van Emde Boas Trees

The Dynamic Predecessor Problem

Goal:

Design a data stucture that maintains a *dynamic* set S of *integers* from a universe $\{0, \ldots, u-1\}$, supporting the following operations:

- Insert(x): Add x into S.
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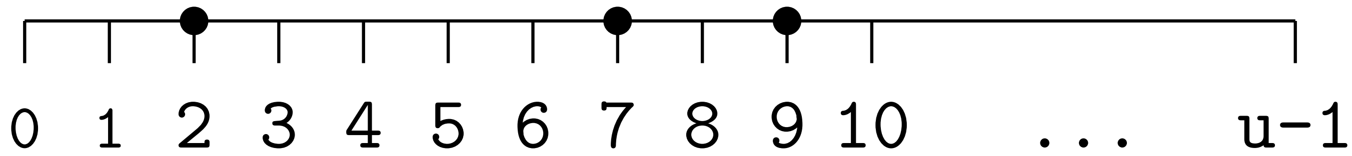
A Geometric Intepretation



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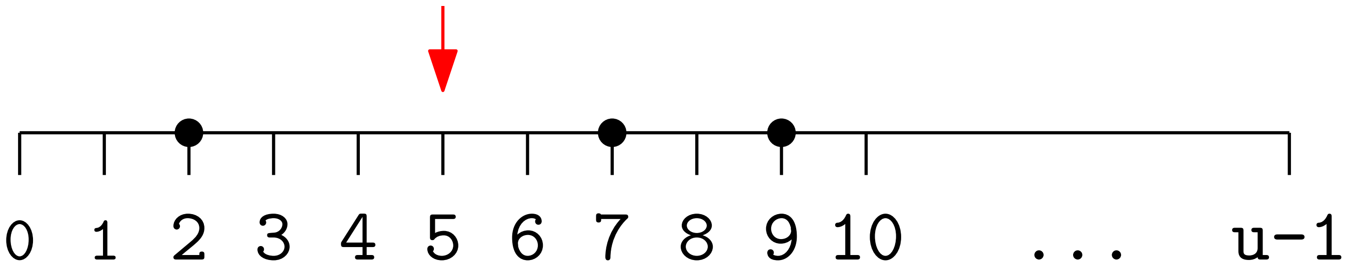

- Insert(5)

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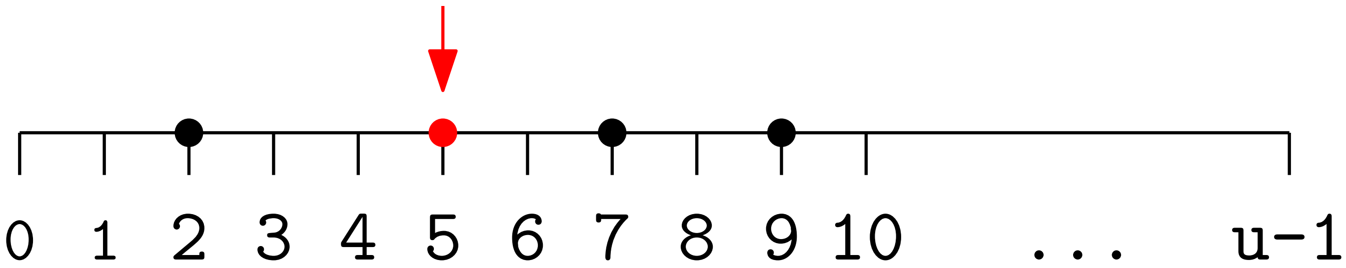

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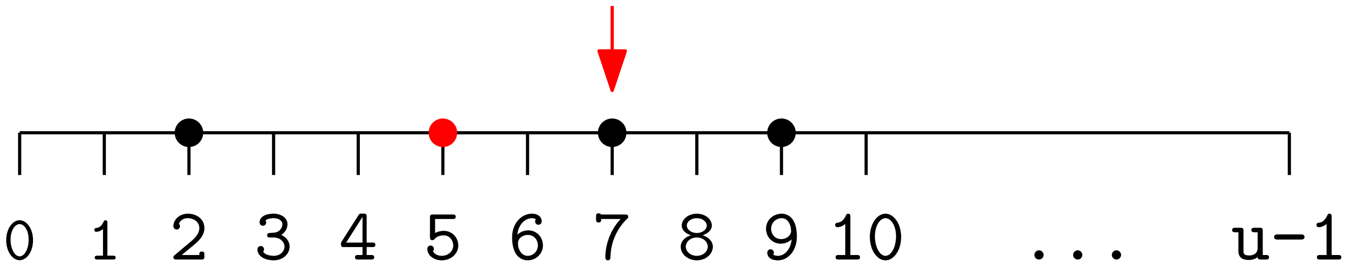

- Insert(5)
- Delete(7)

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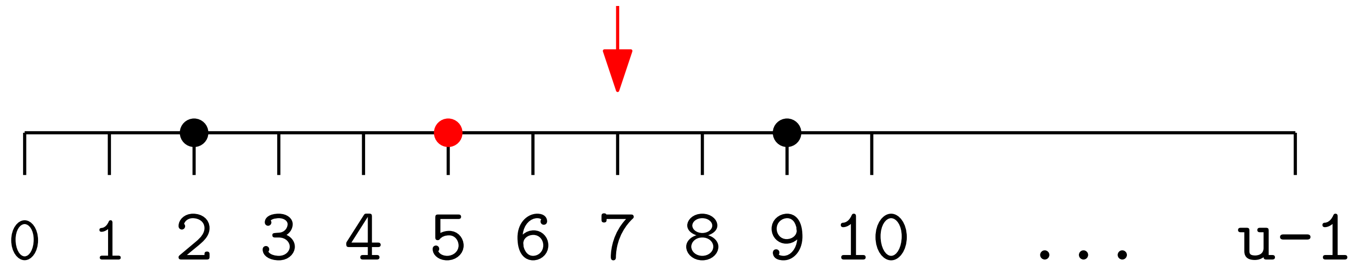

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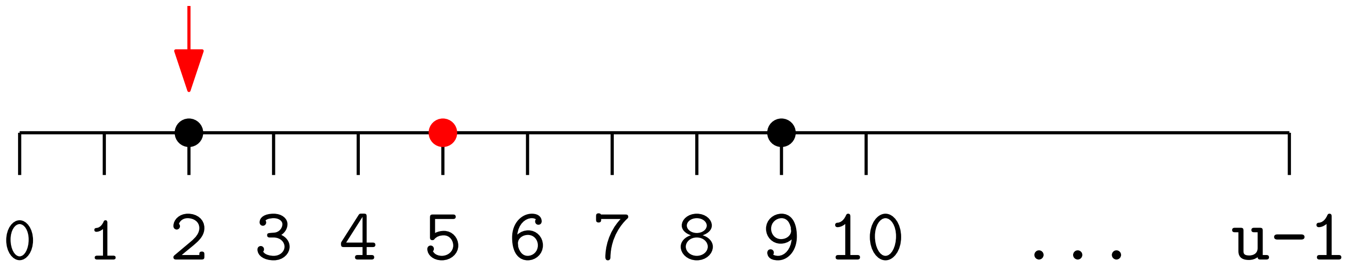

- Insert(5)
- Delete(7)
- Find$(2) = \top$

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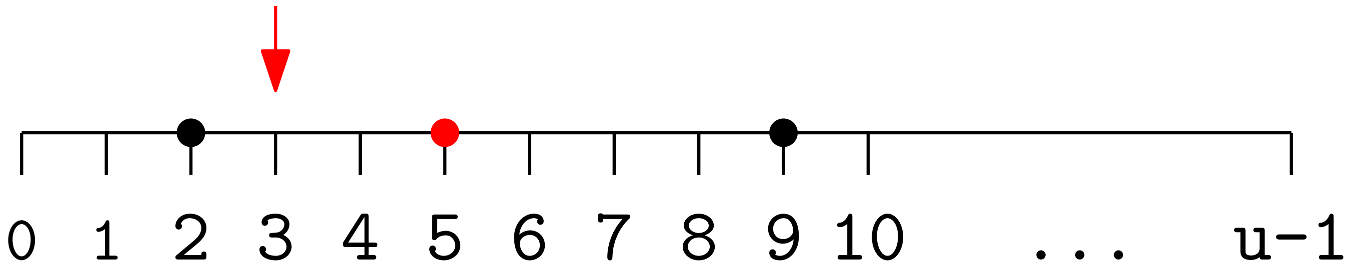

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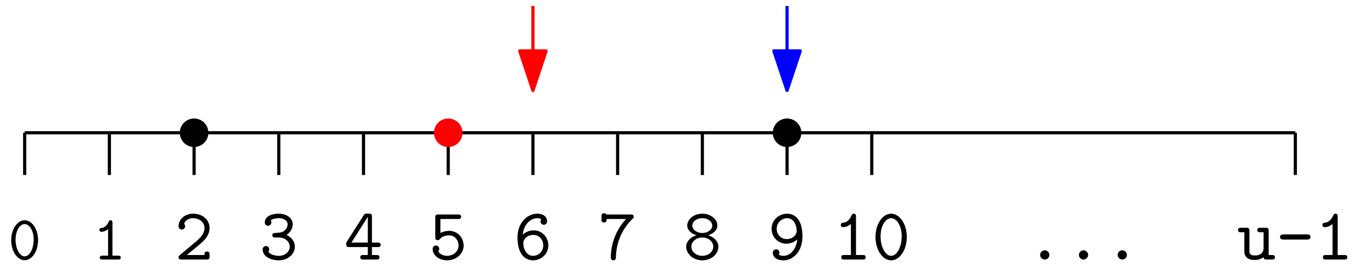

- Insert(5)
- Delete(7)
- Find$(2) = \top$
- Find$(3) = \bot$
- Successor(6)=9

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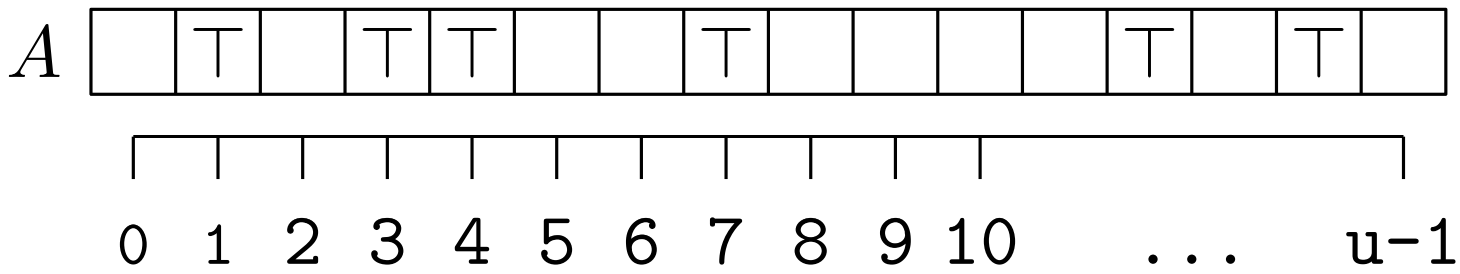
Assume: $u = 2^w$, for some even positive integer w.

A Trivial Solution

Store a Boolean vector $A[0 : u-1]$ where $A[x] = \top$ iff $x \in S$.



