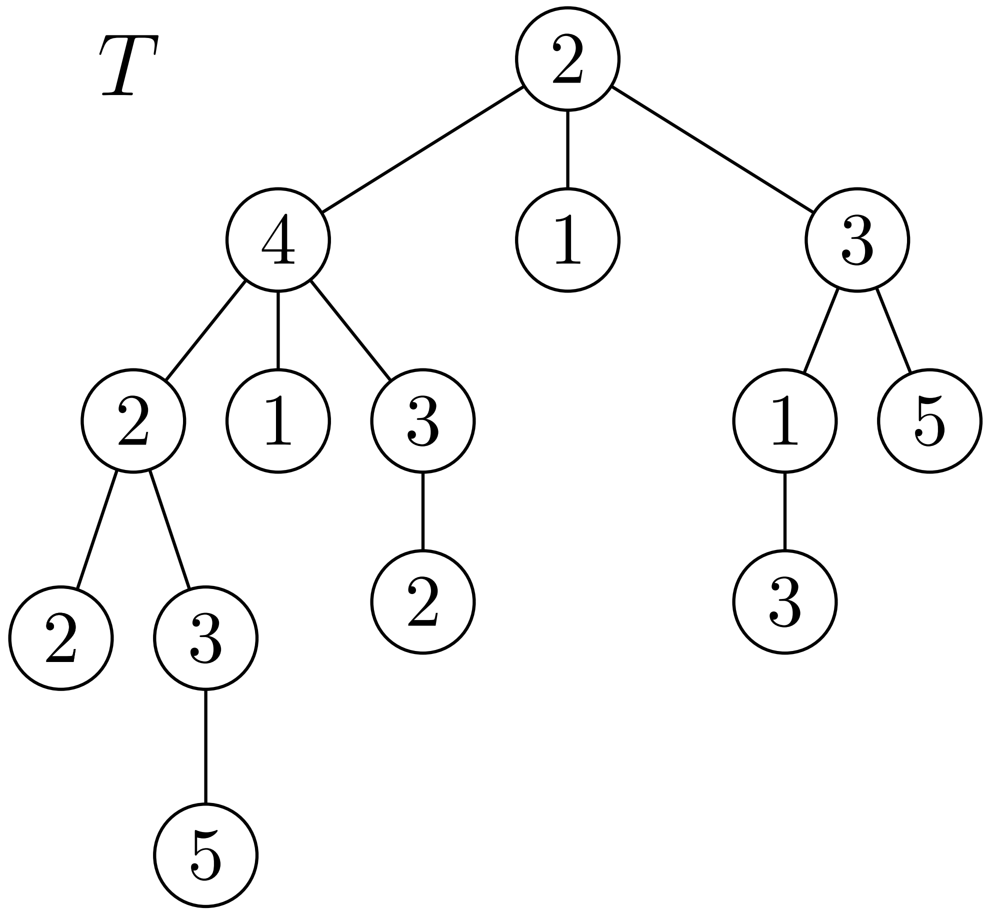
Problem: Given a tree T with integer weights on its vertices, compute the weight of a Max-Weight IS of T.



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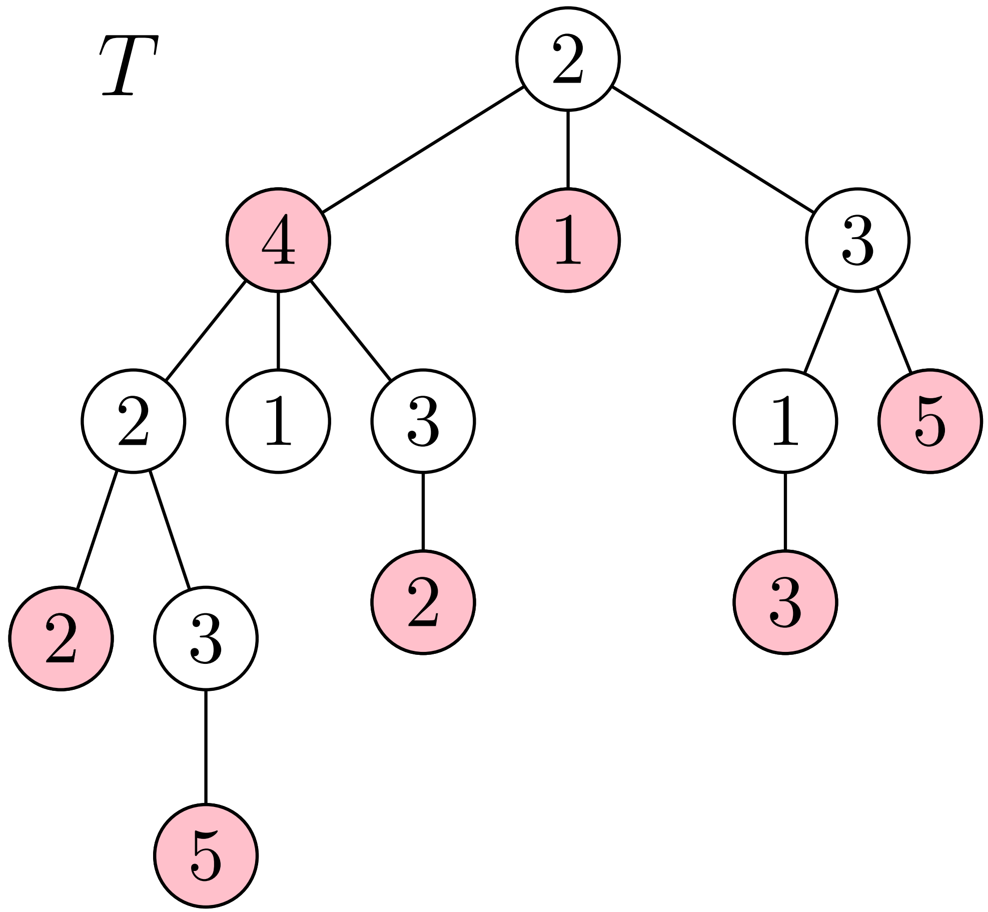

Max-Weight Independent Set on Trees

Given $v \in V(T)$, let T_v be the subtree of T rooted at v, and let $w(v)$ be the *weight* of v.

Subproblems:

• $OPT^+[v] =$ Weight of a maximum-weight IS of T_v with the constraint that v must belong to the IS.

• $OPT^-[v] =$ Weight of a maximum-weight IS of T_v with the constraint that v must *not* belong to the IS.

• $OPT[v] = \max\{OPT^+[v], OPT^-[v]\}$.

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• $OPT[v] = \max\{OPT^+[v], OPT^-[v]\}$.

Base case: If v is a leaf in T, then:

• $OPT^+[v] = w(v)$, and
• $OPT^-[v] = 0$

Max-Weight Independent Set on Trees

Recursive formula(s):

Let $C(v)$ be set of the children of v in T.

- $OPT^+[v] = w(v) + \sum_{u \in C(v)} OPT^-[u]$.
- $OPT^-[v] = \sum_{u \in C(v)} OPT[u] = \sum_{u \in C(v)} \max\{OPT^+[u], OPT^-[u]\}$.

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Optimal solution:

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Order of subproblems: ?

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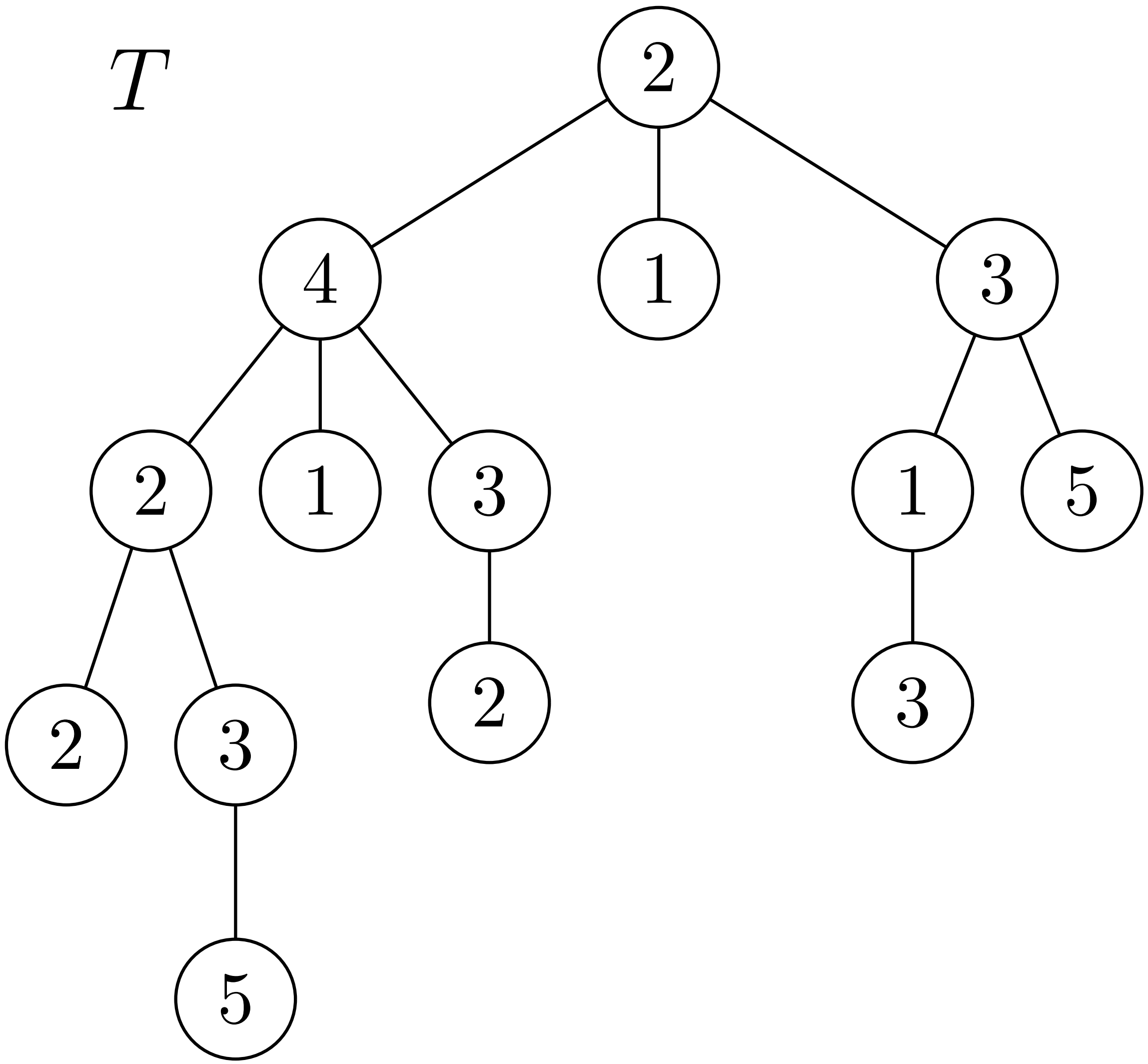

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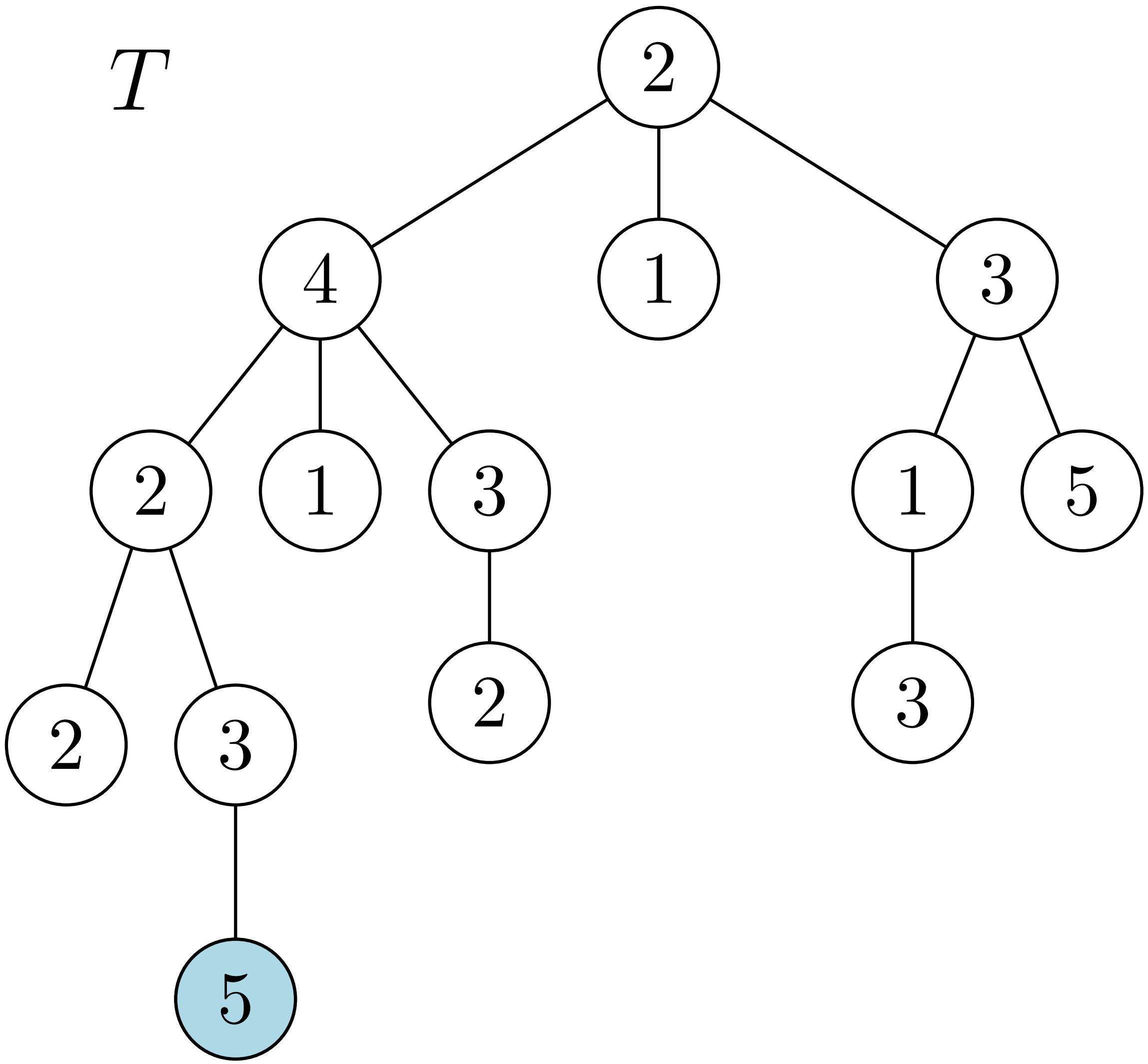

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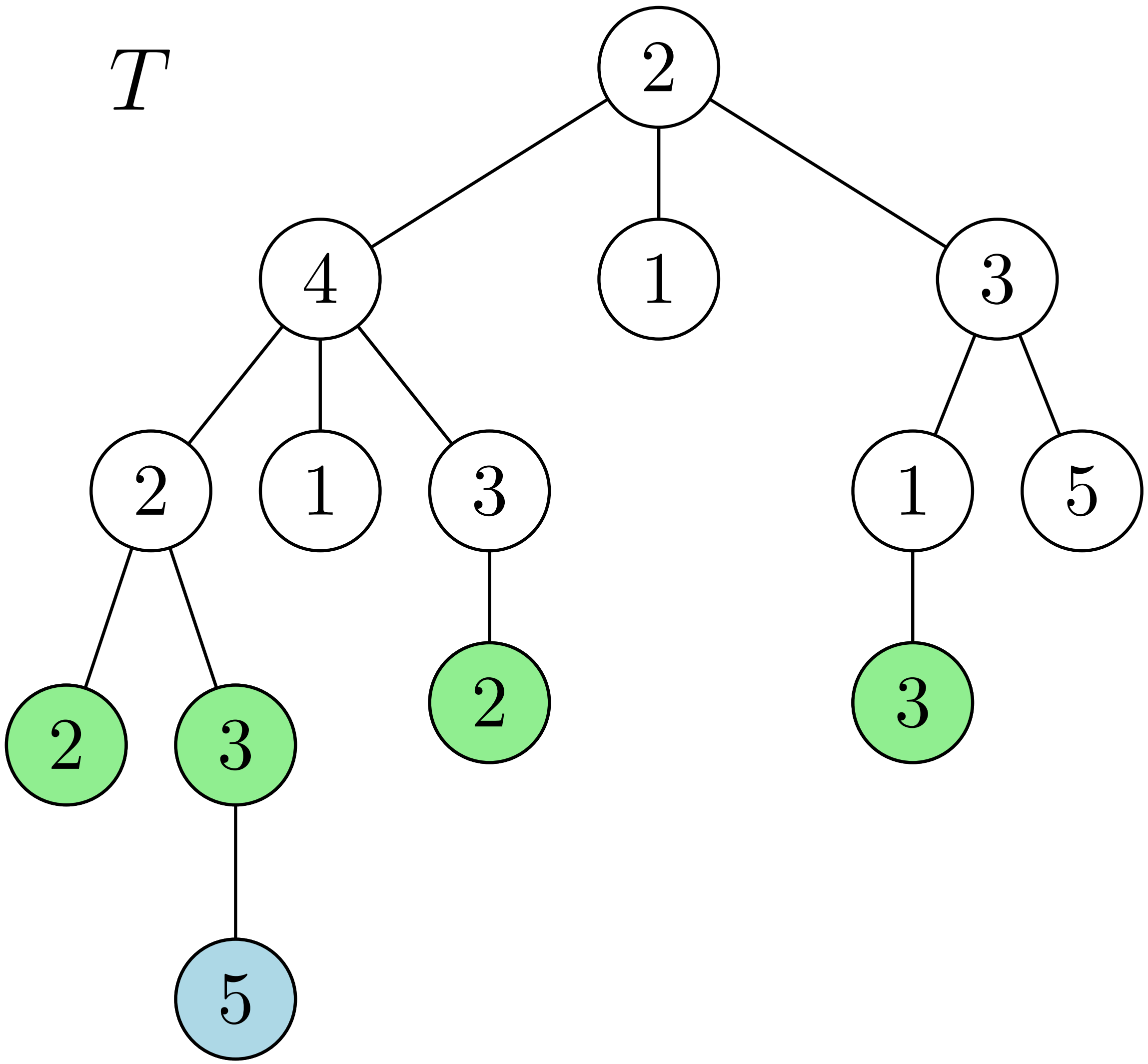

