Subset Sum

Subset Sum

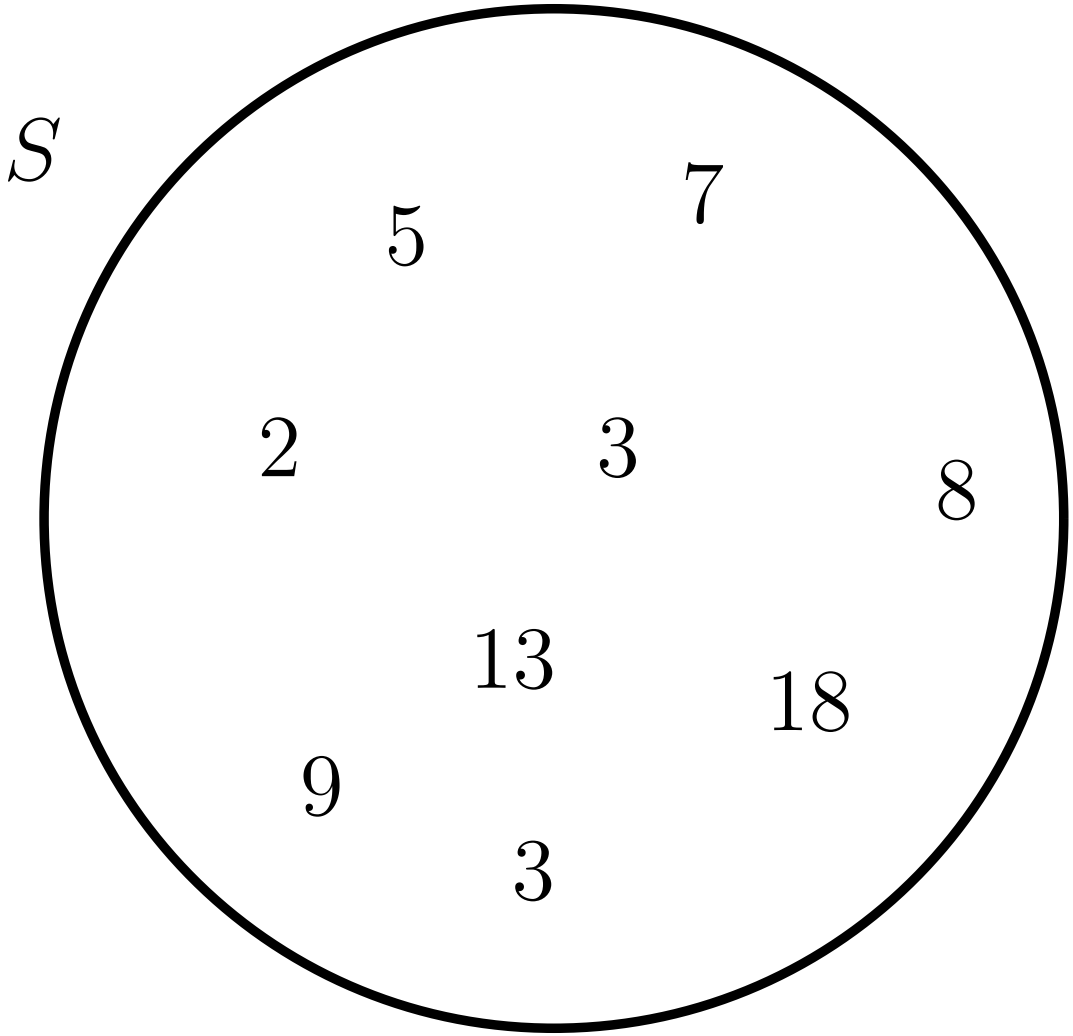
Input

- A (multi)-set $S \subseteq \mathbb{N}^+$ of n positive integers $s_1, \ldots, s_n$.
- A *target value* $T \in \mathbb{N}^+$.