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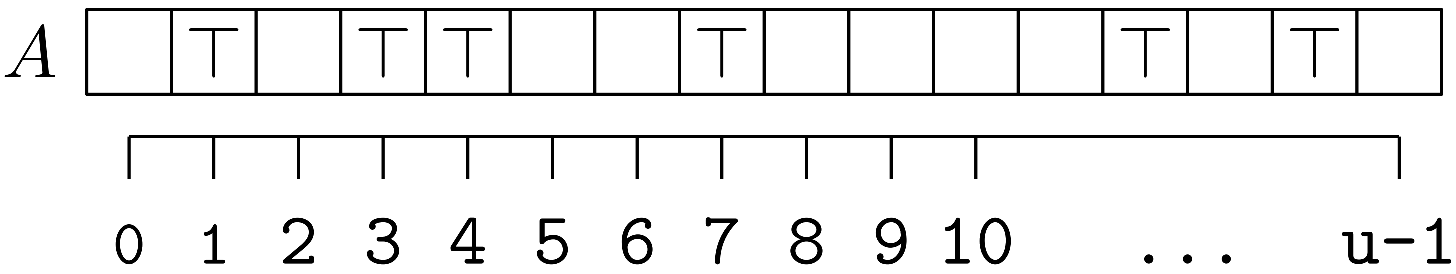

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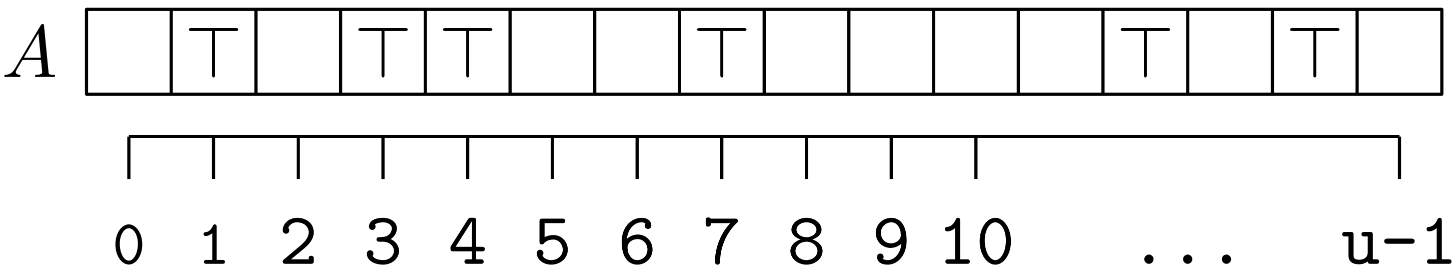

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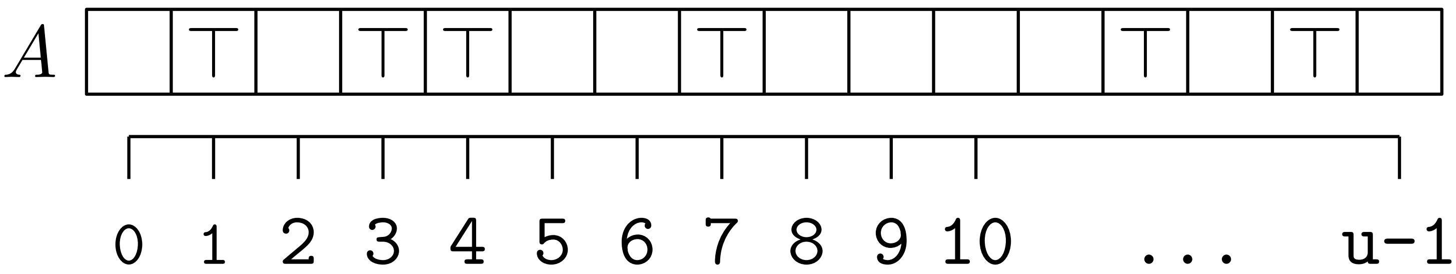
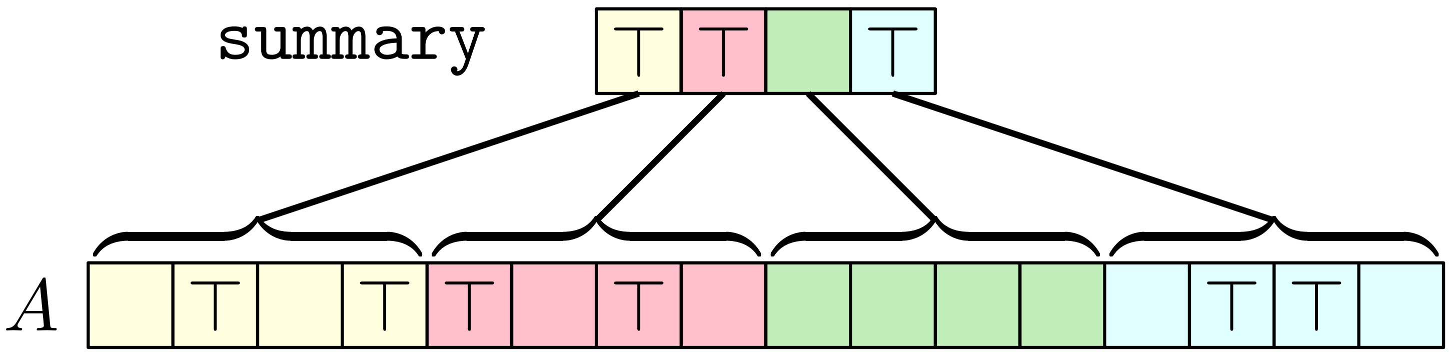


- Insert(x): $A[x] \leftarrow \top$ Time: $O(1)$
- Find(x): Return $A[x]$ Time: $O(1)$
- Successor(x):

Return the smallest $y > x$ with $A[y] = \top$, or $+\infty$ if no such y exists.

Time: $O(u)$

Speeding up Successor Queries



Split A into $\approx k$ clusters of $\approx \frac{u}{k}$ elements each

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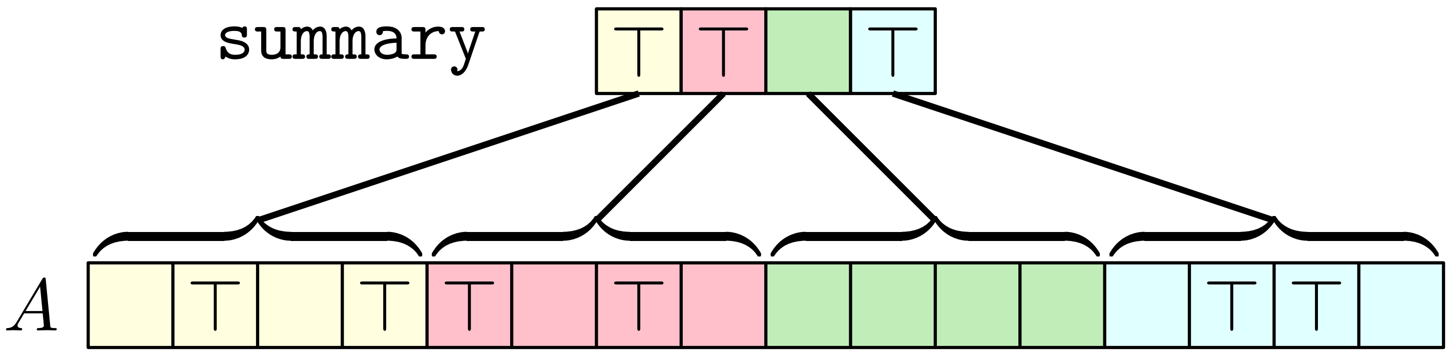

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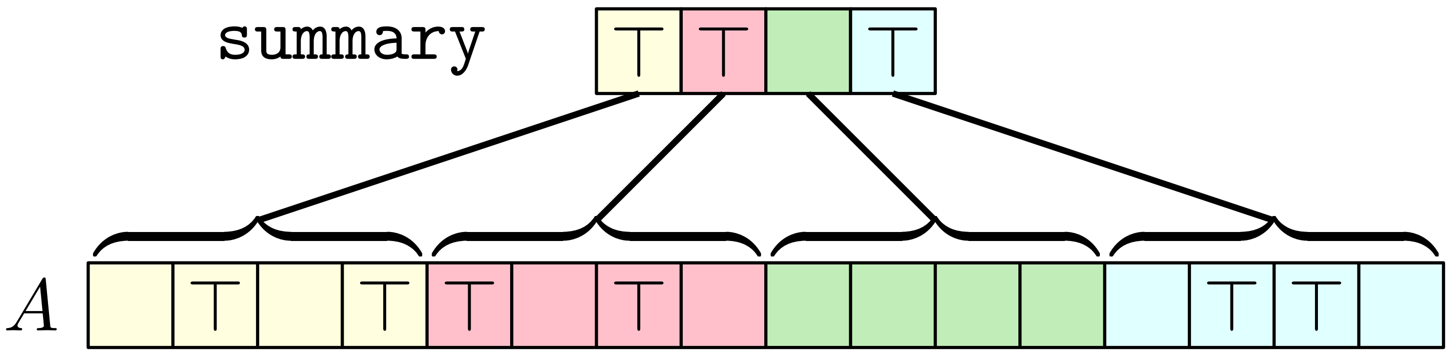

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- Find(x): No changes — Time: $O(1)$
- Insert(x): $A[x] \leftarrow \top$; summary$[i] \leftarrow \top$ — Time: $O(1)$

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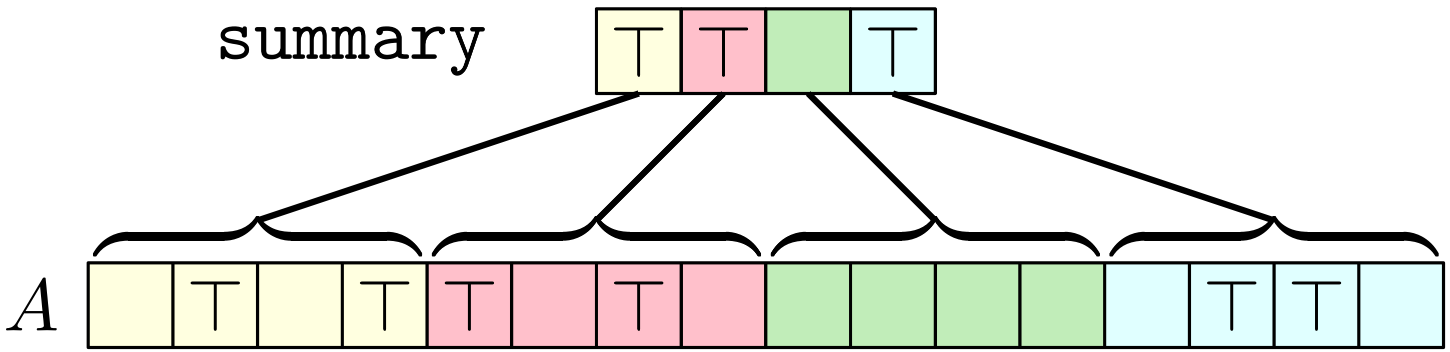

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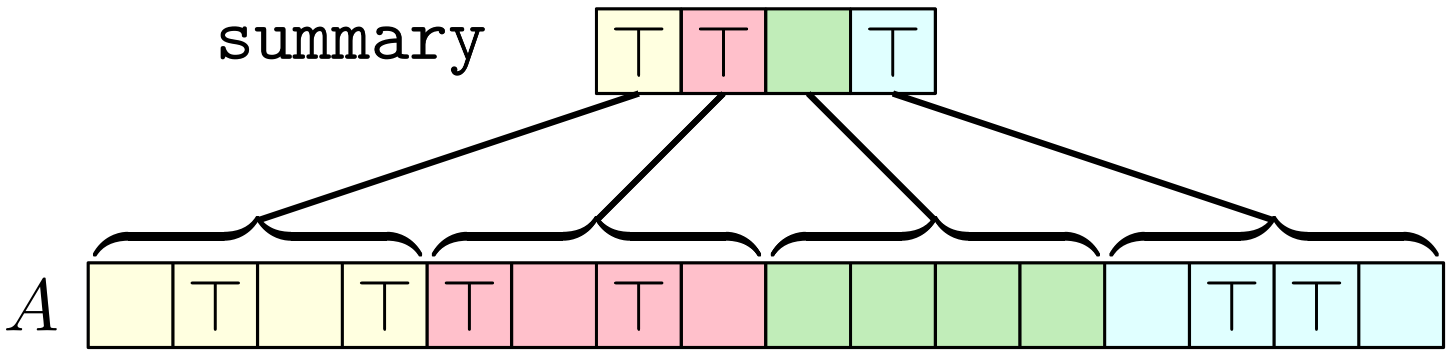

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Time: $O(u/k + k)$

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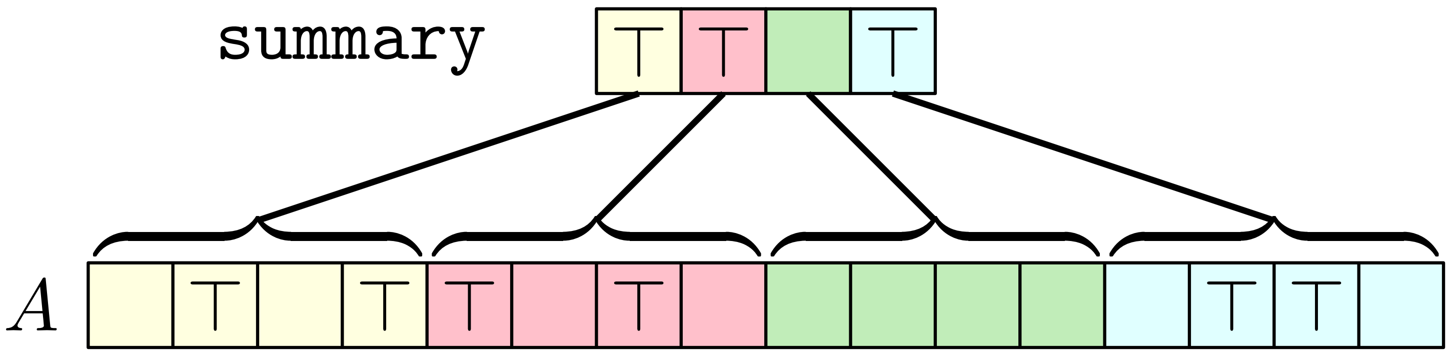

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Time: $O(u/k + k)$ How to optimize k?