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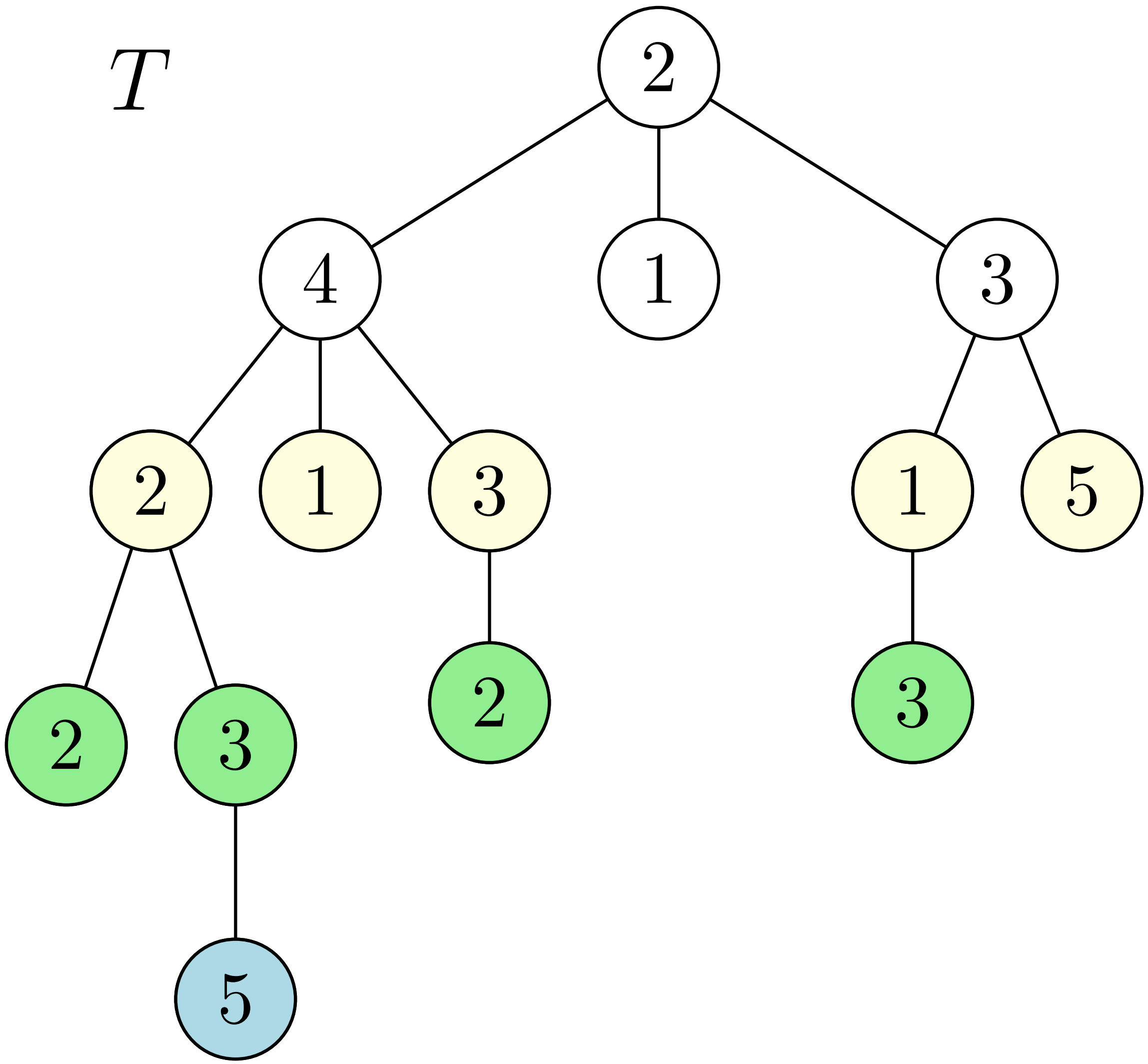

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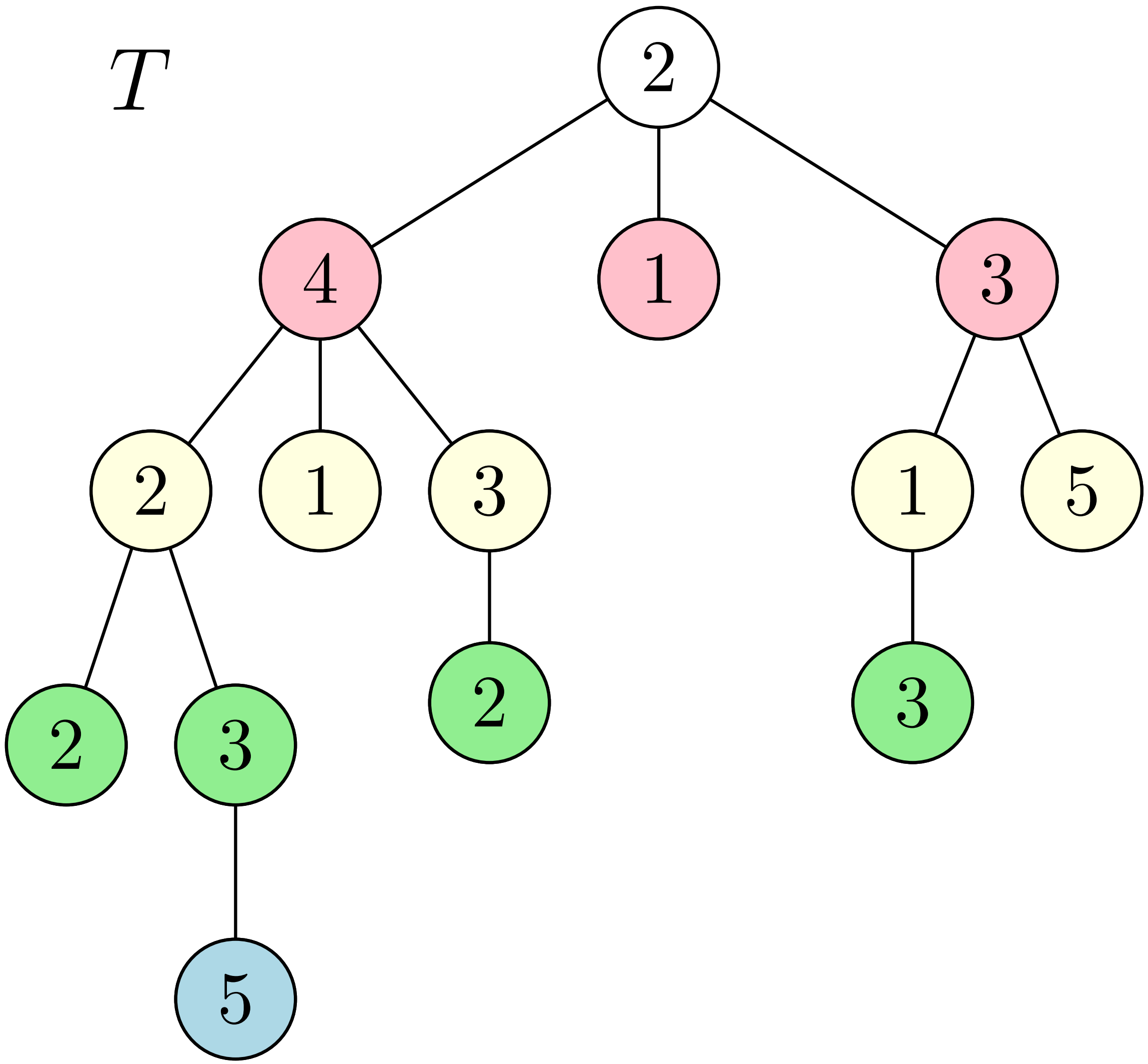

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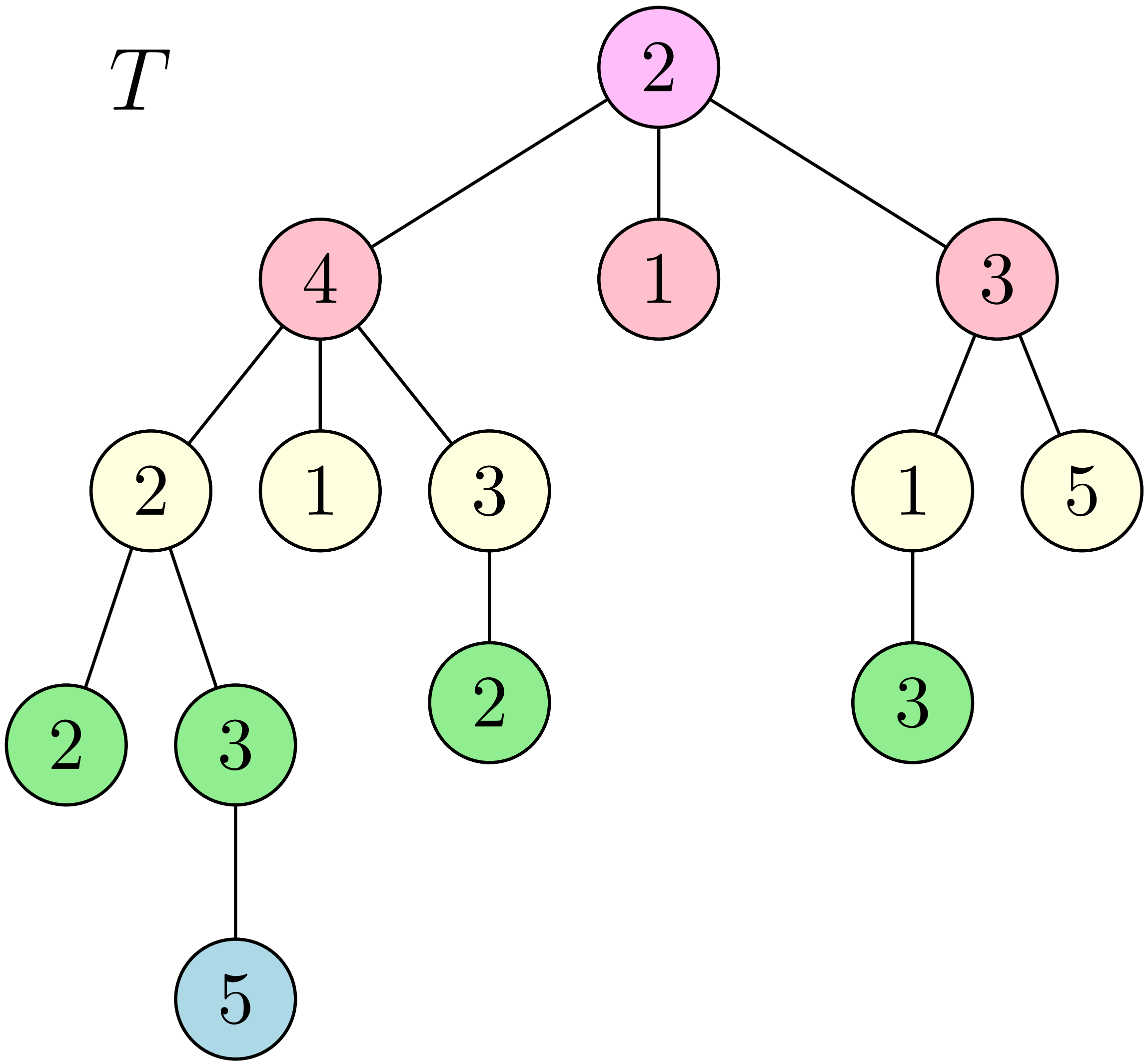
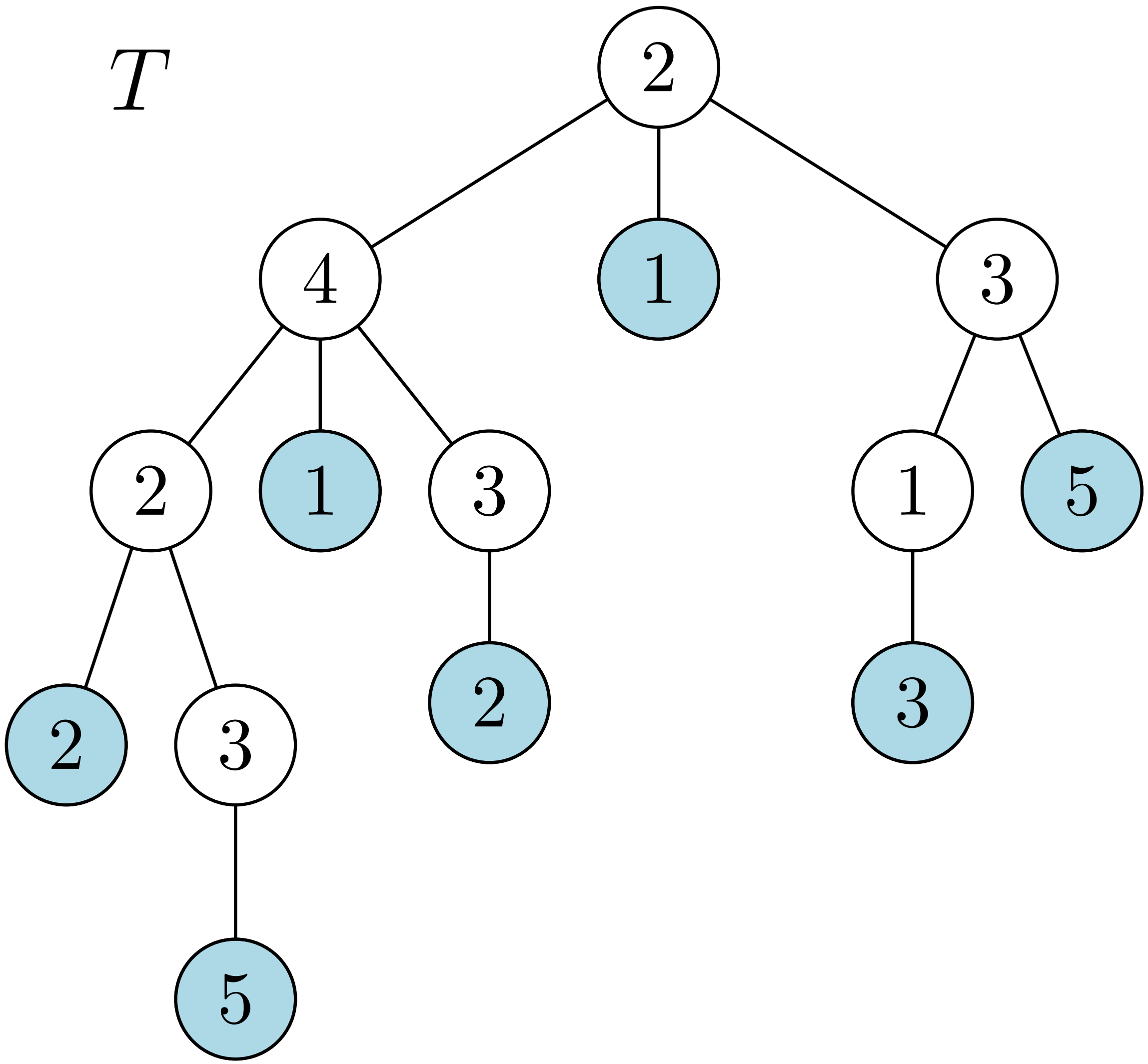