Question

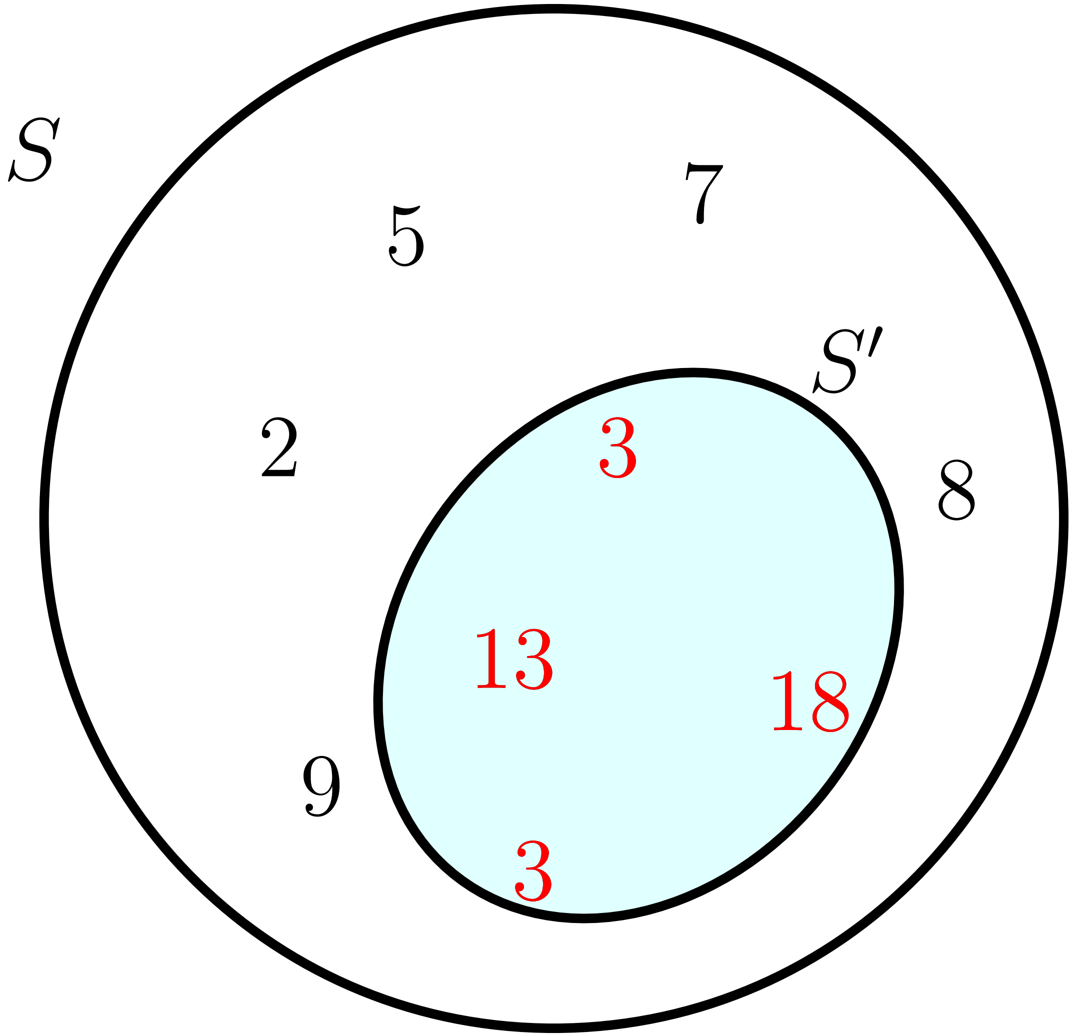
Is there a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = T$?

Example



$T = 37$

Example



$T = 37$

Answer: YES!

A Dynamic Programming Algorithm

Subproblem definition:

$OPT[i,t] =$ `true` iff $\exists S'' \subseteq \{s_1, \ldots, s_i\}$ such that $\sum_{x \in S''} x = t$.

Base cases:

$OPT[0,0] =$ `true`. $\quad$ $OPT[0,t] =$ `false`, for $t > 0$.

Recursive formula:

A Dynamic Programming Algorithm

Subproblem definition:

$$OPT[i,t] = \texttt{true} \text{ iff } \exists S'' \subseteq \{s_1,\ldots,s_i\} \text{ such that } \sum_{x\in S''} x = t.$$

Base cases:

$OPT[0,0] = \texttt{true}$.

$OPT[0,t] = \texttt{false}$, for $t > 0$.

Recursive formula:

- Either we ignore s_i $\quad OPT[i,t] = OPT[i-1,t]$
- Or we include s_i in S''... $\quad OPT[i,t] = OPT[i-1,t-s_i]$ (as long as $t \geq s_i$)

A Dynamic Programming Algorithm

Subproblem definition:

$OPT[i,t] =$ `true` iff $\exists S'' \subseteq \{s_1, \ldots, s_i\}$ such that $\displaystyle\sum_{x \in S''} x = t$.

Base cases:

$OPT[0,0] =$ `true`.