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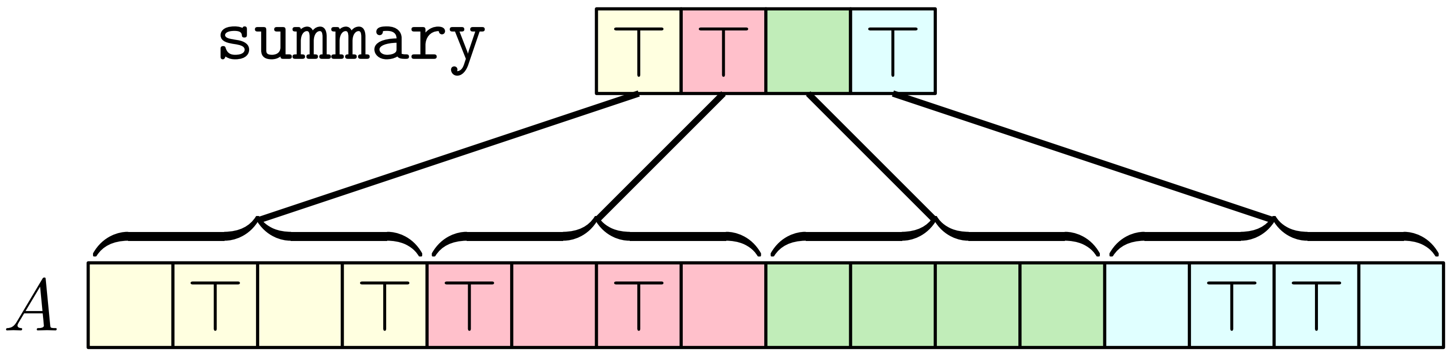

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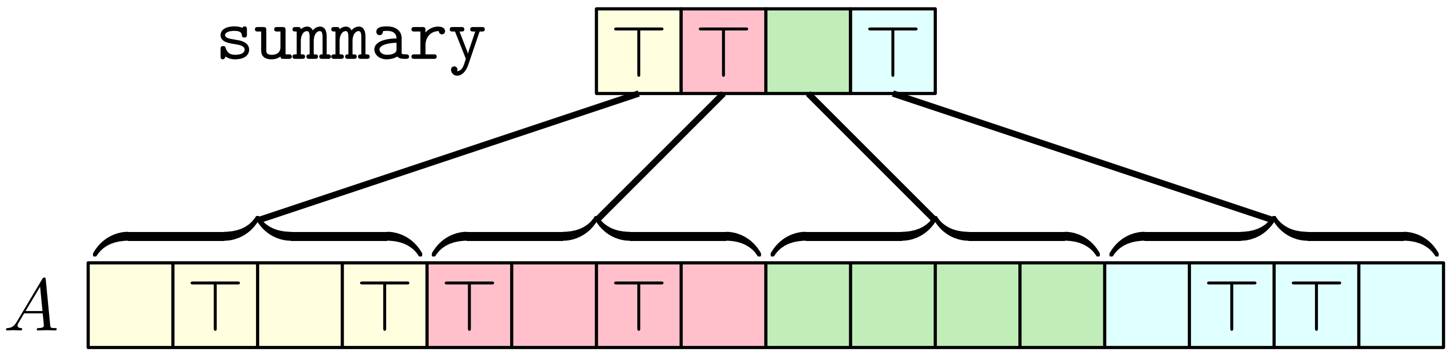
Time: $O(u/k + k)$ How to optimize k? Pick $k = \sqrt{u}$.

Speeding up Successor Queries



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Time: $O(u/k + k) = O(\sqrt{u})$ Pick $k = \sqrt{u}$.

A Binary View

Split A into $\sqrt{u}$ clusters of $\sqrt{u}$ elements each.

$$x = (c_x \cdot \sqrt{u}) + (x \bmod \sqrt{u}) = (c_x \cdot 2^{w/2}) + (x \bmod 2^{w/2}).$$

$x =$ `0 1 0 0 1 1 0 1`

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$$x = \underbrace{\texttt{0 1 0 0}}_{w/2 \text{ bits}} \ \underbrace{\texttt{1 1 0 1}}_{w/2 \text{ bits}}$$

$\text{high}(x) = \lfloor x / \sqrt{u} \rfloor$ Index of the cluster of x

$\text{low}(x) = x \bmod \sqrt{u}$ Index x within its cluster

$\text{idx}(i, j) = i \cdot \sqrt{u} + j$ Gets x from $i = \text{high}(x)$ and $j = \text{low}(x)$

Recursion!

How do we use `summary` ?

$\text{high}(x)$

- We need to be able to add elements from $\{0, \ldots, \sqrt{u} - 1\}$.
- We need to find the next non-empty cluster.

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This is a successor query!

Recursion!

How do we use `summary` ?

- We n… high(x) … $u - 1\}$.
- W… empty cluster.

Replace summary with a (smaller) data structure for the universe $\{0, \ldots, \sqrt{u} - 1\}$.

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What can we do in each cluster?

- Add an element $\{0,\ldots,\sqrt{u}-1\}$
- Find an element low(x)
- Find the next element in the cluster

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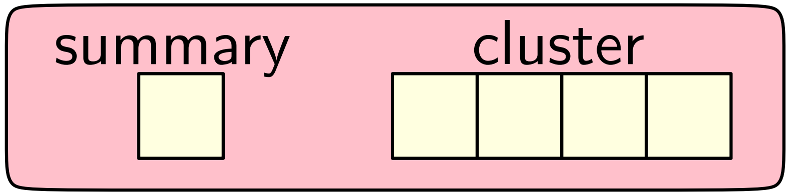
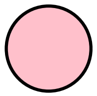
An Example for $u = 16$



 $u = 16$

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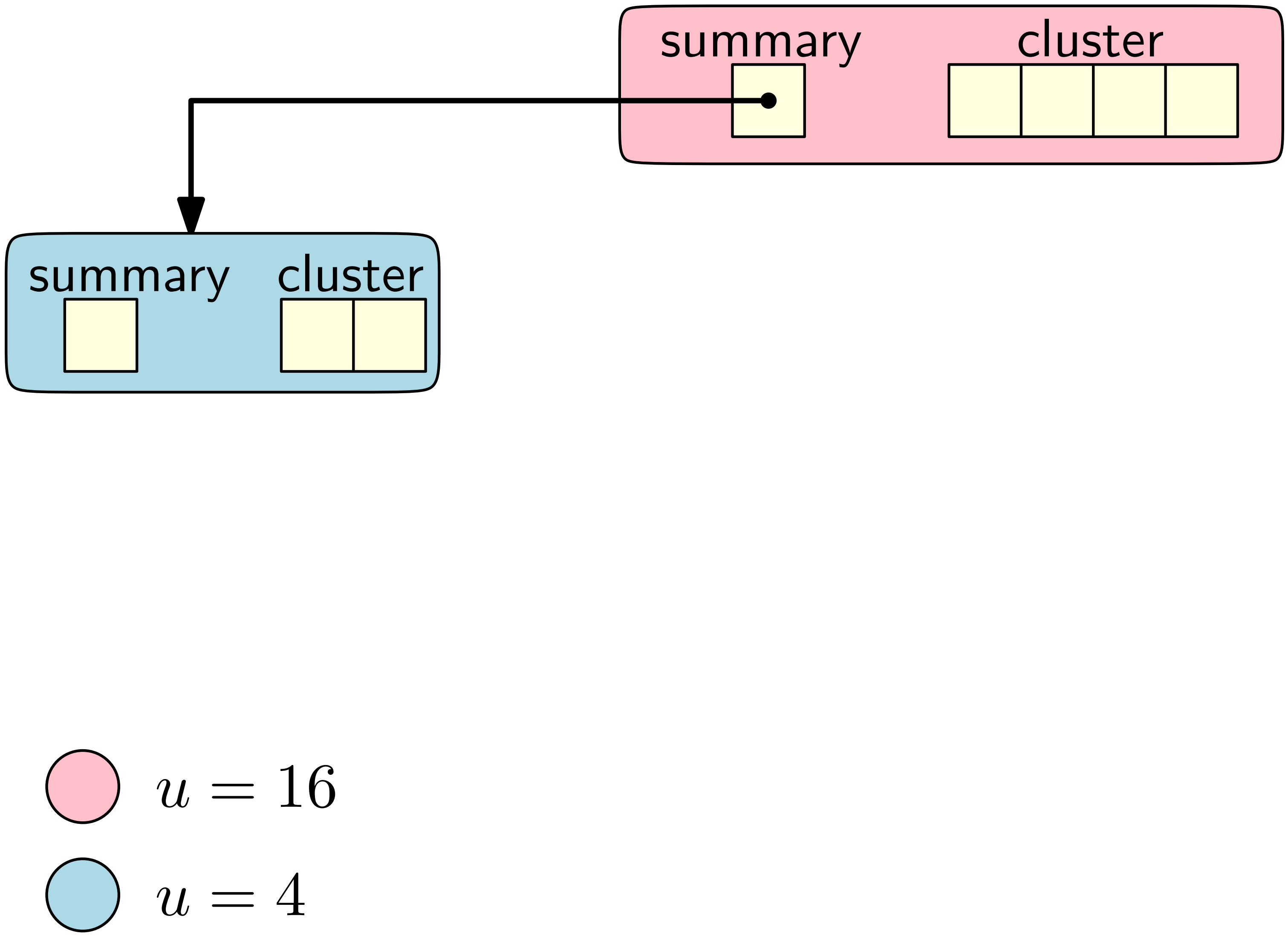