In order of decreasing depth in T

Order of Subproblems



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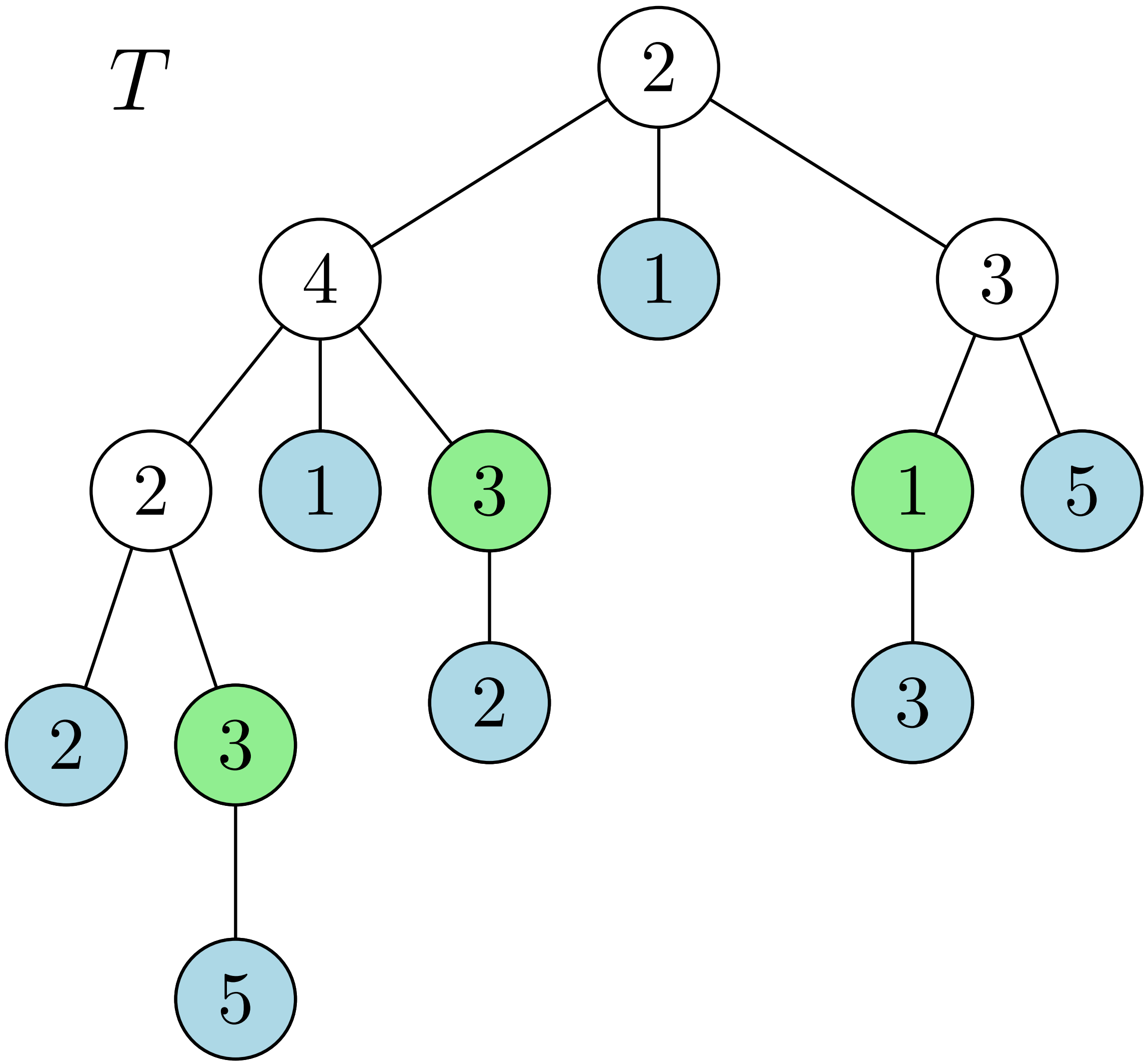

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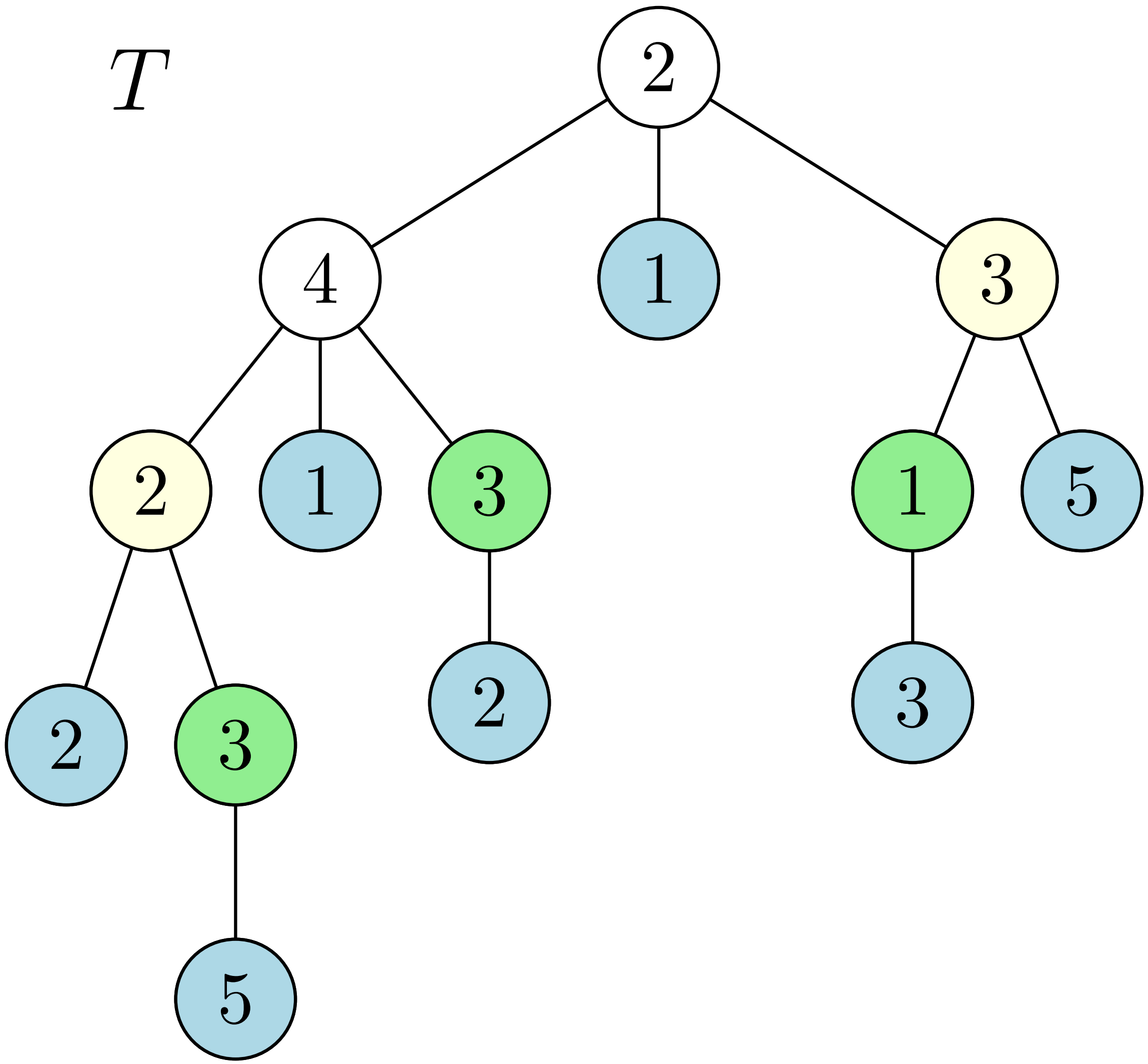

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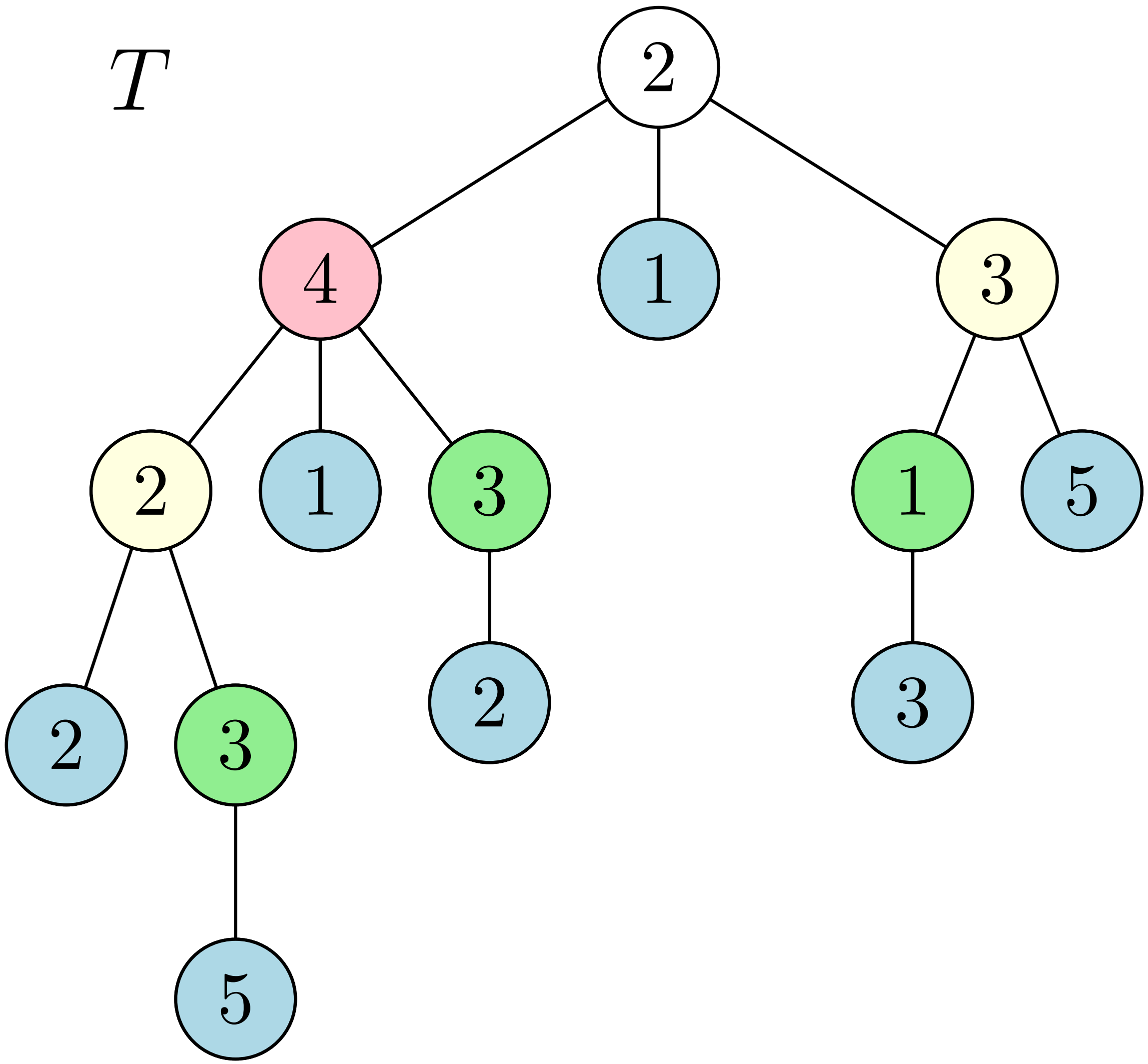

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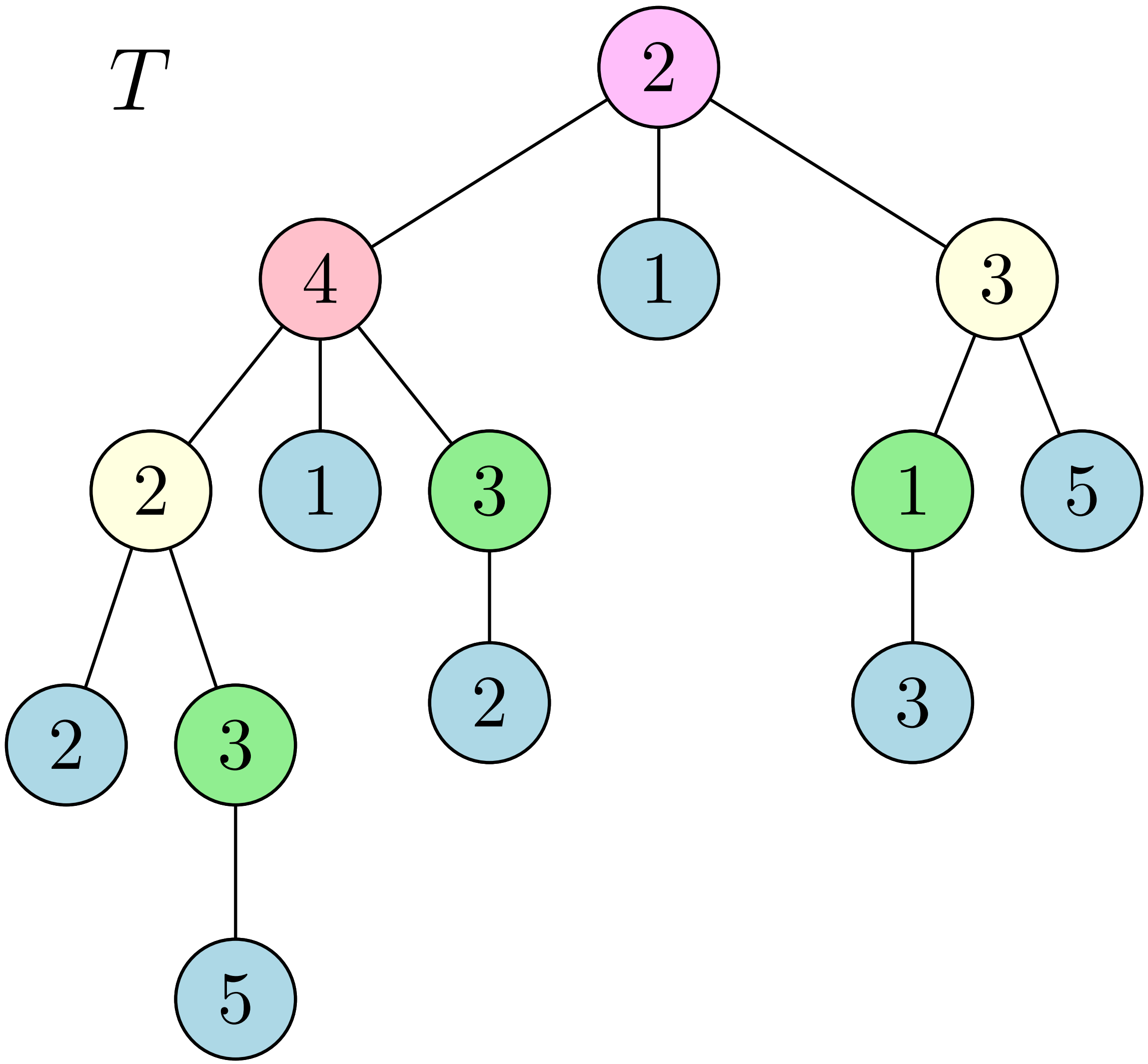
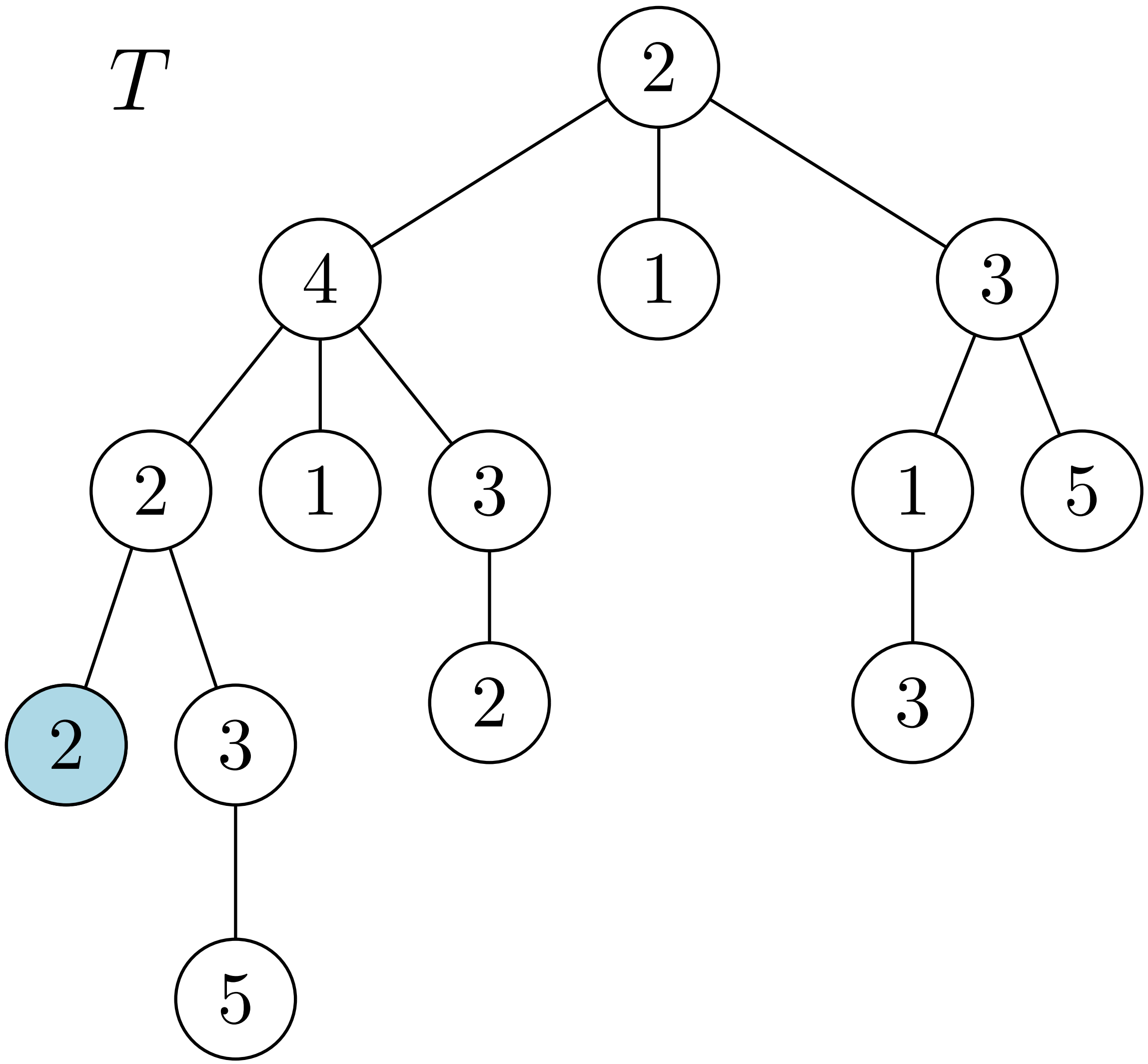