$OPT[0,t] =$ `false`, for $t > 0$.

Recursive formula:

$$OPT[i,t] = \begin{cases} OPT[i-1,t] & \text{if } t < s_i \\ OPT[i-1,t] \vee OPT[i-1,t-s_i] & \text{if } t \geq s_i \end{cases}$$

Time Complexity

- $\Theta(n \cdot T)$ subproblems
- Each problem can be solved in constant time
- **Overall time:** $\Theta(n \cdot T)$

Is this a polynomial-time algorithm?

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- Each problem can be solved in constant time
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Is this a polynomial-time algorithm?

NO!

The input size is $O(n \log T)$ (Under reasonalbe assumptions)

Choose, e.g., $T = 2^n$.

This is called a *pseudo*-polynomial-time algorithm.

Can we do better?

- Subset Sum is a well-known NP-complete problem.
- A polynomial-time algorithm for Subset Sum would imply P=NP. ☹
- Let's give up on polynomial-time algorithms and look at exponential algorithms.
- **Easy exercise:** come up with an algorithm with time complexity $O^*(2^n)$.

  (OK for $n \approx 25$)
- Can the exponent be improved?

Can we do better?

- Subset Sum is a well-known NP-complete problem.
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(OK for $n \approx 25$)

- C

$O^*(2^n)$ is a shorthand for $O(2^n \cdot \text{poly}(n))$.

Split & List

Split & List

Partition S into S_1 and S_2.

Observation: The following two statements are equivalent:

- $\exists S' \subseteq S$ such that $\sum_{x \in S'} x = T$; and
- $\exists S'_1 \subseteq S_1, S'_2 \subseteq S_2$ such that $\sum_{x \in S'_1} x + \sum_{x \in S'_2} x = T$.

Idea: Check whether the second statement hold.

How does this help?

The Algorithm

- Partition S into S_1 and S_2.

- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of S_1.
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of S_2.
- $T_2 \leftarrow$ Sort T_2.

- For each $t \in T_1$
  - Check whether $T - t \in T_2$

The Algorithm

- Partition S into S_1 and S_2. $O(n)$

- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of S_1. $O(2^{|S_1|})$
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of S_2. $O(2^{|S_2|})$
- $T_2 \leftarrow$ Sort T_2. $O(|S_2| \cdot 2^{|S_2|})$

- For each $t \in T_1$ $|T_1| = O(2^{|S_1|})$
  - Check whether $T - t \in T_2$ $O(\log |T_2|) = O(|S_2|)$

$$O\left(|S_2| \cdot 2^{|S_1|} + |S_2| \cdot 2^{|S_2|}\right) = O^*\left(2^{|S_1|} + 2^{|S_2|}\right)$$

The Algorithm

- Partition S into S_1 and S_2. $O(n)$

- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of S_1. $O(2^{|S_1|})$
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of S_2. $O(2^{|S_2|})$
- $T_2 \leftarrow$ Sort T_2. $O(|S_2| \cdot 2^{|S_2|})$

- For each $t \in T_1$ $|T_1| = O(2^{|S_1|})$
  - Check whether $T - t \in T_2$ $O(\log |T_2|) = O(|S_2|)$

Choosing $|S_1| = \lfloor \frac{n}{2} \rfloor$ and $|S_2| = \lceil \frac{n}{2} \rceil$:

$$O^*\left(2^{|S_1|} + 2^{|S_2|}\right) = O^*\left(2^{n/2} + 2^{n/2}\right) = O^*(2^{\frac{n}{2}})$$

Intermission: Generating All Subsets

- Let S be a set of n elements, where n is *small*.
- **Option 1:** use integers to encode the characteristic vectors of all subsets $S' \subseteq S$