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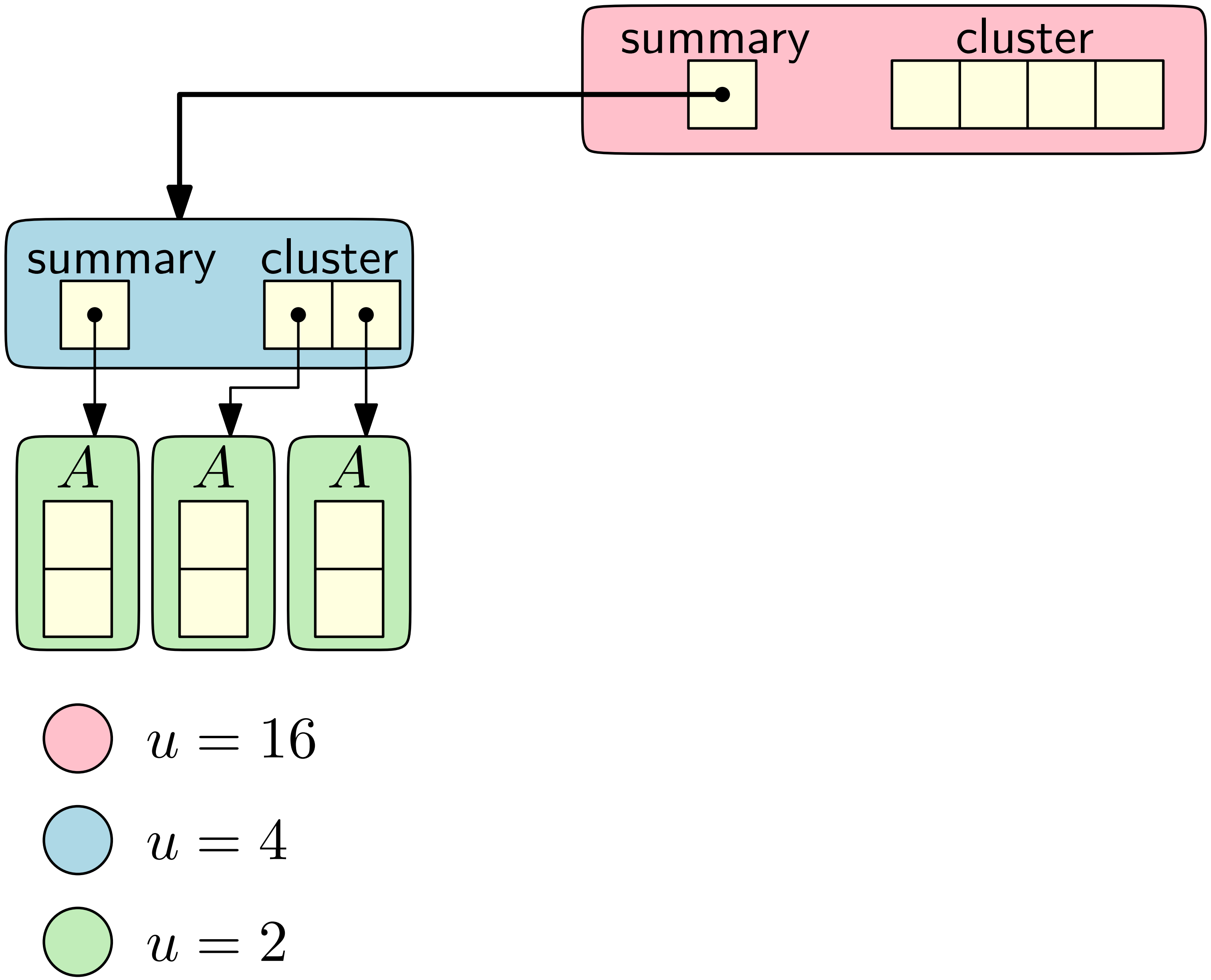

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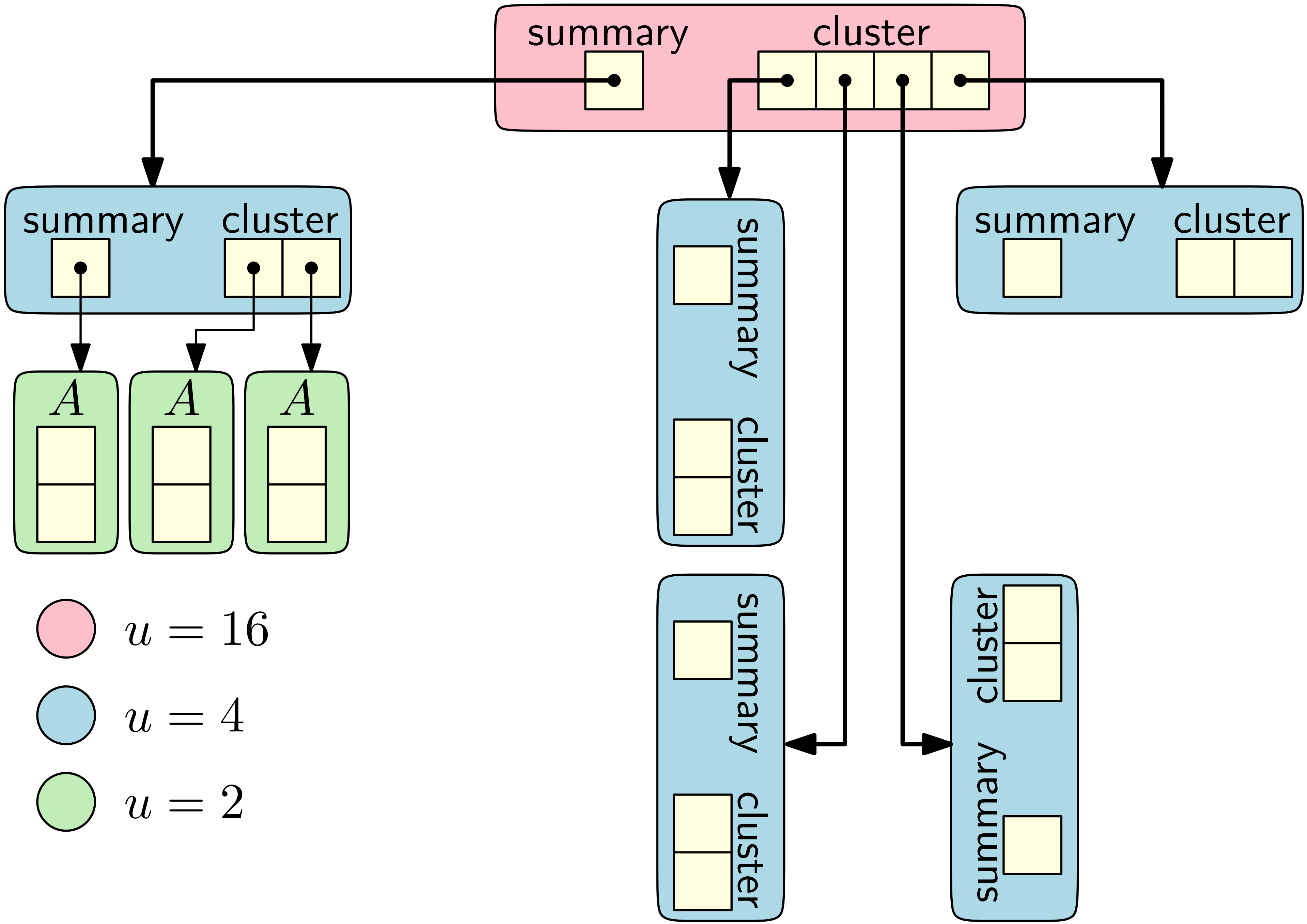

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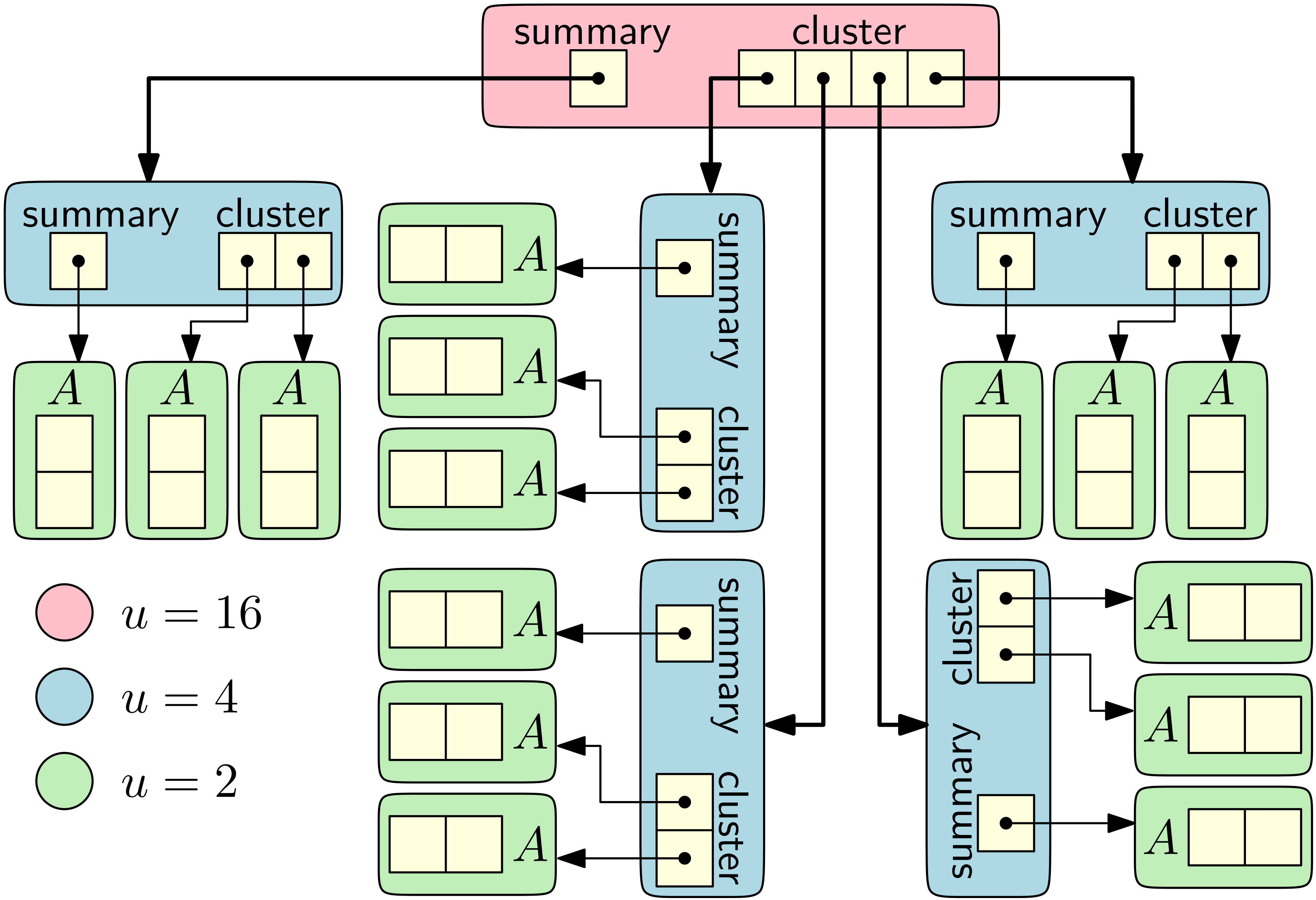