In order of increasing subtree heights

Order of Subproblems



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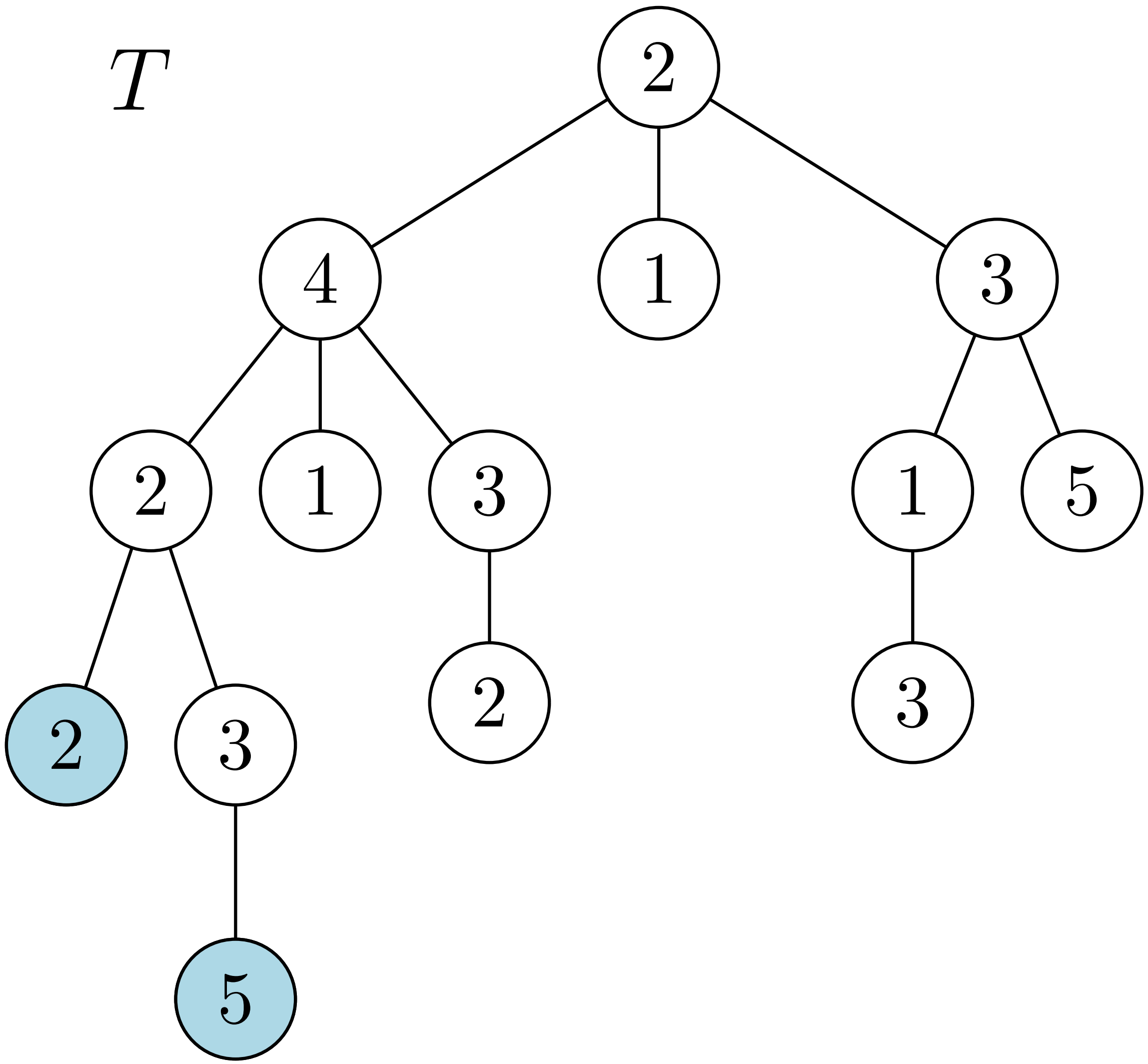

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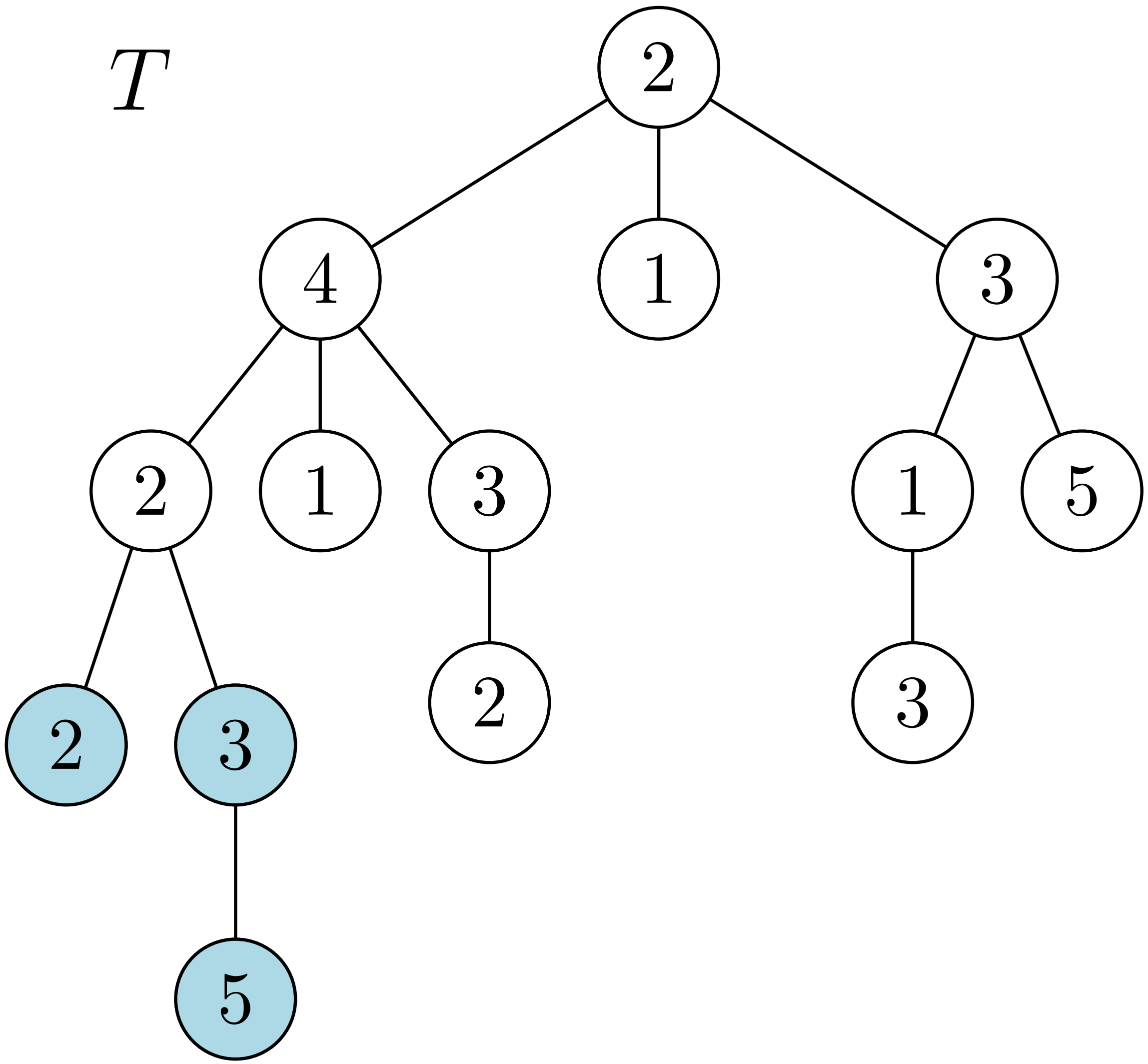

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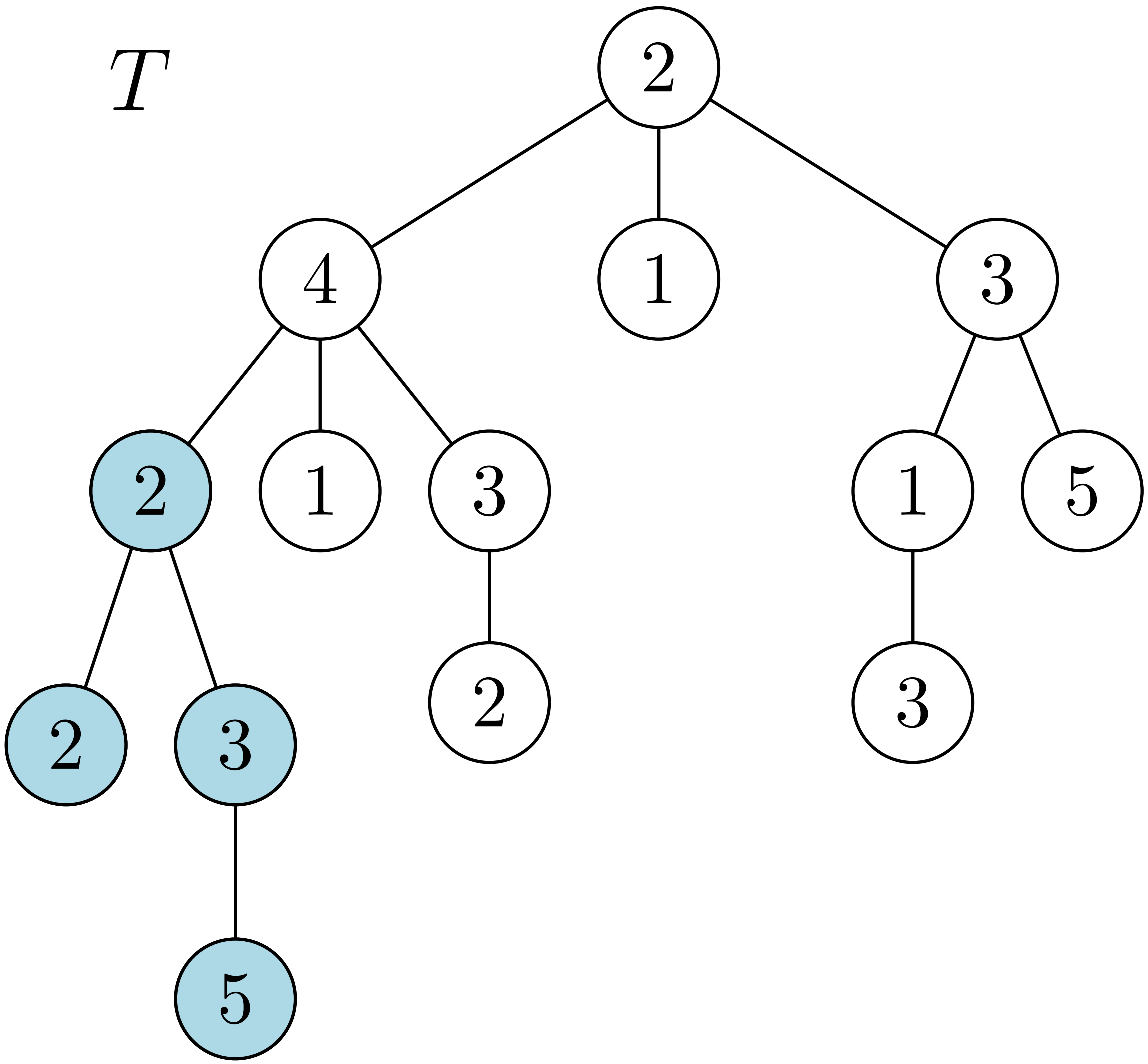

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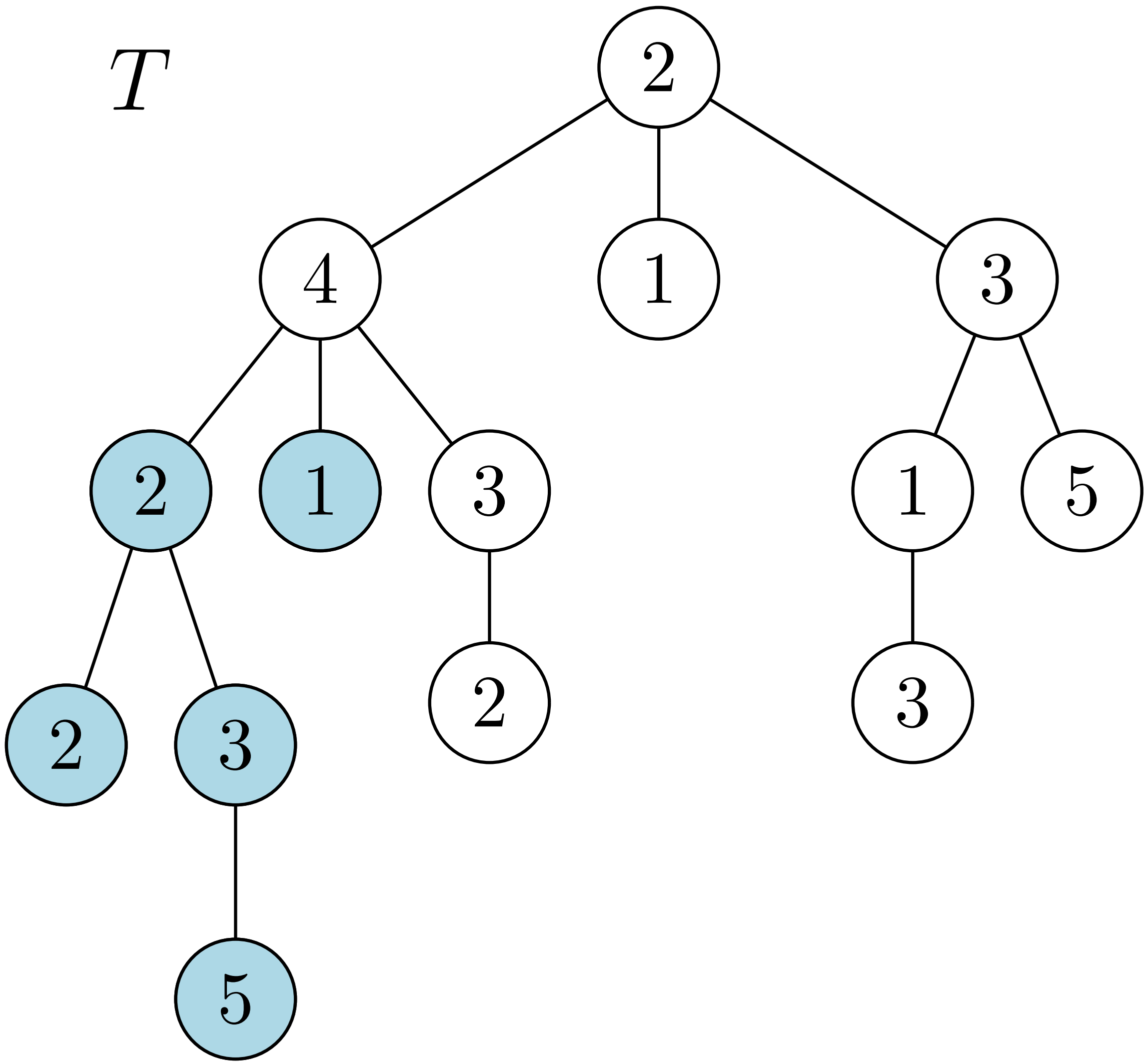