$$S = \{\ 2\ ,\ 5\ ,\ 3\ ,\ 13\ ,\ 7\ ,\ 8\ ,\ 9\ ,\ 18\ ,\ 3\ \}$$

$$x = 0b\quad 0\quad 1\quad 1\quad 0\quad 0\quad 1\quad 0\quad 1\quad 0$$

$$S' = \{\ 5\ ,\ 3\ ,\ 8\ ,\ 18\ \}$$

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> sums(nsums, 0);
for(uint64_t x=0; x<nsums; x++)
    for(unsigned int i=0; i<S.size(); i++)
        sums[x] += ( (x>>i) & 1u )?S[i]:0;
```

Time: $O(n \cdot 2^n)$ $\approx$ 2s for $n = 25$

Intermission: Generating All Subsets

- Let S be a set of n elements, where n is *small*.
- **Option 1:** use integers to encode the characteristic vectors of all subsets $S' \subseteq S$

$$
\begin{array}{rccccccccccc}
S = & \{ & 2, & 5, & 3, & 13, & 7, & 8, & 9, & 18, & 3 & \} \\
x = 0b & & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \\
S' = & \{ & & 5, & 3, & & & 8, & & 18 & & \}
\end{array}
$$

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> sums(nsums, 0);
for(uint64_t x=0; x<nsums; x++)
    for(unsigned int i=0; i<S.size(); i++)
        sums[x] += ( (x>>i) & 1u ) * S[i];
```

Time: $O(n \cdot 2^n)$ $\approx$ 0.75s for $n = 25$

Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ 5\ ,\ 3\ ,\ 13\ ,\ 7\ ,\ 8\ ,\ 9\ ,\ 18\ ,\ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0$$

$\text{sum} = 34$

Intermission: Generating All Subsets

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$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad \color{red}{1}$$

sum $= \color{red}{37}$

Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0$$

sum $= 25$

Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad \color{red}{1} \quad \color{red}{0} \quad \color{red}{0}$$

sum $= \color{red}{25}$

Time complexity?

Intermission: Generating All Subsets

b_{n-1}			$\dots$			b_2	b_1	b_0
0	1	1	0	0	1	0	1	0

- b_0 flips at every iteration
- b_1 flips every 2 iterations
- b_2 flips every 4 iterations
- $\dots$
- b_i flips every 2^i iterations

Total # of operations (including updates to sum) $\propto$ # bit flips

$$\sum_{i=0}^{n-1} \frac{2^n}{2^i} = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2 = \Theta(2^n).$$

Generating All Subsets

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> &sums = *new std::vector<int>(nsums);
std::vector<bool> bits(S.size());

for(uint64_t i=0; i<nsums; i++)
{
    int j=0;
    while(bits[j])
    {
        bits[j] = 0;
        sums[i] -= S[j];
        j++;
    }