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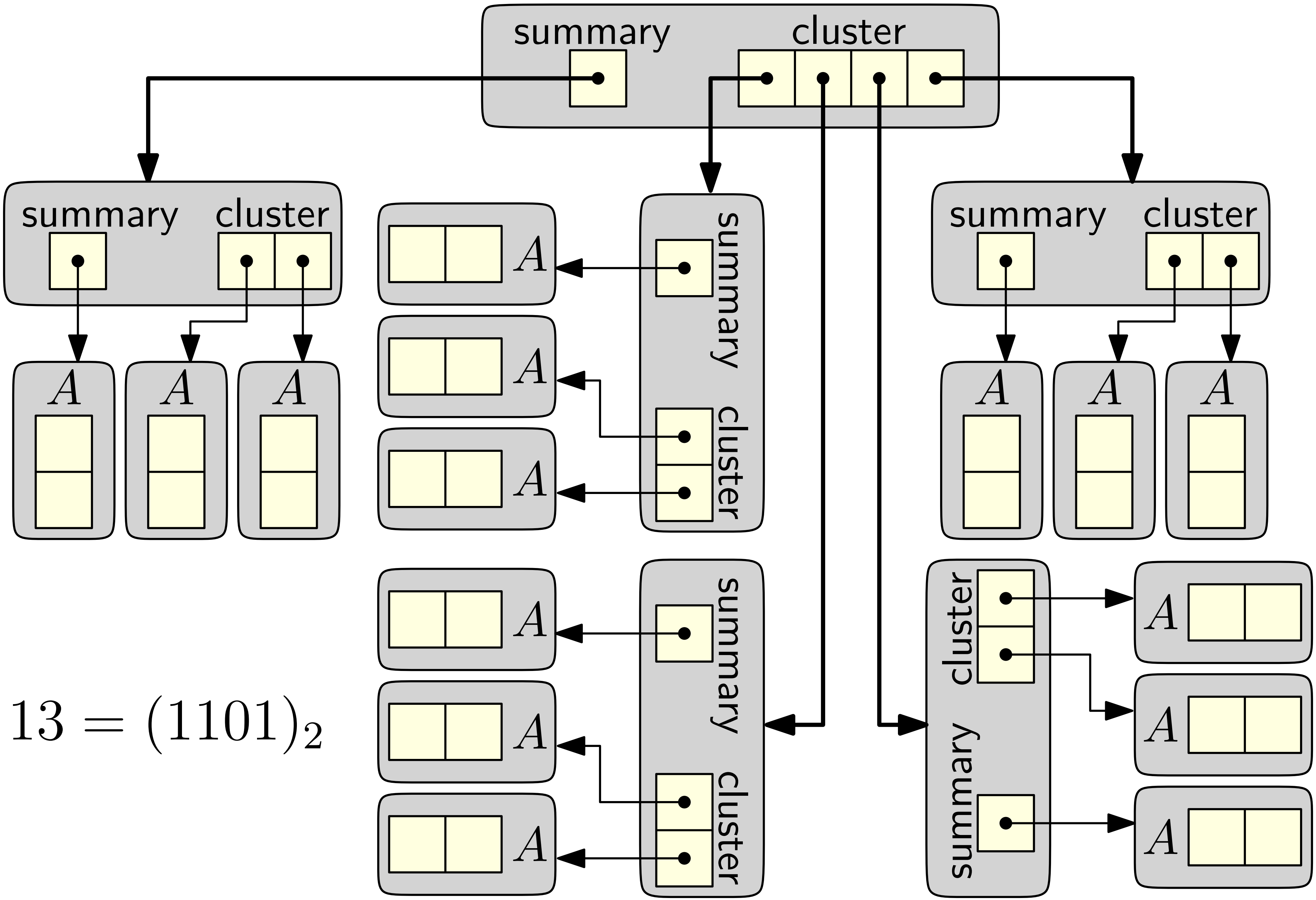

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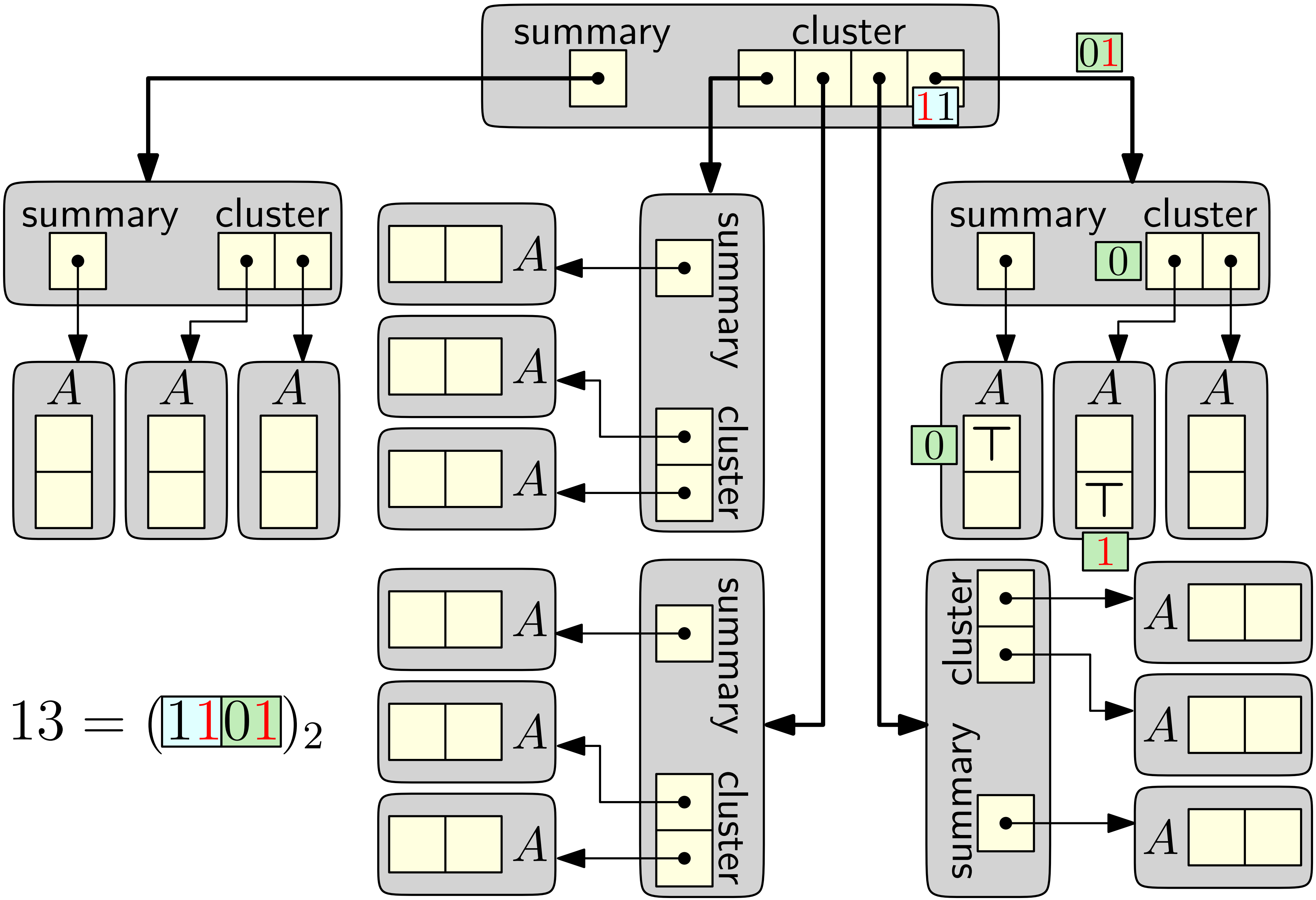

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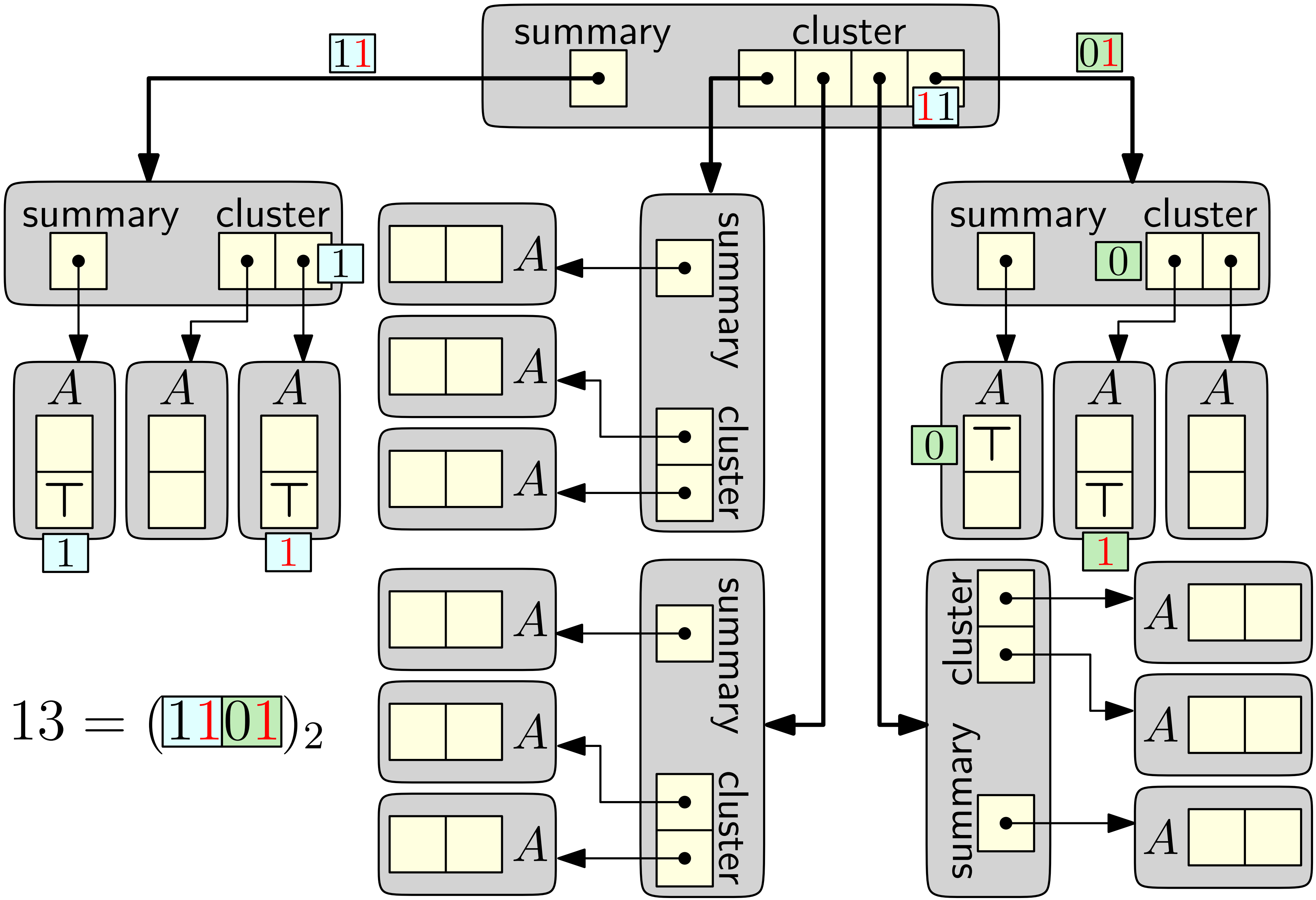

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Space Usage

Let $S(u)$ be the space usage of a structure for the universe $\{0, \ldots, u-1\}$

- `summary` occupies $S(\sqrt{u})$ space.
- `cluster` occupies $\sqrt{u} \cdot S(\sqrt{u})$ space.
- $O(1)$ additional information (e.g., the value of u)
- (Charge pointers to the inner structures)

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$$S(u) = O(1) \text{ when } u = O(1)$$

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Solution: $S(u) = O(u)$

Space Usage

There is a c such that:

- $S(u) \leq c \quad$ for $u < 9$
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We want to show that $S(u) = O(u)$.