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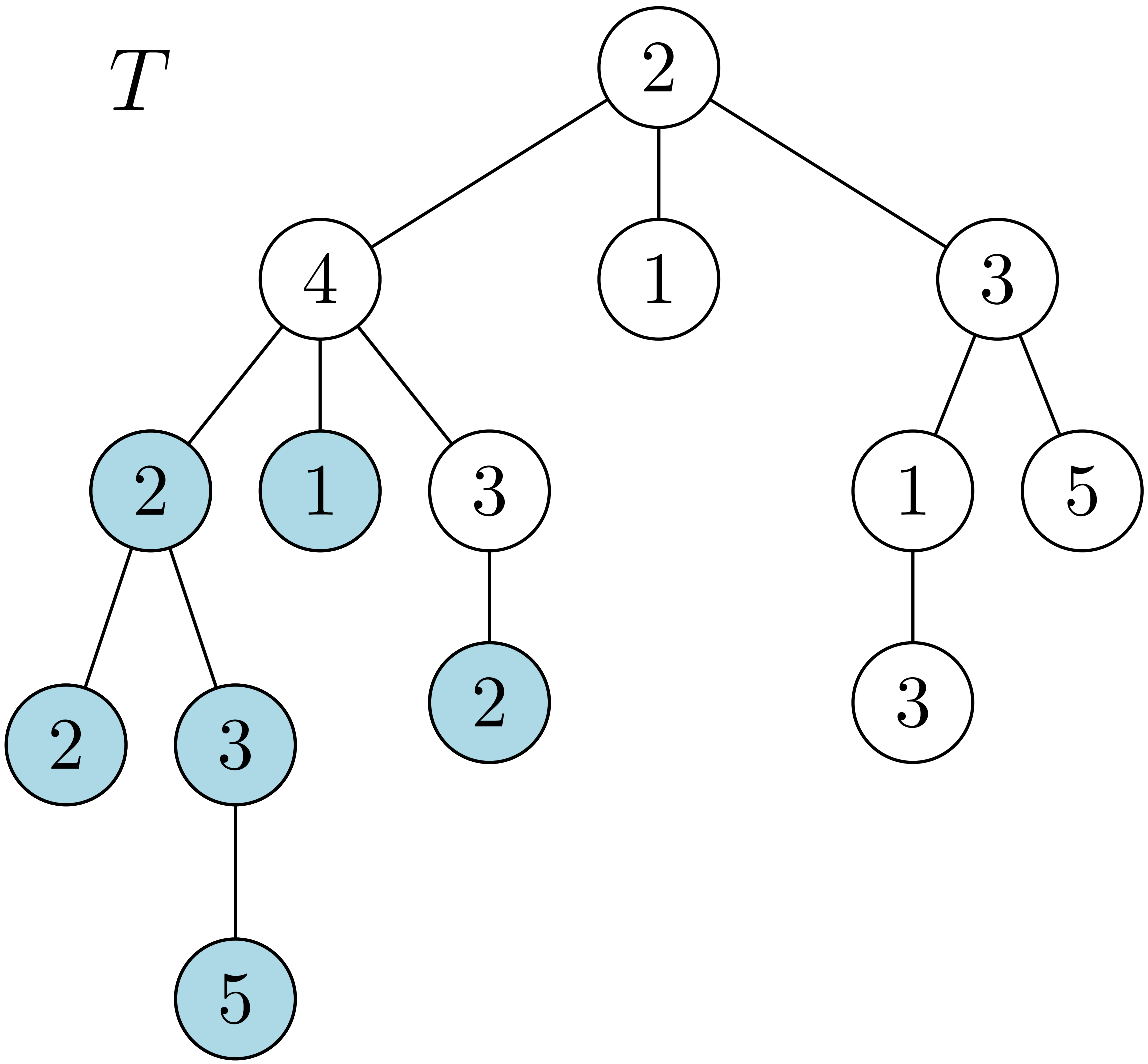

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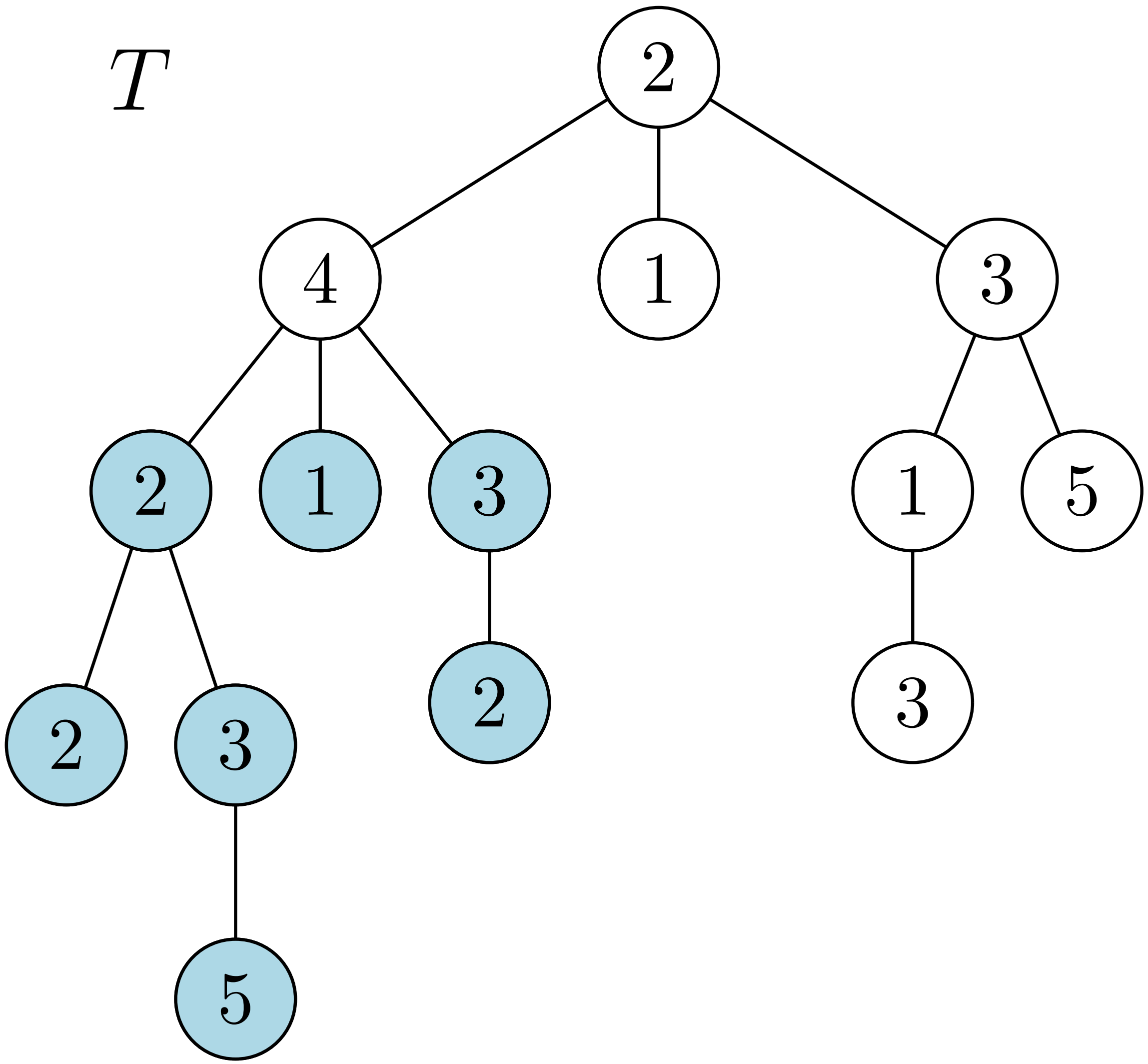

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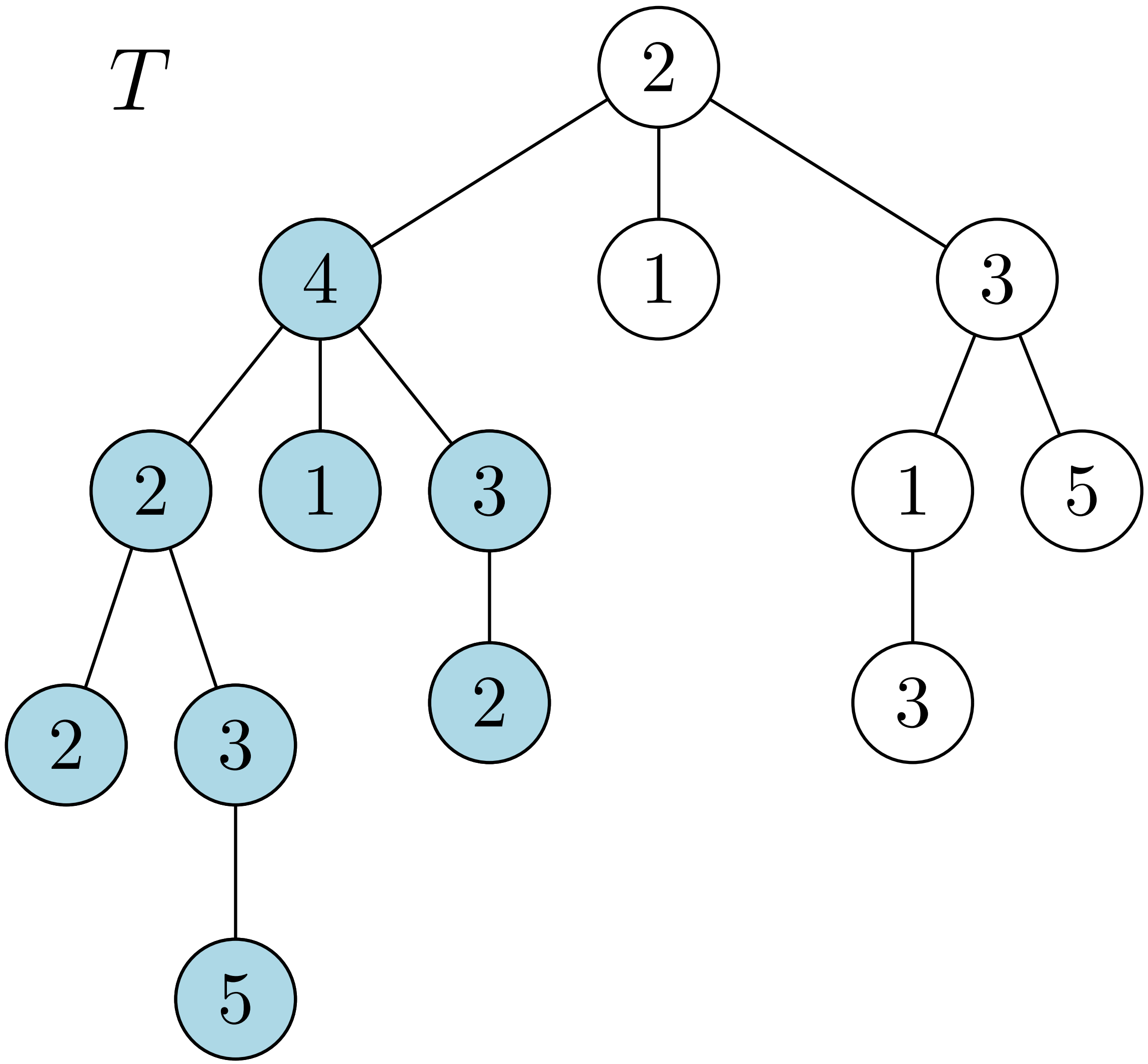

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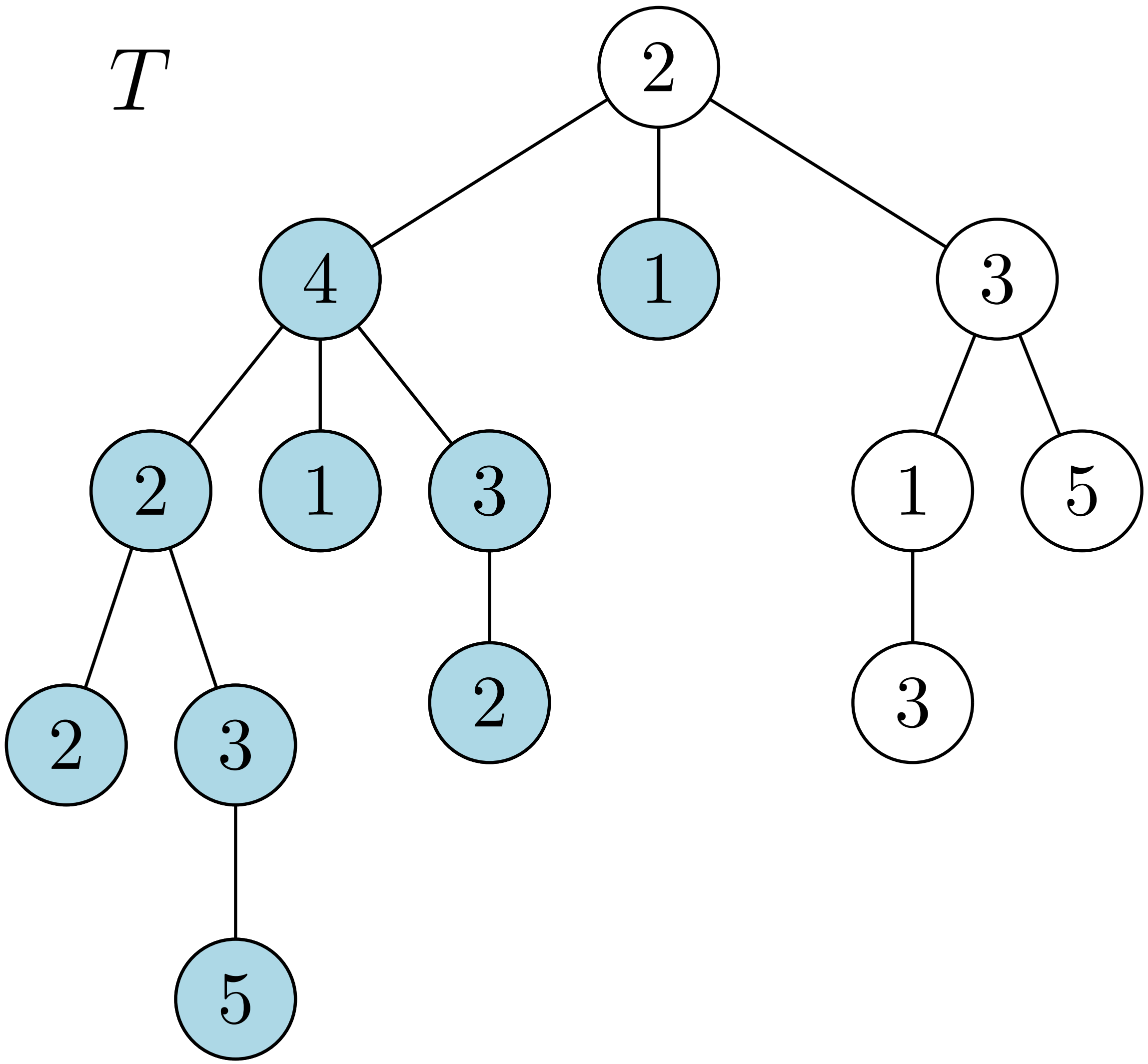

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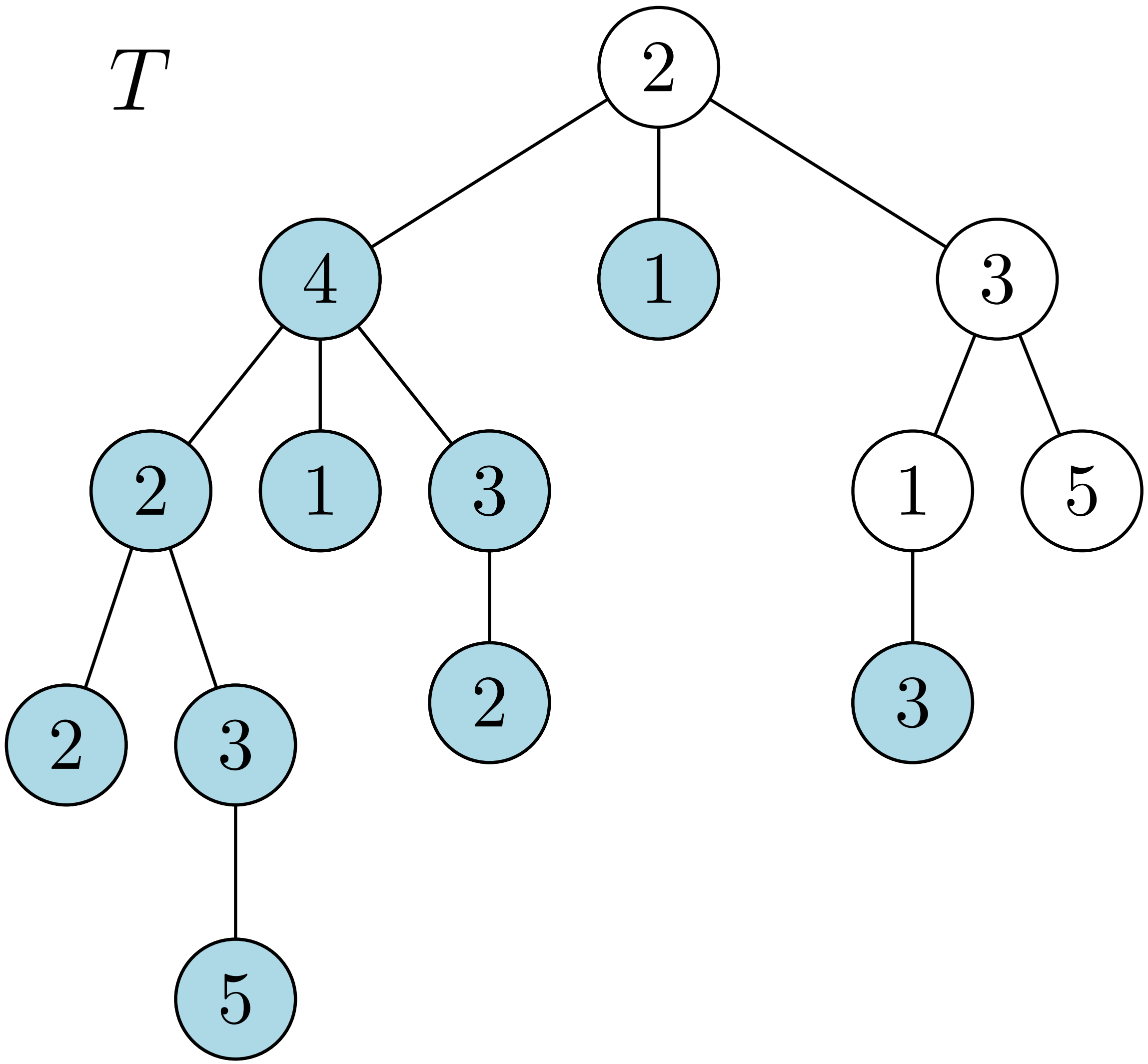