    bits[j]=1;
    sums[i] += bits[j];
}
```

$\approx$ 0.2s for $n = 25$

Back to Subset Sum: Which Algorithm?

Dynamic Programming

$O(n \cdot T)$

$T \leq 2^{\frac{n}{2}}$

OK for "small" T

Split and List

$O(n \cdot 2^{\frac{n}{2}})$

$T \geq 2^{\frac{n}{2}}$

OK for $n \leq 50$, regardless of T

Split & List

- Split input into two sets S_1, S_2
- Explicitly compute all possible (partial) solutions w.r.t. S_1 and S_2

Brute force!

- Combine the solutions of S_1 with those of S_2

Quicker than brute force

Can we split into 3 sets?

1-in-3 positive SAT

1-in-3 positive SAT

Input: A formula ϕ consisting of

- A set of n boolean variables $x_1, \ldots, x_n$
- A collection of m clauses $C_1, \ldots, C_m$, i.e., triples of variables $C_j = (c_j^{(1)}, c_j^{(2)}, c_j^{(3)}) \in \{x_1, \ldots, x_n\}^3$

A truth assignment is a function $\tau : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$

- A clause $C_j = (c_j^{(1)}, c_j^{(2)}, c_j^{(3)})$ is *satisfied* by τ iff *exactly* one of $\tau(c_j^{(1)})$, $\tau(c_j^{(2)})$, and $\tau(c_j^{(3)})$ is $\top$.
- ϕ is satisfied iff all m clauses $C_1, \ldots, C_m$ are satisfied.

Question: Is there a truth assigment that satisfies ϕ?

Example

Formula

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

Example

Formula

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

Satisfying assignment:

$$x_1 = \bot \quad x_2 = \bot \quad x_3 = \top \quad x_4 = \top \quad x_5 = \bot$$

Example

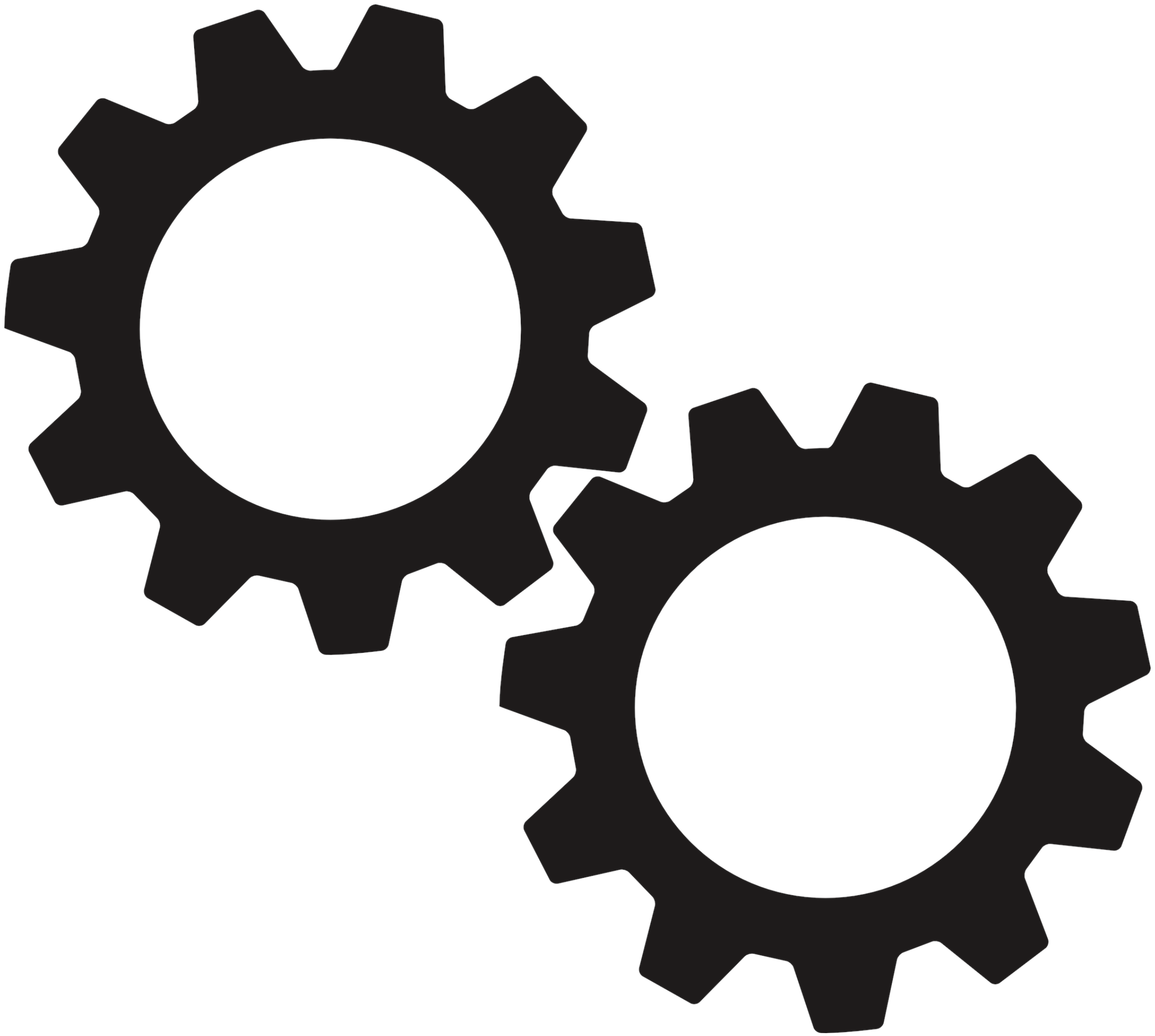
Formula

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

Satisfying assignment:

$$x_1 = \bot \quad x_2 = \bot \quad x_3 = \top \quad x_4 = \top \quad x_5 = \bot$$

Trivial solution $O^*(2^n)$



An Algorithm Based on Split & List

- Split the n boolean variables into two sets S_1, S_2 of size $\approx \frac{n}{2}$
- For each possible truth assigment τ_1 of the variables in S_1
  - If τ_1 sets ≥ 2 variables in the same clause to $\top$, discard it.
  - Otherwise, store in X_1 the characteristic vector $\chi(\tau_1) = (\chi_1, \chi_2, \ldots, \chi_m) \in \{\top, \bot\}^m$ of the satisfied clauses, where $\chi_j = \top$ iff τ_1 satisfies C_j.
- Compute X_2 in a similar way.
- Sort X_2 (e.g., w.r.t. the lexicographic order)
- For each vector $\chi = (\chi_1, \ldots, \chi_m) \in X_1$
  - $\overline{\chi} \leftarrow (\overline{\chi_1}, \ldots, \overline{\chi_m})$
  - Binary search for $\overline{X}$ in X_2