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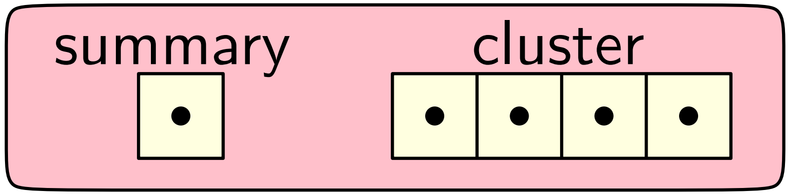
- For $u \ge 9$: $S(u) \le (2 + \sqrt{u}) \cdot c(\sqrt{u} - 2) + c$
  $$\le c(u - 4) + c < c(u - 2).$$

Implementing Find



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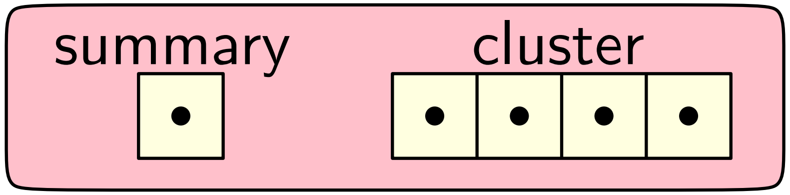

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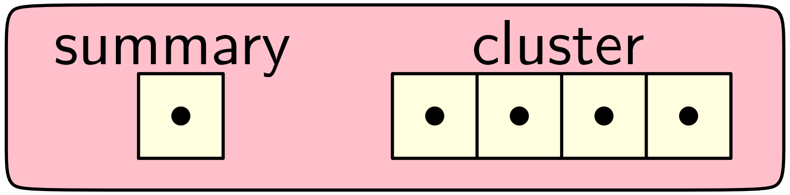
Time: $T(u) = T(\sqrt{u}) + O(1)$

Implementing Find



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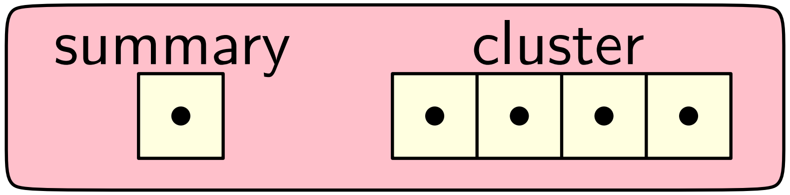

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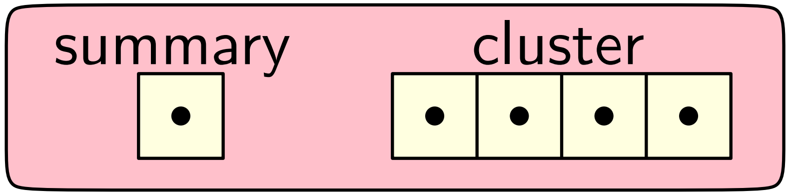

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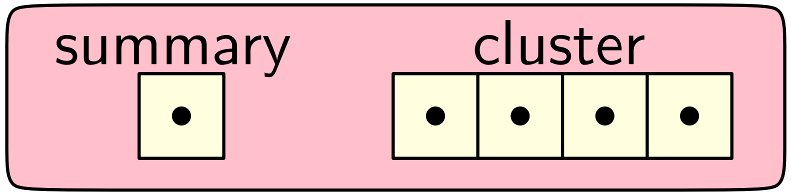

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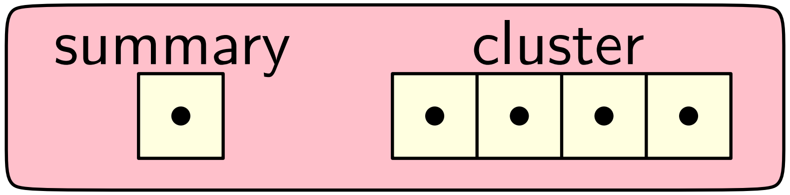
$$T(u) = W(\log u) = \Theta(\log \log u)$$ ☺

Implementing Insert



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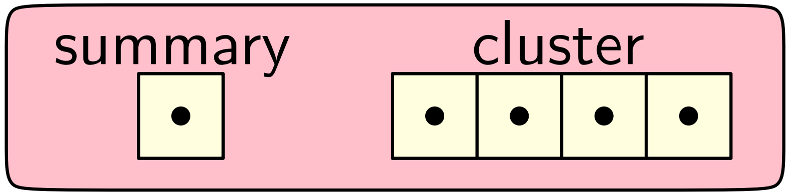

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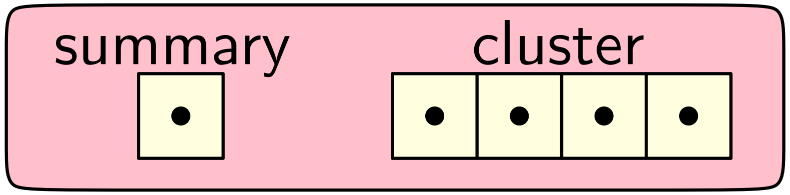
Implementing Insert



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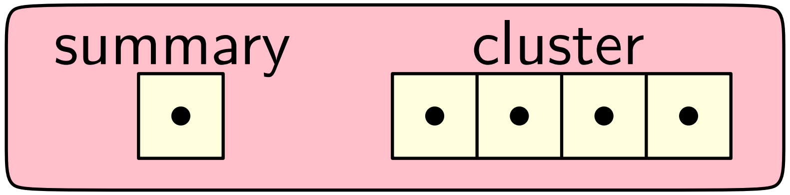

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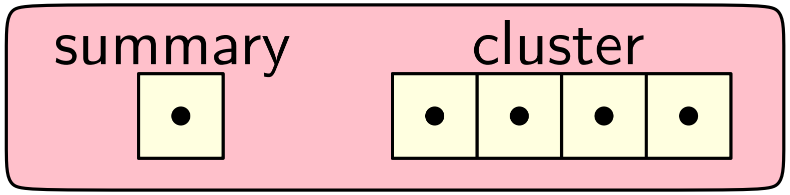

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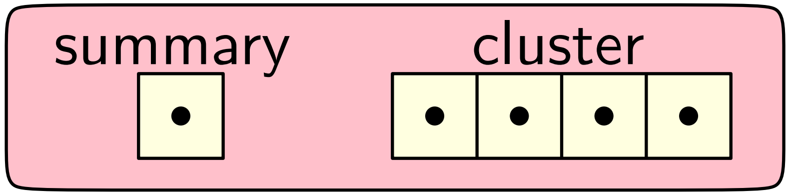

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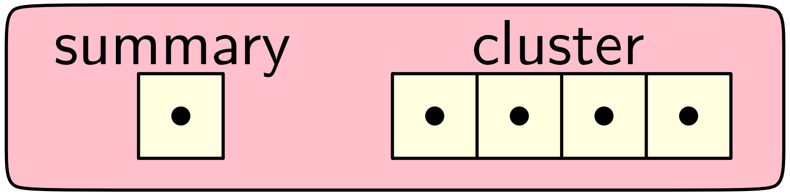

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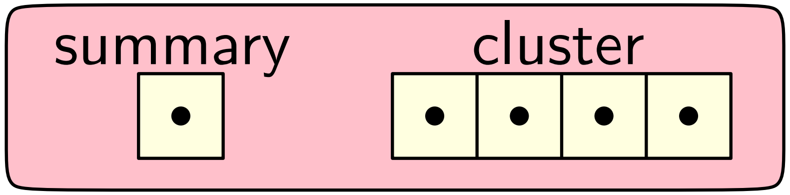
Implementing Successor



Successor(x):

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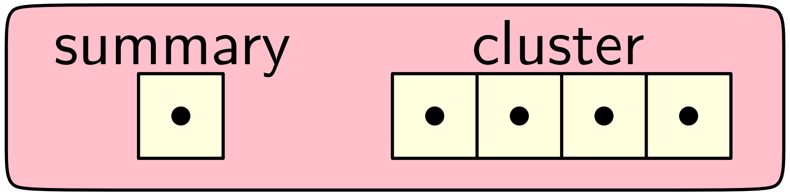

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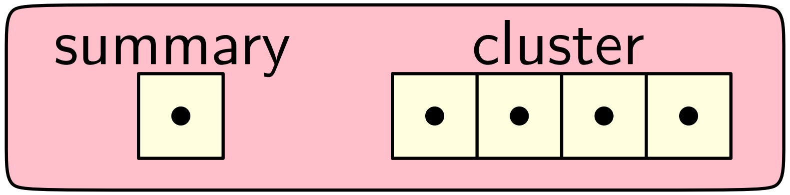
Implementing Successor



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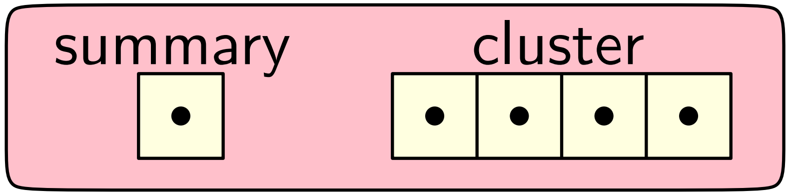
find(0) or successor(0)

Implementing Successor



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Implementing Successor

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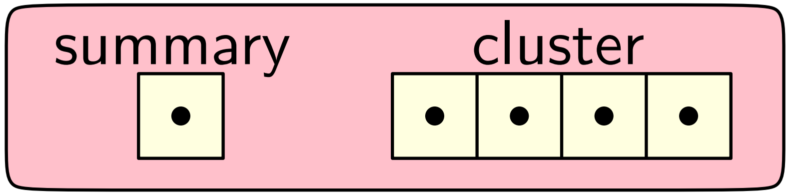

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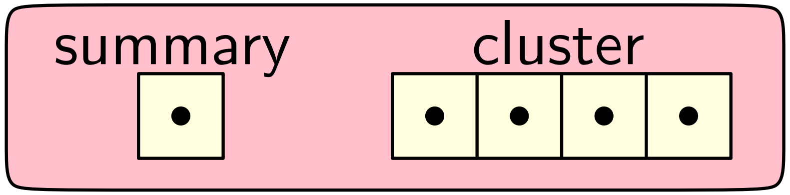
Implementing Successor



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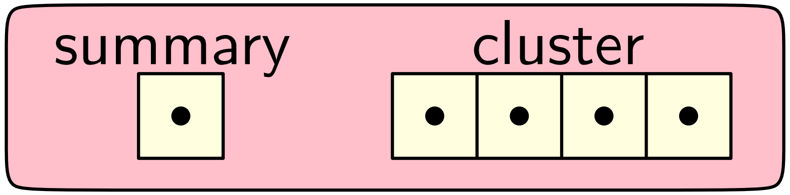
find(0) or successor(0)

Time: $T(u) = 3T(\sqrt{u}) + O(\log \log u)$

$W(z) = \Theta\left(z^{\log_2 3}\right)$

Implementing Successor



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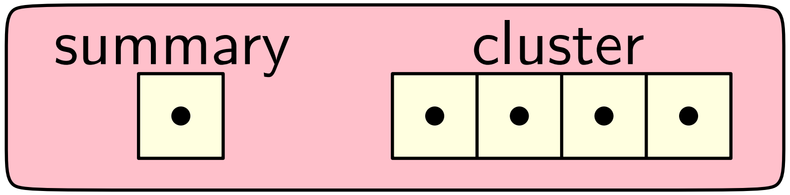
Implementing Successor



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Speeding Up the Operations (again)

Idea 1: Maintain the minimum separately!

- Store a new field `min`
- `min` is no longer stored in `cluster`
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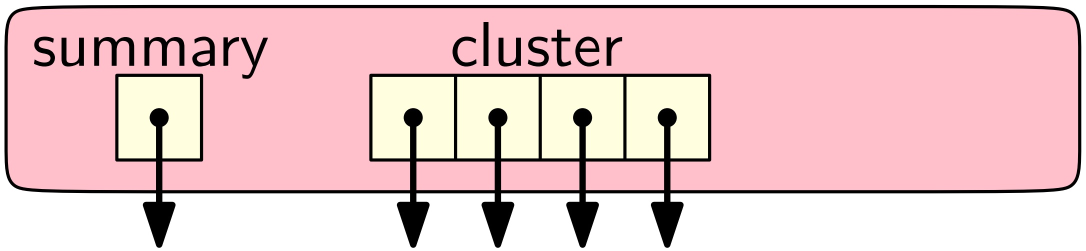
- (If S is empty, `min` $= +\infty$)



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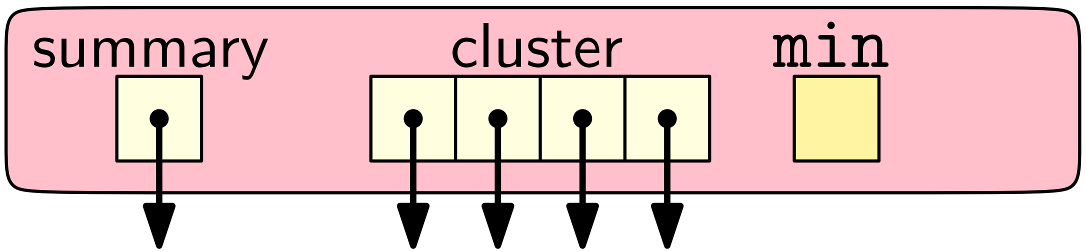

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Idea 2: Keep track of the maximum.