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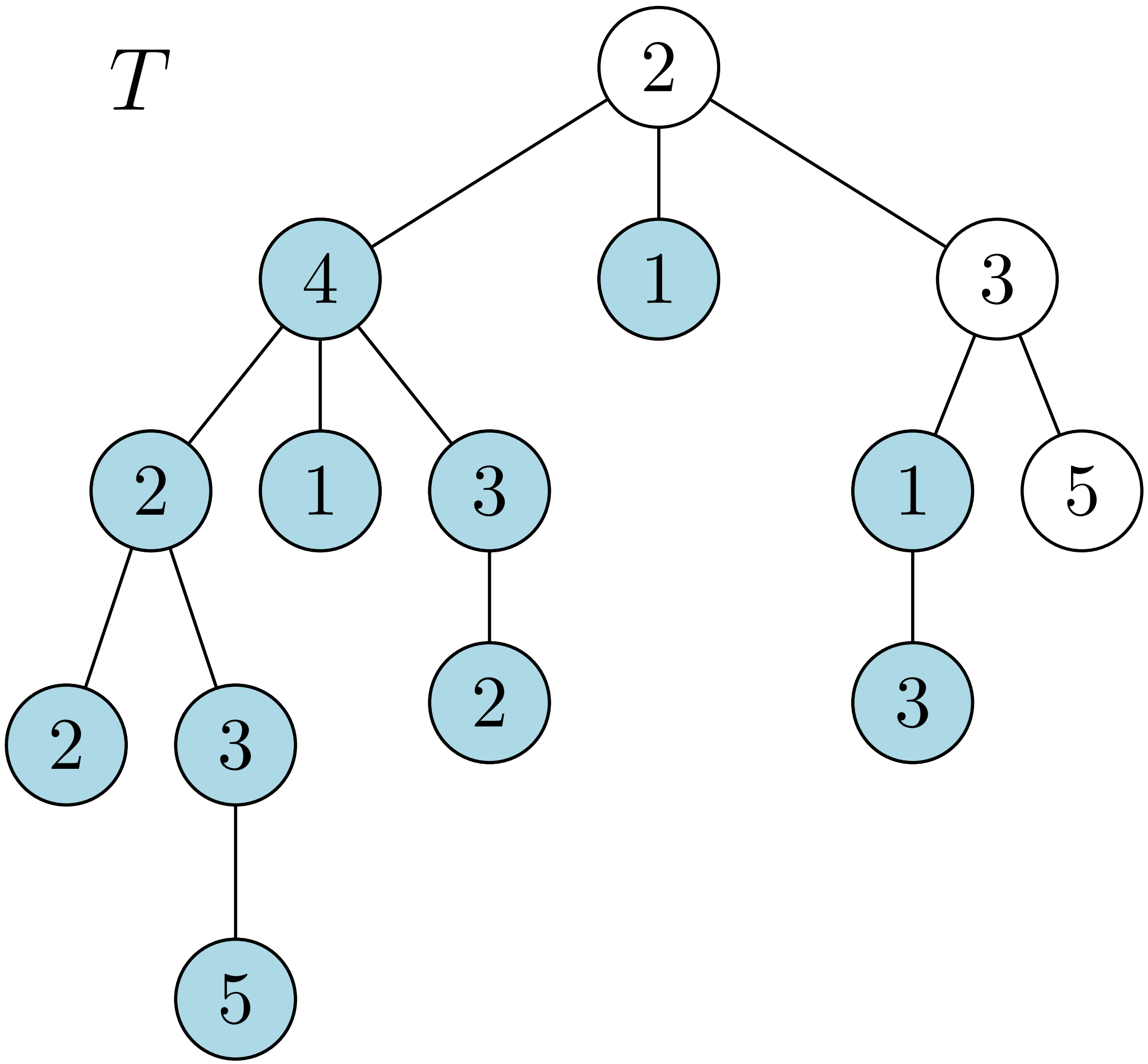

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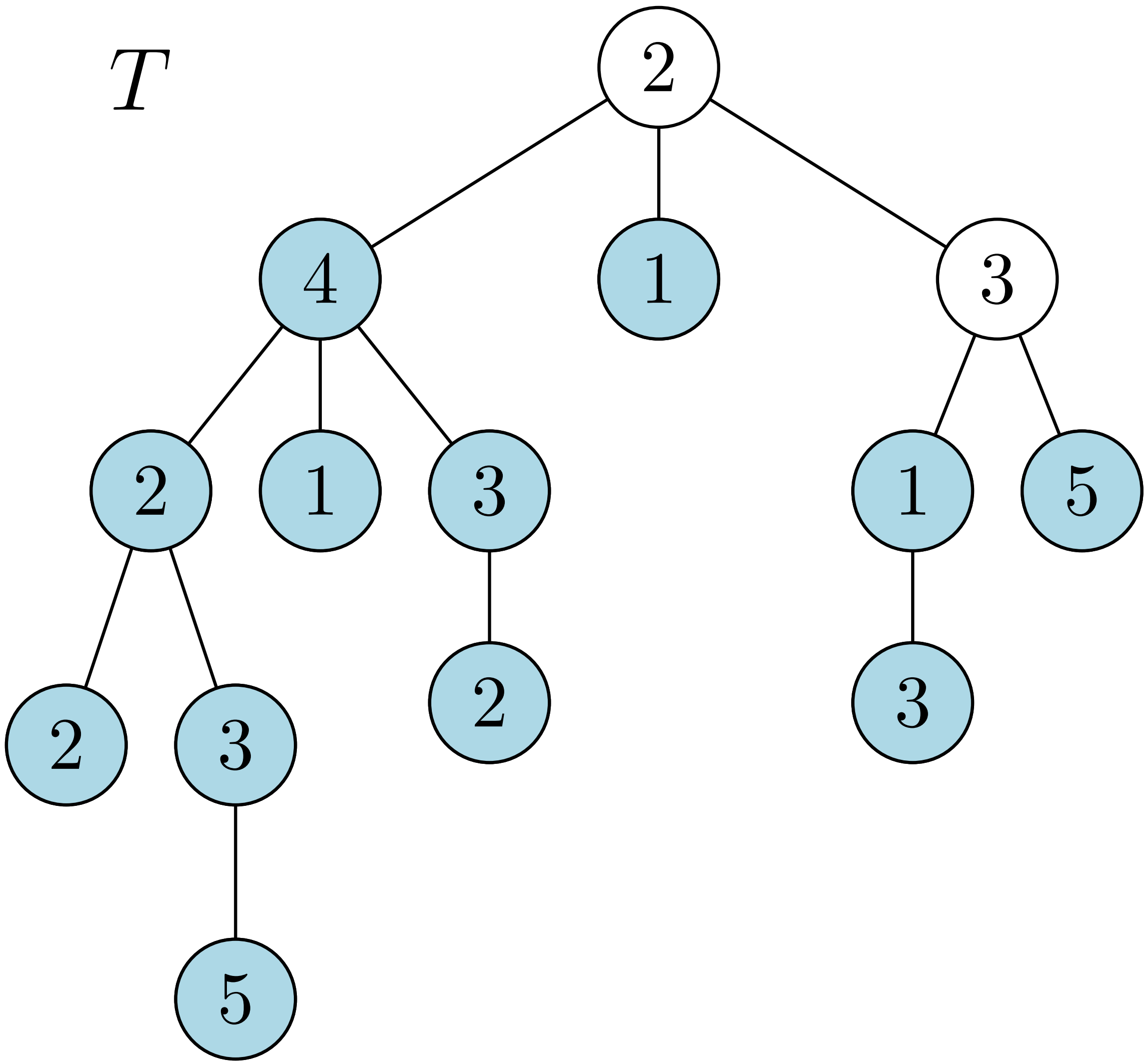

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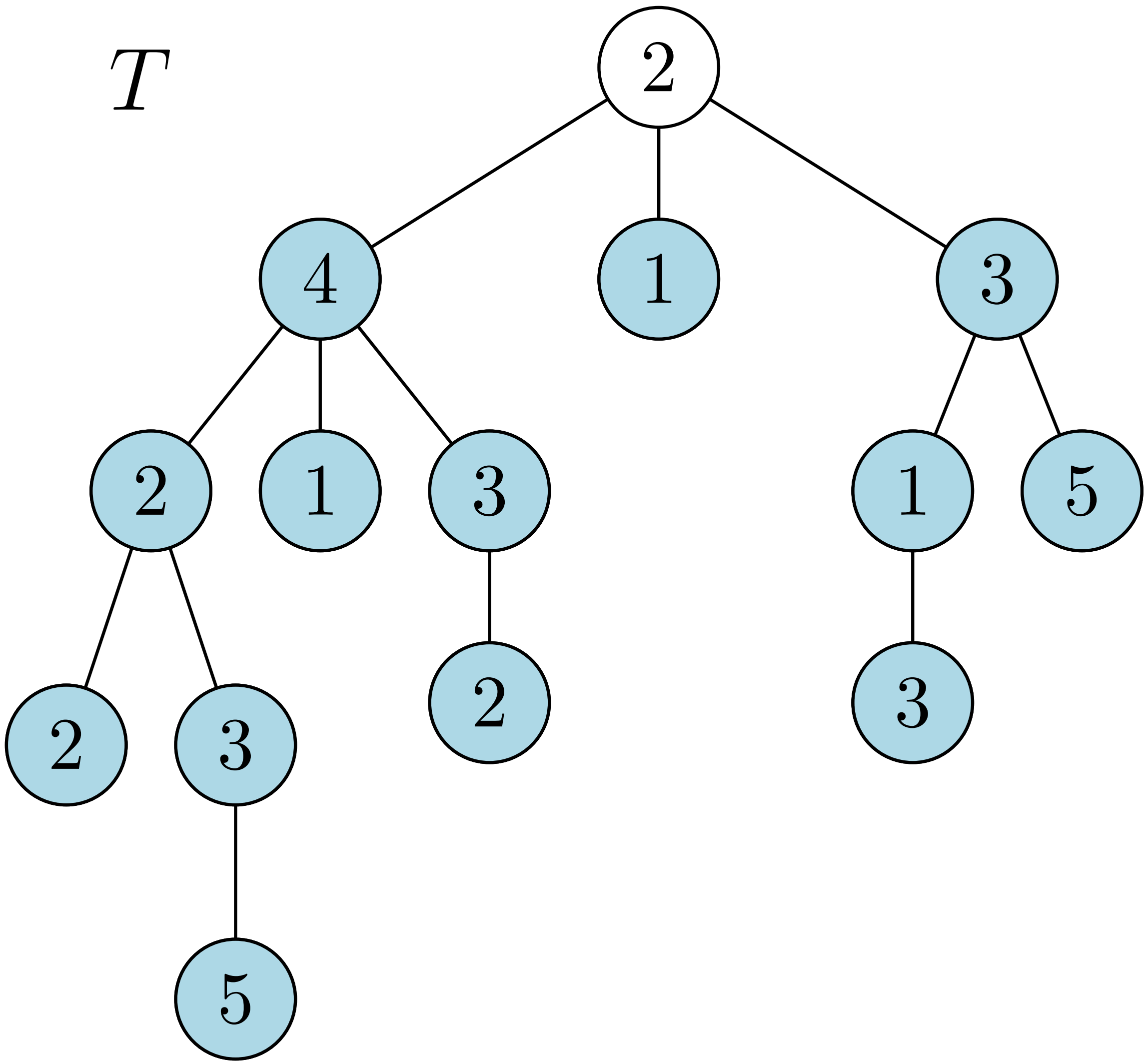

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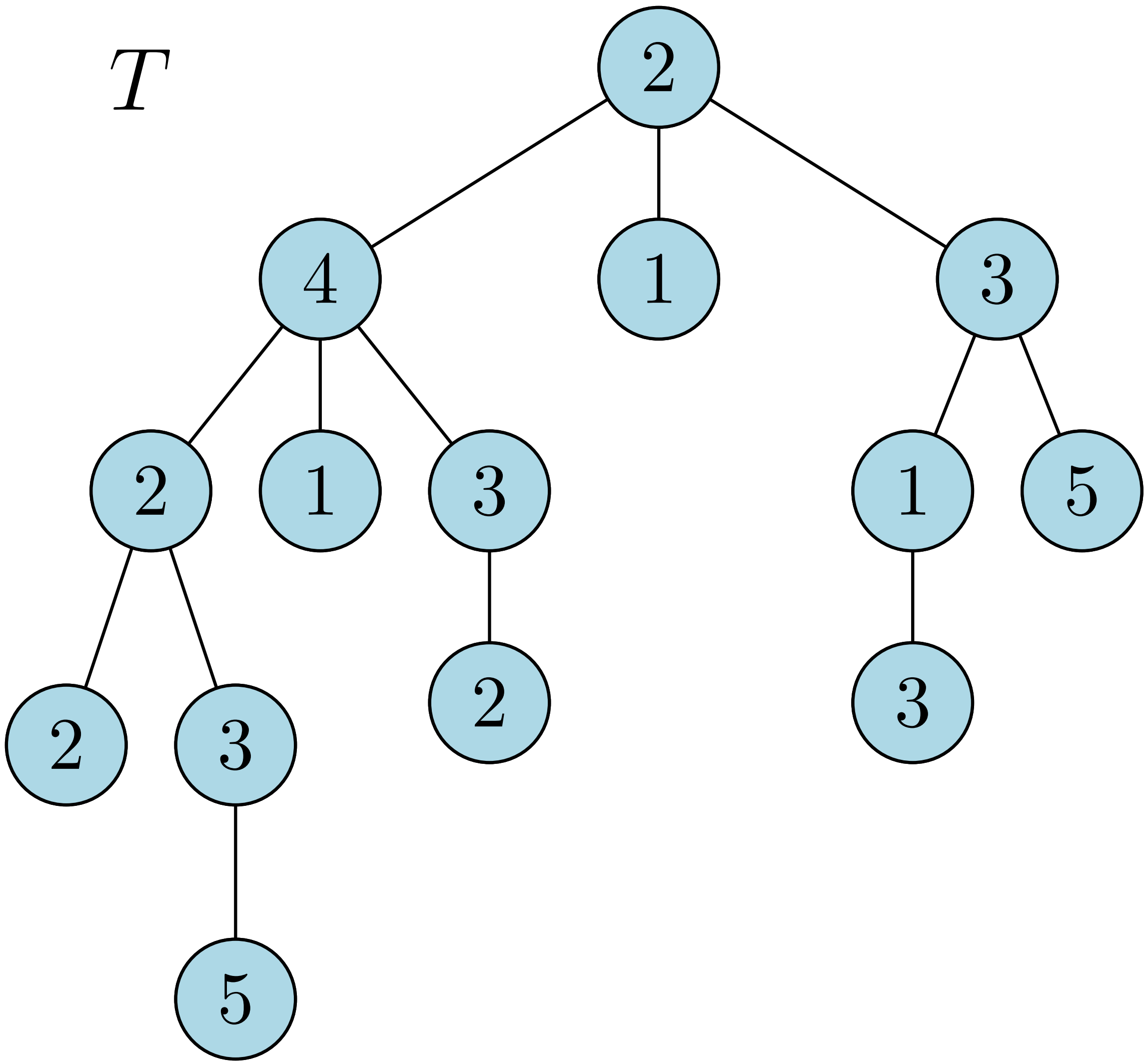


In DFS postoder

Time Complexity

- Suppose we can find a suitable order in $O(n)$ time.
- The time spent on vertex v is: $O(1 + |C(v)|)$
- Overall time complexity, up to multiplicative constants:

$$\sum_{v \in V(T)} (1 + |C(v)|) = n + \sum_{v \in V(T)} |C(v)| = n + (n-1) = O(n)$$

A possible implementation with DFS

```cpp
struct Node
{
    int weight;
    std::vector<Node*> children;
};

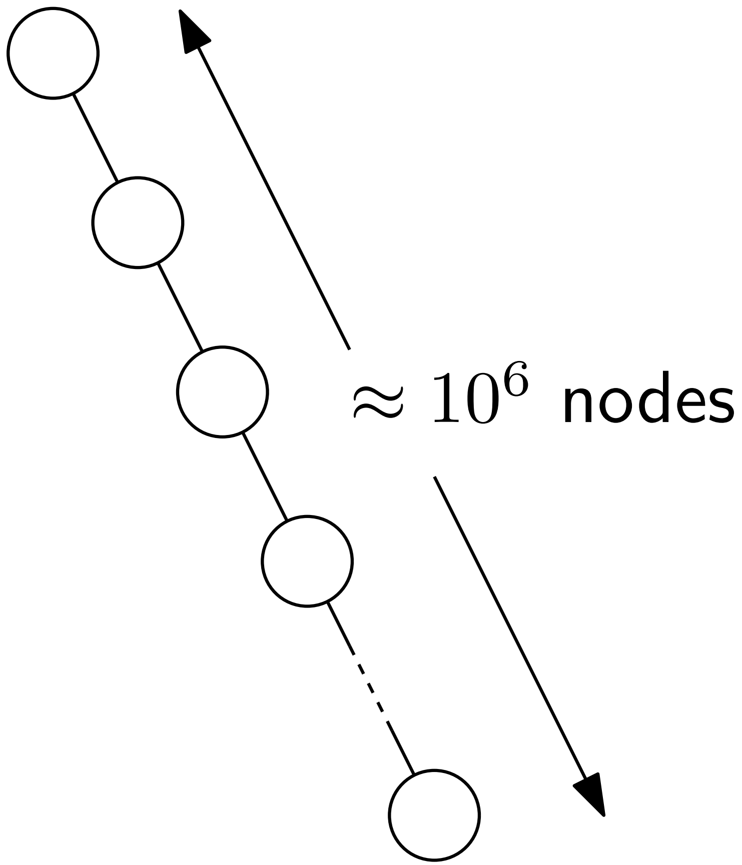
std::pair<int,int> dfs(Node* v)
{
    int opt_plus = v->weight;
    int opt_minus = 0;
    for(Node *u : v->children)
    {
        std::pair<int,int> opt_u = dfs(u);
        opt_plus += opt_u.second;
        opt_minus += std::max(opt_u.first, opt_u.second);
    }

    return std::make_pair(opt_plus, opt_minus);
}

Node* root = load_tree(); //Read T. Return a pointer to its root.
std::pair<int,int> opt = dfs(root);
std::cout << std::max(opt.first, opt.second) << "\n";
```

A Nasty Instance

What happens if the previous code is run on this tree?