- Store a new field `max`
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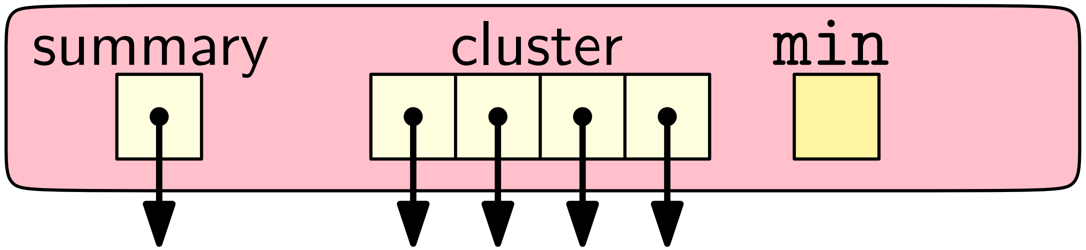

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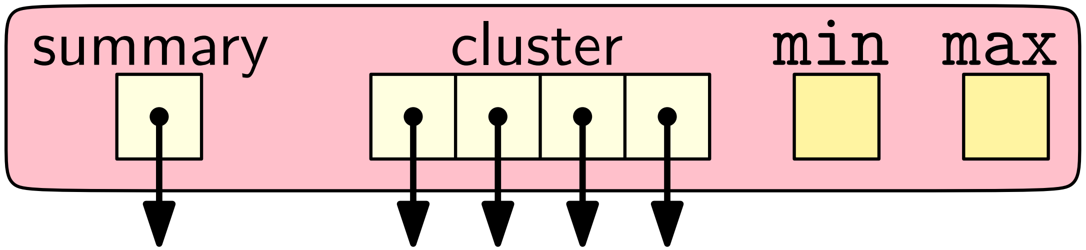
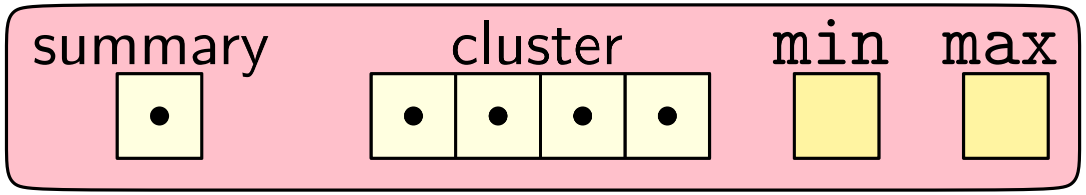


Implementing Find (again)



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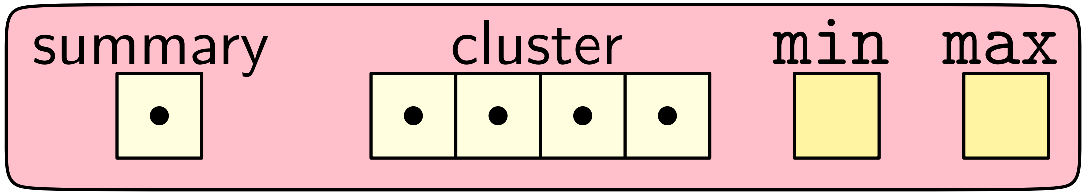
Implementing Find (again)



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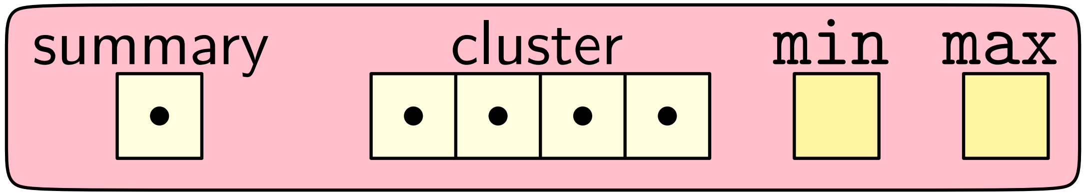
Implementing Insert (again)



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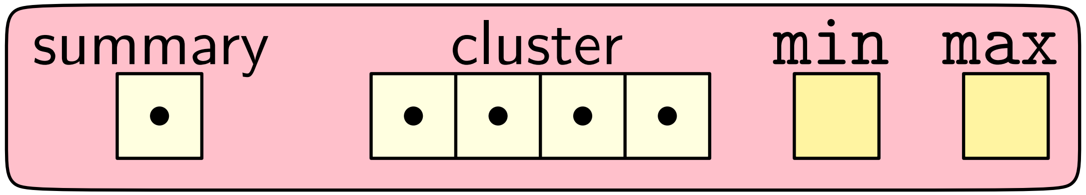

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- If $x > \texttt{max}$: $\texttt{max} \leftarrow x$
- If $x < \texttt{min}$: swap x and $\texttt{min}$
- $h, \ell \leftarrow$ high(x), low(x)
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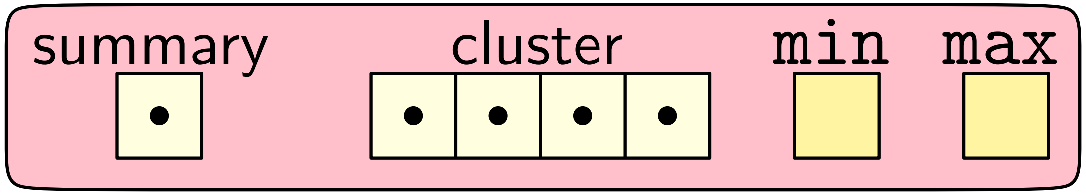

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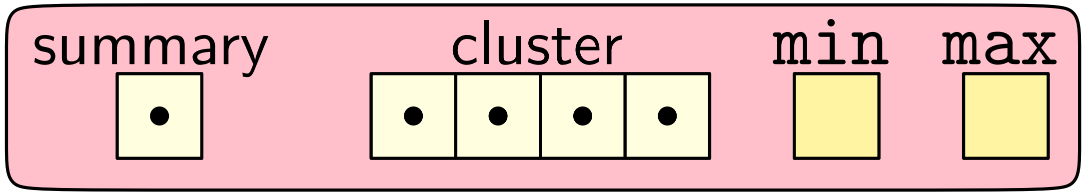

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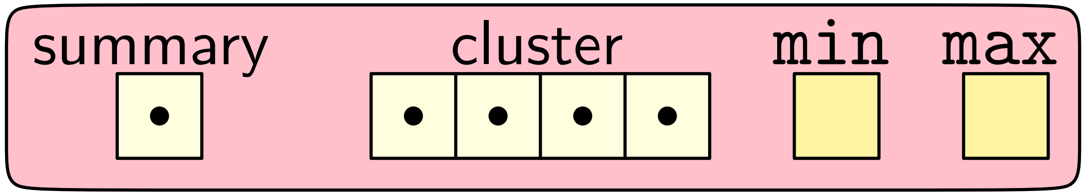

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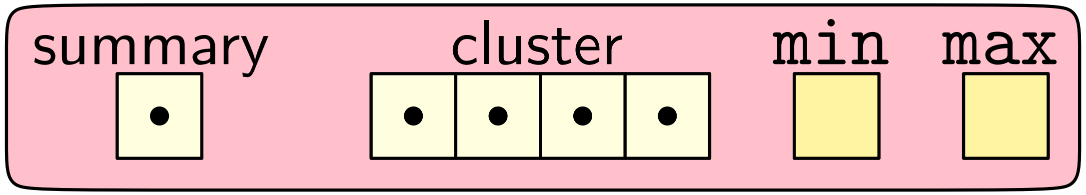
Implementing the Operations (again)



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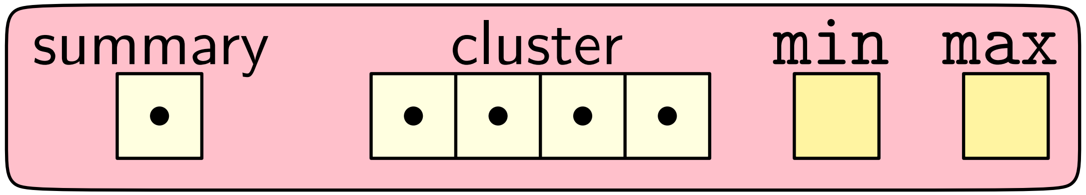
Implementing the Operations (again)



Successor(x):

- $h, \ell \leftarrow$ high(x), low(x)
- If $\ell <$ `cluster`$[h]$.`max`:
  - Return idx(h, `cluster`$[h]$.successor(ℓ))
- $h' \leftarrow$ `summary`.successor(h)
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  - $\ell' \leftarrow$ `cluster`$[h']$.successor($-\infty$)
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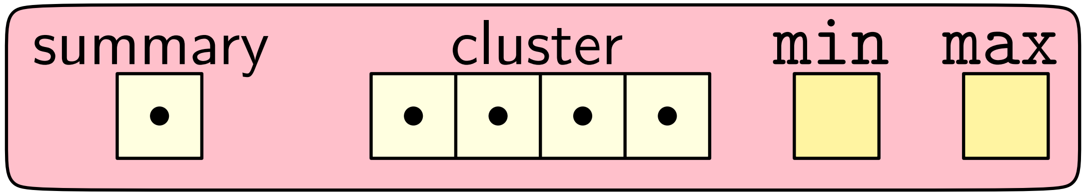

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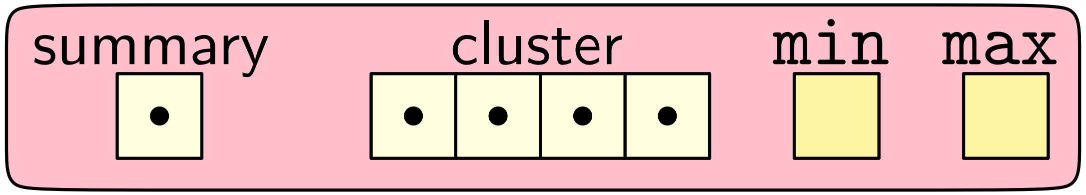

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Time: $T(u) = T(\sqrt{u}) + O(1) = \Theta(\log \log u)$ ☺

The Dynamic Predecessor Problem

Goal:

Design a data stucture that maintains a *dynamic* set S of *integers* from a universe $\{0, \ldots, u-1\}$, supporting the following operations:

- Insert(x): Add x into S.
- Delete(x): Remove x from S.
- Find(x): report whether $x \in S$.
- Predecessor(x): return the largest integer $y < x$ in S (if any).
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Assume: $u = 2^w$, for some positive even integer w.

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Delete(x):

`// Last element?`

- If $x = \texttt{min} = \texttt{max}$: $\texttt{min} \leftarrow \texttt{max} \leftarrow +\infty$. Return

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  - $\texttt{min} \leftarrow x \leftarrow \text{idx}(\texttt{summary.min}, \texttt{cluster}[\texttt{summary.min}].\texttt{min})$

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- $h, \ell \leftarrow \text{high}(x), \text{low}(x)$ // Actual deletion
- $\texttt{cluster}[h].\text{delete}(\ell)$
- If $\texttt{cluster}[h].\texttt{min} = +\infty$:
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- $\texttt{cluster}[h].\text{delete}(\ell)$ // ...then this took $O(1)$ time
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- $\texttt{cluster}[h].\text{delete}(\ell)$ `// ...then this took` $O(1)$ `time`
- If $\texttt{cluster}[h].\texttt{min} = +\infty$:
  - $\texttt{summary}.\text{delete}(h)$ `// If we execute this line...`

`// Recompute max from scratch`

- If $\texttt{summary.max} = +\infty$: $\texttt{max} \leftarrow \texttt{min}$. Return
- $\texttt{max} \leftarrow \text{idx}(\texttt{summary.max}, \texttt{cluster}[\texttt{summary.max}].\texttt{max})$

Recap

van Emde Boas trees: maintain a dynamic collection of integers from the universe $\{0, \ldots, u-1\}$

- Insert $O(\log\log u)$
- Delete $O(\log\log u)$
- Successor/Predecessor $O(\log\log u)$
- Min/Max $O(1)$
- Space $O(u)$ Not $O(n)$!!
- Supports satellite data

(n is the number of elements currently in the collection)

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van Emde Boas trees: maintain a dynamic collection of integers from the universe $\{0, \ldots, u-1\}$

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(n is the number of elements currently in the collection)

Space can be improved to $O(n)$.