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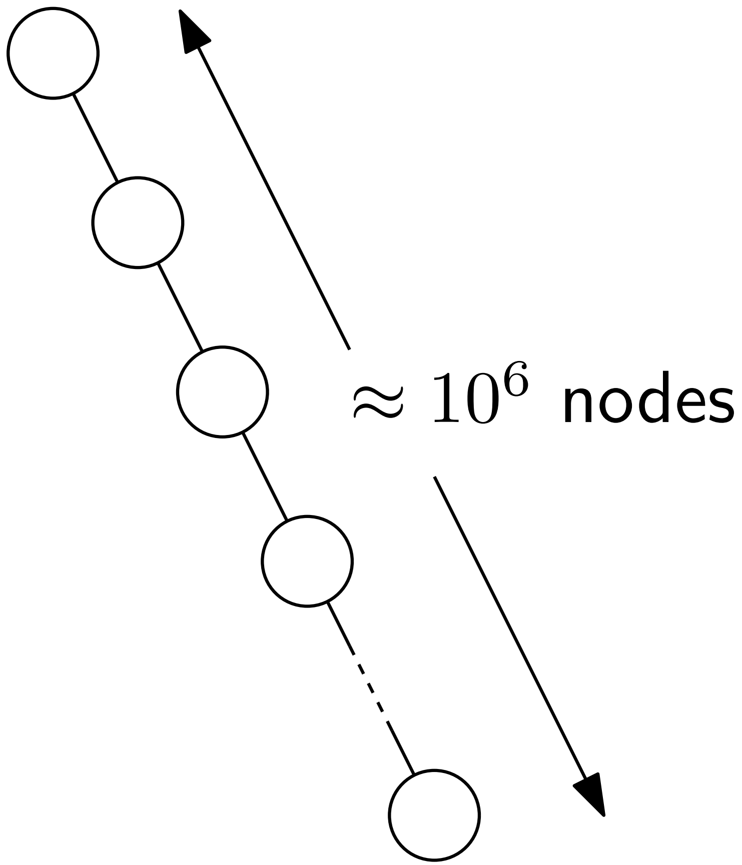


```
$ ./max_weight_is < nasty_instance.in
$ Segmentation fault
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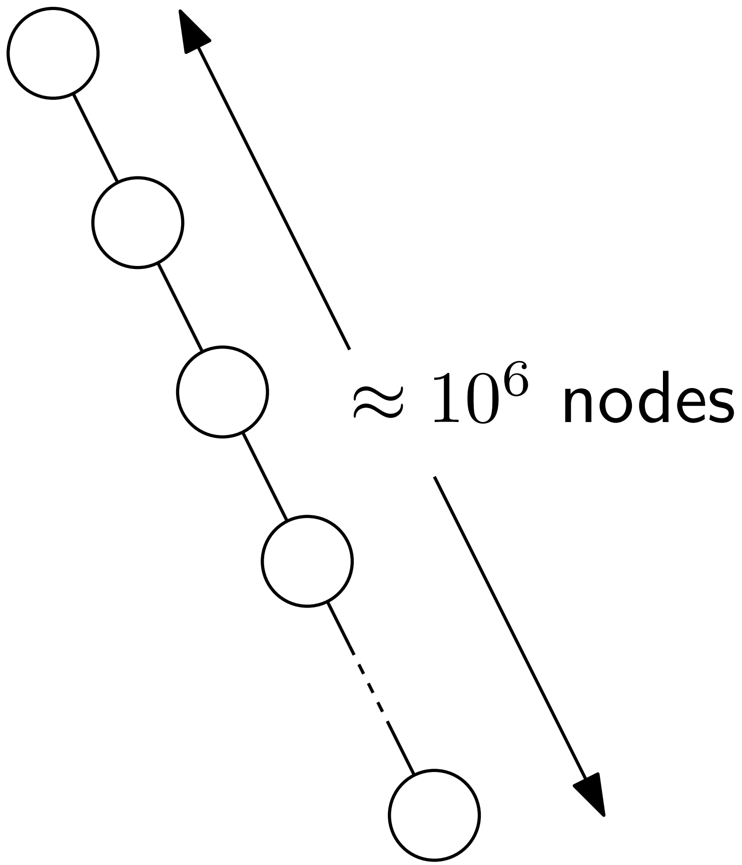


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Why?

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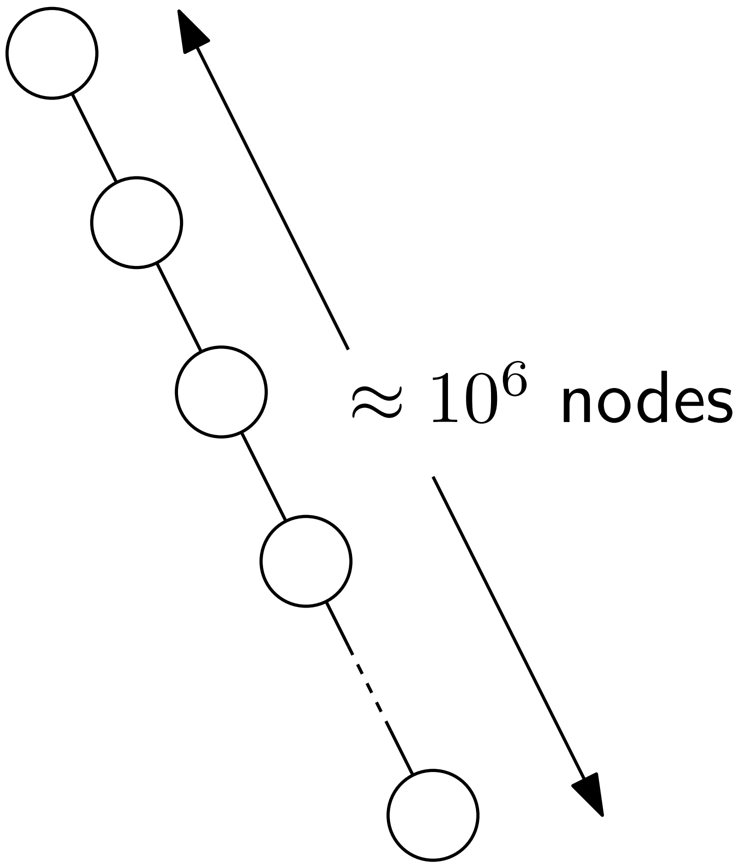

Solutions

- Non recursive DFS
- Different order
  (use BFS to construct levels)
- Explicitly manage DFS stack

```
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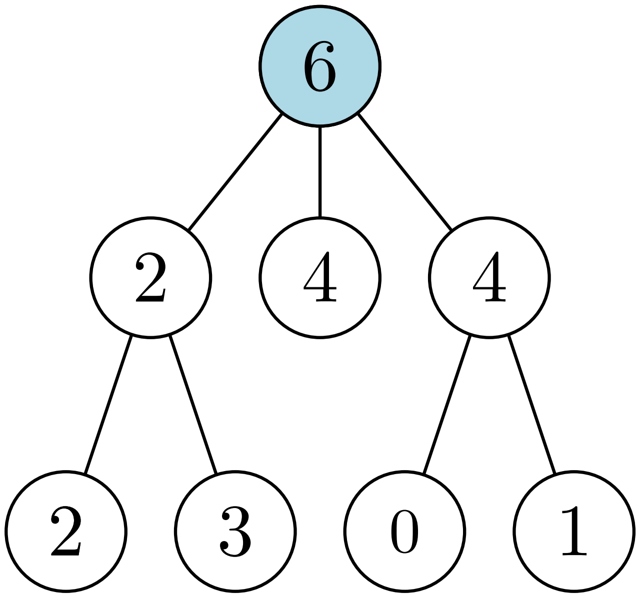
```

Max-Weight Independent Set on Trees
+ Budget Constraints

Budgeted Max-Weight IS on Trees

Input: A tree T with integer weights on its vertices, a *budget* $B \in \mathbb{N}$.

Output: The maximum weight of an independent set S of T such that $|S| \leq B$.



$B = 1$

Budgeted Max-Weight IS on Trees

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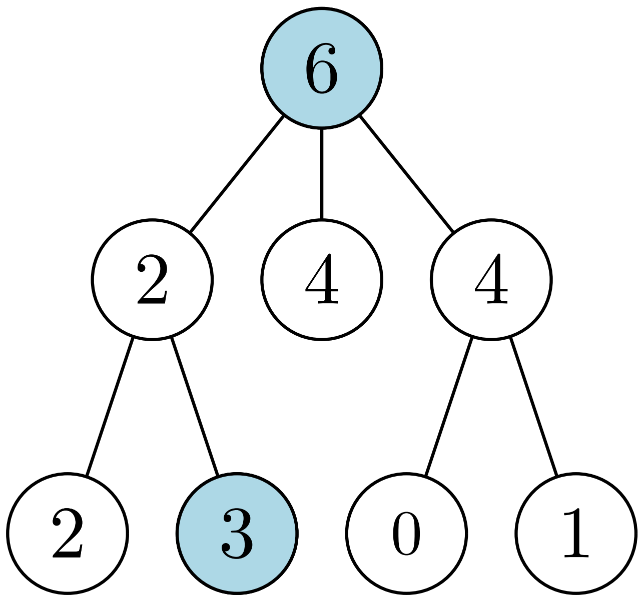

$B = 2$

Budgeted Max-Weight IS on Trees

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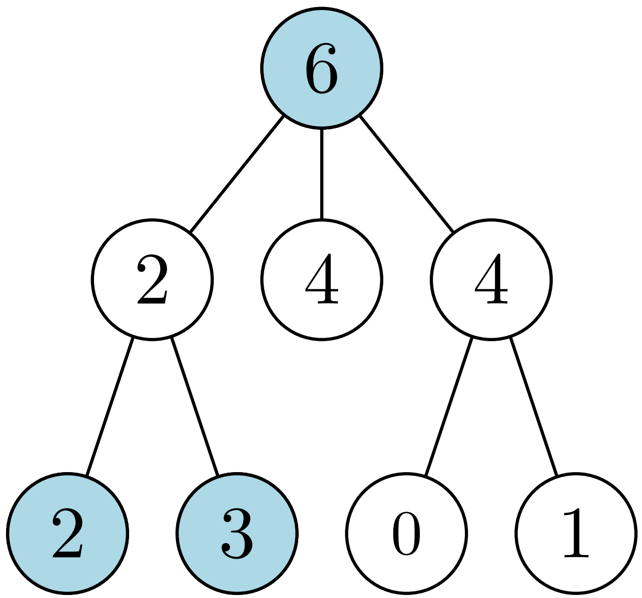

$B = 3$

Budgeted Max-Weight IS on Trees

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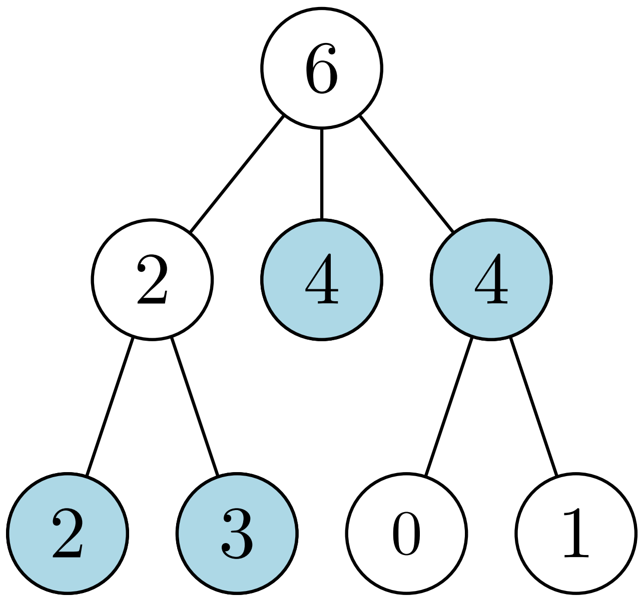

$B = 4$

Budgeted Max-Weight IS on Trees

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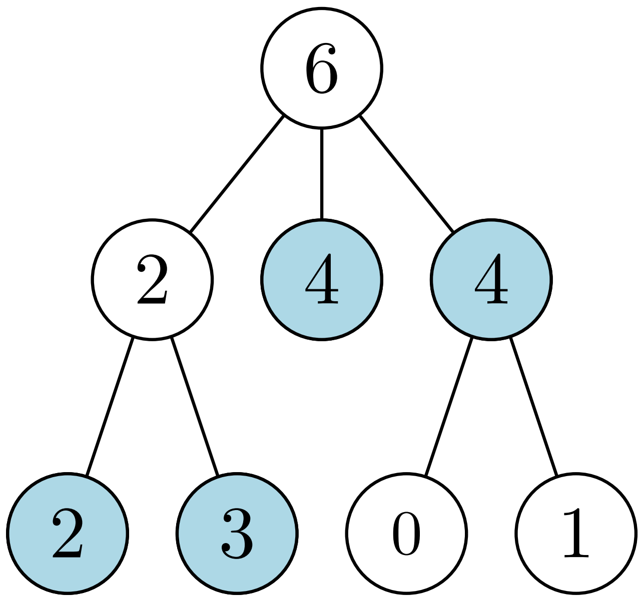

$B = 5$

Budgeted Max-Weight IS on Trees

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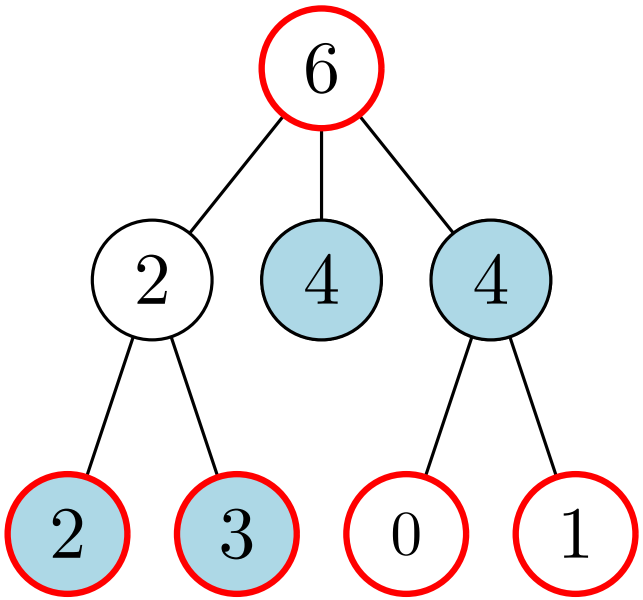

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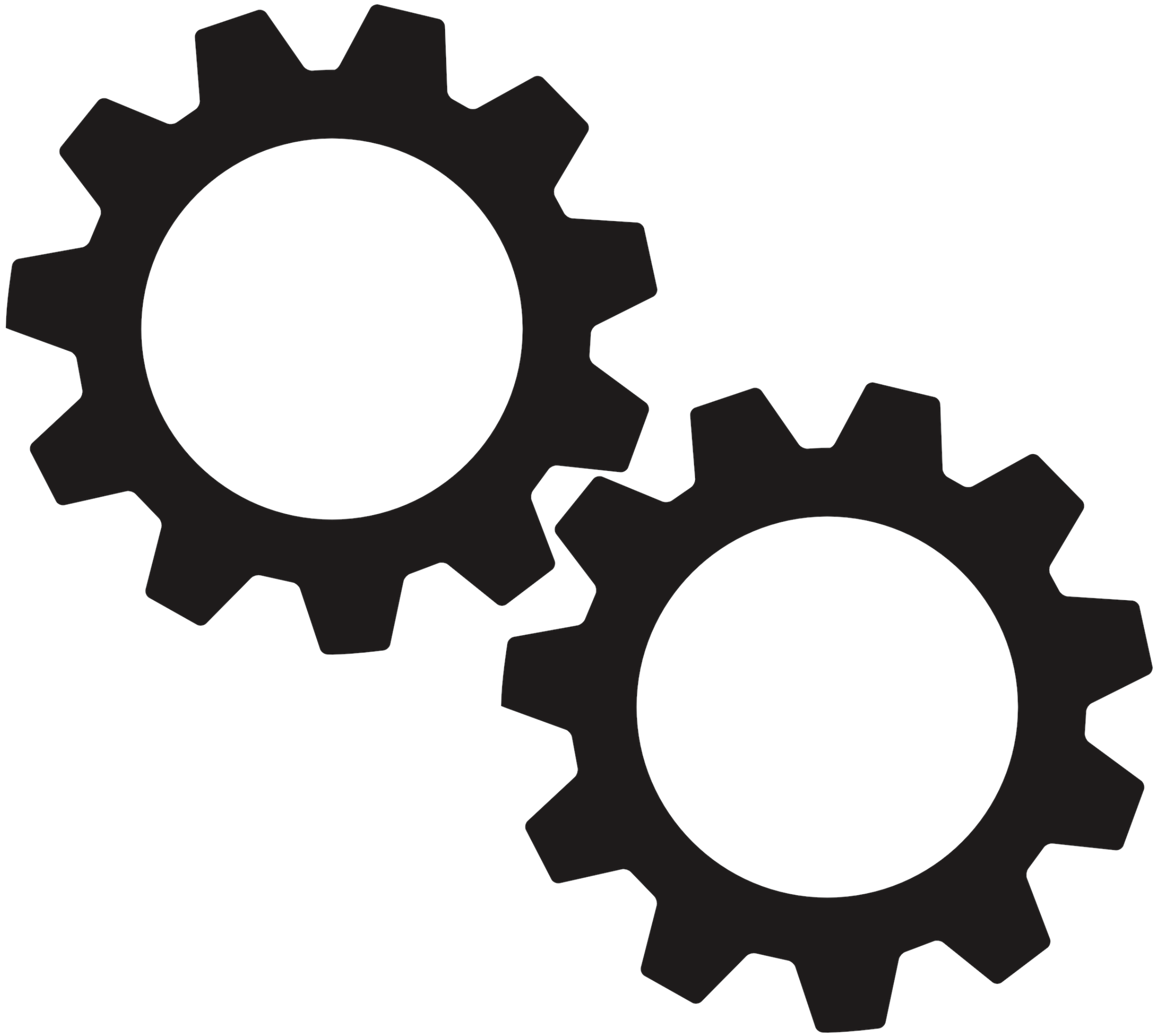

Budgeted Max-Weight IS on Trees

Subproblem definition:

$OPT^+[v,b] =$ Maximum Weight of an IS of T_v that contains v and has size at most b.

$OPT^-[v,b] =$ Maximum Weight of an IS of T_v that does not contain v and has size at most b.

Base cases: v is a leaf of T.

$$OPT^+[v,b] = \begin{cases} w(v) & \text{if } b \geq 1 \\ -\infty & \text{if } b = 0 \end{cases}$$

Constrains can't be satisfied!

$$OPT^-[v,b] = 0$$

Max-Weight IS on Trees w. Budget

Recursive Formula:

- Let's consider $OPT^+[v, b]$.
- If $b = 0$, then $OPT^+[v, b] = -\infty$.
- If $b > 0$, we need to "distribute" $b - 1$ units of budget among $C(v) = \{u_1, u_2, \ldots, u_k\}$
- We want to choose $b_1, b_2, \ldots, b_k \in \mathbb{N}$ such that $b_1 + \cdots + b_k \leq b - 1$ and they maximize:

$$OPT^-[u_1, b_1] + OPT^-[u_2, b_2] + \cdots + OPT^-[u_k, b_k]$$

Max-Weight IS on Trees w. Budget

Recursive Formula (First Attempt):

- "Guess" the correct combination of $b_1, b_2, \ldots, b_k$:

$$OPT^+[v, b] = w(v) + \max_{\substack{b_1, b_2, \ldots, b_k \in \mathbb{N} \\ b_1 + \cdots + b_k \leq b-1}} \sum_{i=1}^{k} OPT^-[u_i, b_i]$$

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Will this work?

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Will this work? Yes!

How long will this take?

Stars and Bars!

- How many possible choices of $b_1, b_2, \ldots, b_k \in \mathbb{N}$ such that $b_1 + \cdots + b_k = x$?

Stars and Bars!

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- How many different ways to arrange x stars and $k-1$ bars?

$$\star\star \mid \star\star\star \mid \quad \mid \star\star\star\star$$

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$$\underbrace{\star\star}_{b_1=2} \mid \underbrace{\star\star\star}_{b_2=3} \mid \underbrace{\quad}_{b_3=0} \mid \underbrace{\star\star\star\star}_{b_4=4}$$

Stars and Bars!

- How many possible choices of $b_1, b_2, \ldots, b_k \in \mathbb{N}$ such that $b_1 + \cdots + b_k = x$?

- How many different ways to arrange x stars and $k-1$ bars?

$$\underbrace{\star\star}_{b_1=2} \mid \underbrace{\star\star\star}_{b_2=3} \mid \underbrace{\quad}_{b_3=0} \mid \underbrace{\star\star\star\star}_{b_4=4}$$

$$\frac{(x+k-1)!}{x!(k-1)!} = \binom{x+k-1}{k-1} = \Omega\left(\left(\frac{x}{k}\right)^k\right)$$

Stars and Bars!

- How many possible choices of $b_1, b_2, \ldots, b_k \in \mathbb{N}$ such that $b_1 + \cdots + b_k = x$?

Too slow!

$$\frac{(x+k-1)!}{x!(k-1)!} = \binom{x+k-1}{k-1} = \Omega\left(\left(\frac{x}{k}\right)^k\right)$$

Recursive Formula: Second Attempt

Let's consider a more abstract problem.

- **Input:** $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{R}$ and $B \in \mathbb{N}$.

- **Output:** $x_1, \ldots, x_k \in \mathbb{N}$ such that $\sum_i x_i \leq B$ and $\sum_i f_i(x_i)$ is maximized.

(Assume that each f_i can be evaluated in constant time).

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How do we solve this problem?

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(Assume that each f_i can be evaluated in constant time).

How do we solve this problem?

Dynamic Programming!

Distributing Budget Optimally

Subproblem Idea

$D[j, b]$ = Best way to distribute b units of budget among the first j functions.

More Formally:

$$D[j, b] = \max_{\substack{x_1, \ldots, x_j \in \mathbb{N} \\ x_1 + \cdots + x_j \leq b}} \sum_{i=1}^{j} f_i(x_i)$$

Base Case: If $j = 1$, explictly check the $b + 1$ possible choices.

$$D[1, b] = \max\{f_1(0), f_1(1), f_1(2), \ldots, f_1(b)\}$$

Distributing Budget Optimally

Recursive Formula

"Guess" how much budget b' will be assigned to f_j.

$$D[j, b] = \max_{b' \in \{0,\ldots,b\}} \{D[j-1, b-b'] + f_j(b')\}$$

At most $O(B)$ choices.

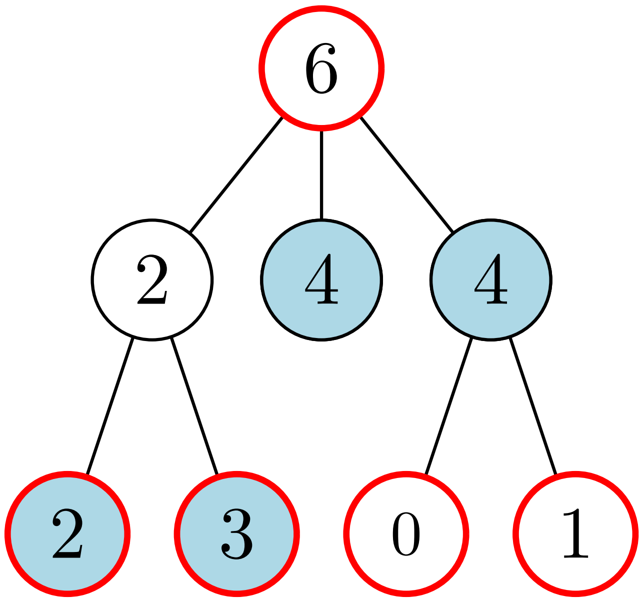
Time Complexity: $k(B+1) \cdot O(B) = O(kB^2)$

(Value of the) Optimal Solution: $D[k, B]$

Back to the Original Problem

Input: A tree T with integer weights on its vertices, a *budget* $B \in \mathbb{N}$.

Output: The maximum weight of an independent set S of T such that $|S| \leq B$.



$B = 5$

Max-Weight IS on Trees w. Budget

Base cases: v is a leaf of T.

$$OPT^+[v,b] = \begin{cases} w(v) & \text{if } b \geq 1 \\ -\infty & \text{if } b = 0 \end{cases}$$

$$OPT^-[v,b] = 0$$

Recursive formula for $OPT^+[v,b]$

- Let $C(v) = \{u_1, \ldots, u_k\}$.
- Compute $D[k, b-1]$ for $f_i(x) = OPT^-[u_i, x]$.

$$OPT^+[v,b] = w(v) + D[k, b-1]$$

Max-Weight IS on Trees w. Budget

Base cases: v is a leaf of T.

$$OPT^{+}[v,b] = \begin{cases} w(v) & \text{if } b \geq 1 \\ -\infty & \text{if } b = 0 \end{cases}$$

$$OPT^{-}[v,b] = 0$$

Nested DP!

Recursive formula for $OPT^{+}[v,b]$

- Let $C(v) = \{u_1, \ldots, u_k\}$.
- Compute $D[k, b-1]$ for $f_i(x) = OPT^{-}[u_i, x]$.

$$OPT^{+}[v,b] = w(v) + \underline{D[k, b-1]}$$

Max-Weight IS on Trees w. Budget

Base cases: v is a leaf of T.

$$OPT^+[v,b] = \begin{cases} w(v) & \text{if } b \geq 1 \\ -\infty & \text{if } b = 0 \end{cases}$$

$$OPT^-[v,b] = 0$$

Recursive formula for $OPT^-[v,b]$

Nested DP!

- Let $C(v) = \{u_1, \ldots, u_k\}$.
- Compute $D[k,b]$ for

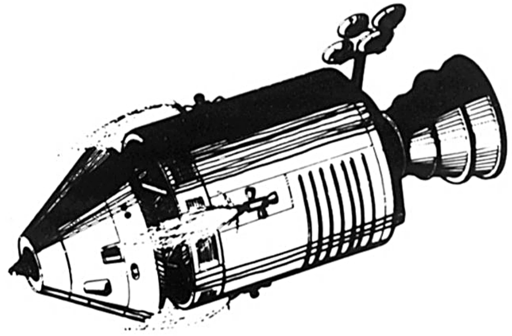
$$f_i(x) = \max\{OPT^-[u_i,x], OPT^+[u_i,x]\}.$$

$$OPT^-[v,b] = \underline{D[k,b]}$$

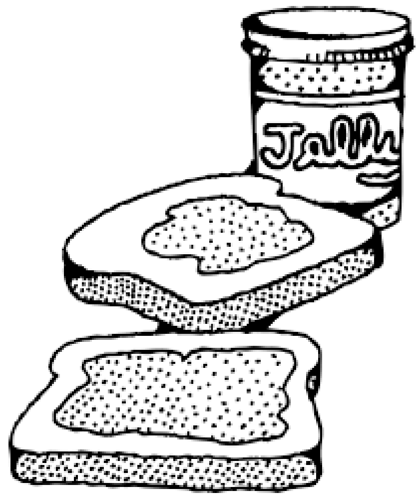
Edit Distance

Edit Distance

"Next NASA mission is going to land on ~~toast~~ mars"



- Autocorrect / Spell checking
- Unix `diff`
- Bioinformatics (DNA alignment)
- Plagiarism detection
- Speech recognition
- ...



Edit Distance

Input: Two strings $S = s_1 s_2 \ldots s_n$, and $T = t_1 t_2 \ldots t_m$.

Output: The *edit distance* between S and T.

Definition: The *edit distance* between S and T is the minimum number of *edits* required to turn S into T, where an edit is one of:

- **Insertion:** Inserting a new character at some position of S.

  `MARS` $\rightarrow$ `MARKS`

- **Deletion:** Removing one of the characters in S.

  `MARS` $\rightarrow$ `MAS`

- **Substitution:** Replacing one character of S with another.

  `MARS` $\rightarrow$ `CARS`

A Dynamic Programming Algorithm

Subproblem definition. For $0 \leq i \leq n$ and $0 \leq j \leq m$:

$OPT[i, j] =$ Edit distance between $S^{(i)} = s_1, \ldots, s_i$ and $T^{(j)} = t_1, \ldots, t_j$.

Note: $S^{(0)} = T^{(0)} = \varepsilon$, where ε is the empty string.

Base case:

$OPT[0, 0] =$ Minum number of operations needed to transform $S^{(0)} = \varepsilon$ into $T^{(0)} = \varepsilon$.

$OPT[0, 0] = 0$

A Dynamic Programming Algorithm

Recursive formula

If $i, j > 0$:

$$OPT[i,j] = \min \begin{cases} 1 + OPT[i-1,j] & \text{(deletion)} \\ 1 + OPT[i,j-1] & \text{(insertion)} \\ \mathbb{1}_{(s_i \neq t_j)} + OPT[i-1,j-1] & \text{(substitution)} \end{cases}$$

If $i = 0$ or $j = 0$:

$$OPT[i,j] = \begin{cases} 1 + OPT[0,j-1] & \text{if } i = 0 \\ 1 + OPT[i-1,0] & \text{if } j = 0 \end{cases} = \max\{i,j\}$$

Example

$\texttt{MARS} \rightarrow \texttt{TOAST}$

		0	1	2	3	4	5
	$i \backslash j$	ε	T	O	A	S	T
0	ε	0					
1	M						
2	A						
3	R						
4	S						

Example

$\texttt{MARS} \rightarrow \texttt{TOAST}$

		0	1	2	3	4	5
	$i \backslash j$	ε	T	O	A	S	T
0	ε	0	1	2	3	4	5
1	M	1					
2	A	2					
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$$\texttt{MARS} \to \texttt{TOAST}$$

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$i \backslash j$	ε	T	O	A	S	T
0 ε	0	1	2	3	4	5
1 M	1					
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3 R	3					
4 S	4					

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		0	1	2	3	4	5
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0	ε	0	1	2	3	4	5
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		0	1	2	3	4	5
	$i \backslash j$	ε	T	O	A	S	T
0	ε	0	1	2	3	4	5
1	M	1	1	2			
2	A	2					
3	R	3					
4	S	4					

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MARS $\rightarrow$ TOAST

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	$i \backslash j$	ε	T	O	A	S	T
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2	A	2	2				
3	R	3					
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3	R	3					
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0	ε	0	1	2	3	4	5
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2	A	2	2	2	2		
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		0	1	2	3	4	5
	$i \backslash j$	ε	T	O	A	S	T
0	ε	0	1	2	3	4	5
1	M	1	1	2	3	4	5
2	A	2	2	2	2	3	4
3	R	3	3	3	3	3	4
4	S	4	4	4	4	3	4

Edit distance: 4

Example

MARS $\rightarrow$ TOAST

		0	1	2	3	4	5
	$i \backslash j$	ε	T	O	A	S	T
0	ε	0	1	2	3	4	5
1	M	1	1	2	3	4	5
2	A	2	2	2	2	3	4
3	R	3	3	3	3	3	4
4	S	4	4	4	4	3	4

Edit distance: 4 **Time:** ?

Example

$$\texttt{MARS} \rightarrow \texttt{TOAST}$$

	0	1	2	3	4	5
$i \backslash j$	ε	T	O	A	S	T
0 ε	0	1	2	3	4	5
1 M	1	1	2	3	4	5
2 A	2	2	2	2	3	4
3 R	3	3	3	3	3	4
4 S	4	4	4	4	3	4

Edit distance: 4 **Time:** $O(nm)$

A Possible Implementation

```
int edit_distance(std::string &s, std::string &t)
{
    std::array<std::array<int, t.size()+1>, s.size()+1> OPT;

    for(int i=0; i<=s.size(); i++) OPT[i][0] = i;
    for(int j=1; j<=t.size(); j++) OPT[0][j] = j;

    for(int i=1; i<=s.size(); i++)
        for(int j=1; j<=t.size(); j++)
            OPT[i][j] = std::min({OPT[i-1][j]+1, OPT[i][j-1]+1,
                              OPT[i-1][j-1] + ((s[i]==t[j])?0:1)});

    return OPT[s.size()][t.size()];
}
```

Reconstructing Solutions

- **Option 1:** Retrace optimal choices backwards.

	ε	T	O	A	S	T
ε	0	1	2	3	4	5
M	1	1	2	3	4	5
A	2	2	2	2	3	4
R	3	3	3	3	3	4
S	4	4	4	4	3	4

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R	3	3	3	3	3	4
S	4	4	4	4	3	4

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S	4	4	4	4	3	4

Reconstructing Solutions

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	ε	T	O	A	S	T
ε	$\boxed{0}$	1	2	3	4	5
M	1	1	2	3	4	5
A	2	2	2	2	3	4
R	3	3	3	3	3	4
S	4	4	4	4	3	4

```
int i=s.size(), j=t.size();
while(i!=0 || j!=0)
{
    //Do something
    if(i>0 && OPT[i][j]==OPT[i-1][j]+1) i--; //Deletion
    else if(j>0 && OPT[i][j]==OPT[i][j-1]+1) j--; //Insertion
    else { i--; j--; } //Substitution
}
```

Reconstructing Solutions

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R	3	3	3	3	3	4
S	4	4	4	4	3	4

	MARS
• Change M to T	TARS
• Insert O	TOARS
• (Leave A unchanged)	TOARS
• Delete R	TOAS
• (Leave S unchanged)	TOAS
• Insert T	TOAST

Reconstructing Solutions

- **Option 2:** Esplicitly store (any of) the optimal choice(s) for each subproblem while filling the table.

		0	1	2	3	4	5
	$i\backslash j$	ε	T	O	A	S	T
0	ε	0	1	2	3	4	5
1	M	1	1	2	3	4	5
2	A	2	2	2	2	3	4
3	R	3	3	3	3	3	4
4	S	4	4	4	4	3	4

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		0	1	2	3	4	5
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0	ε	0	1	2	3	4	5
1	M	1	1	2	3	4	5
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3	R	3	3	3	3	3	4
4	S	4	4	4	4	3	4