(but the data structure becomes randomized & update times expected)

Reducing the Space Usage

Idea: Only store non-empty clusters!

- Replace `clusters` with a hash table
- Keys are high$(\cdot)$
- Values are pointers to the data-structures representing the clusters

$\ldots$ + additional tricks.

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Space: $O(n)$ (with *amortized expected* update times)

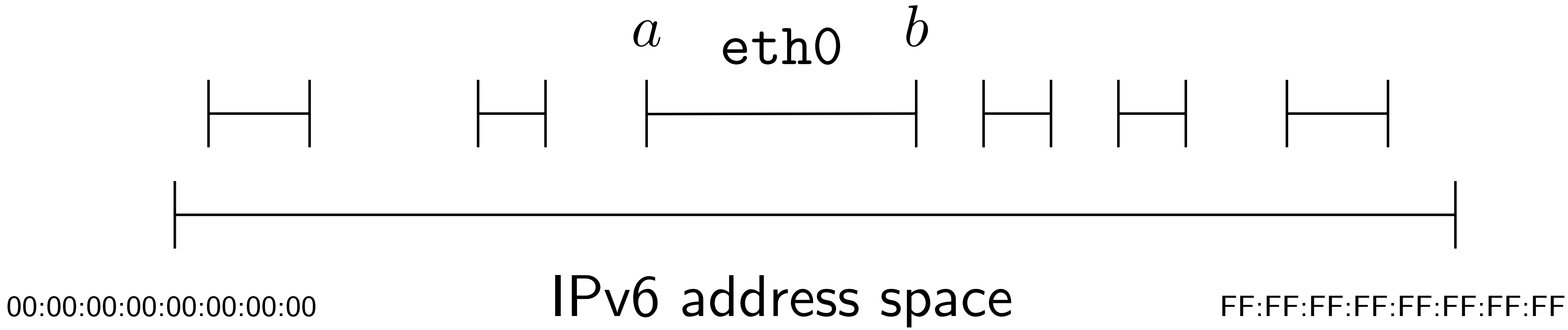
$\Omega(\log \log u)$ query time is needed when space is $O(n \operatorname{polylog} n)$

- Even when S is *static* and only Successor() is needed

In the cell probe model [M. Pătrașcu & M. Throup, STOC'06 & SODA'07]

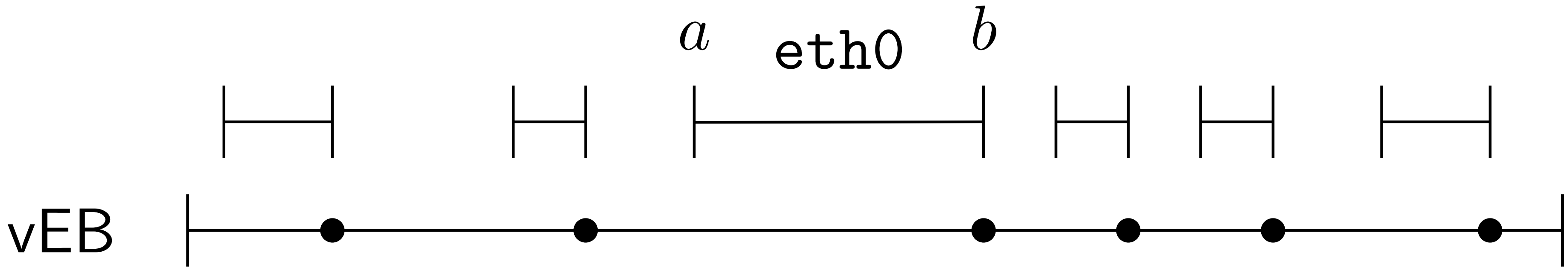
An Application

IP routing



An Application

IP routing



00:00:00:00:00:00:00:00 IPv6 address space FF:FF:FF:FF:FF:FF:FF:FF

An Application

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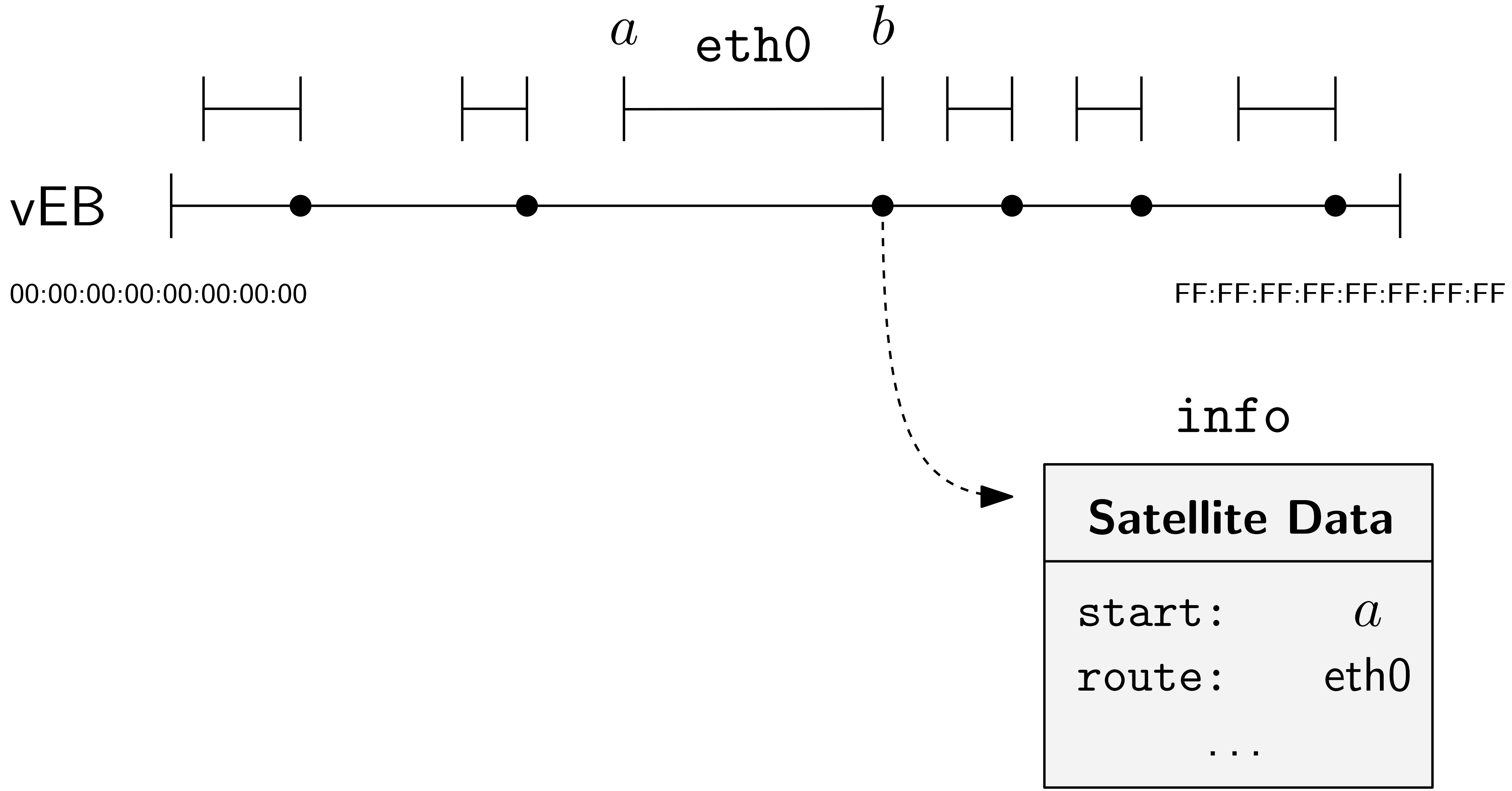

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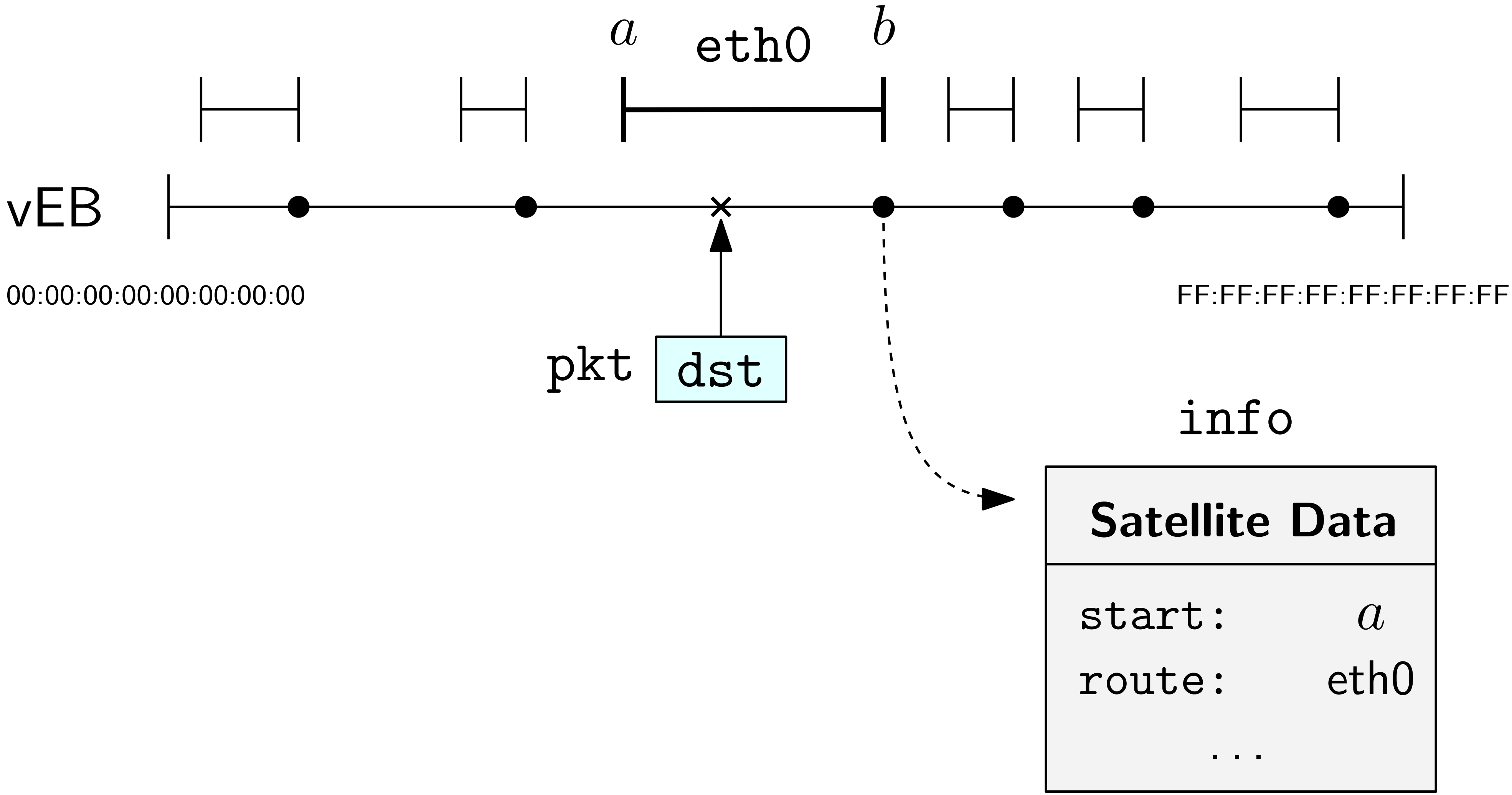

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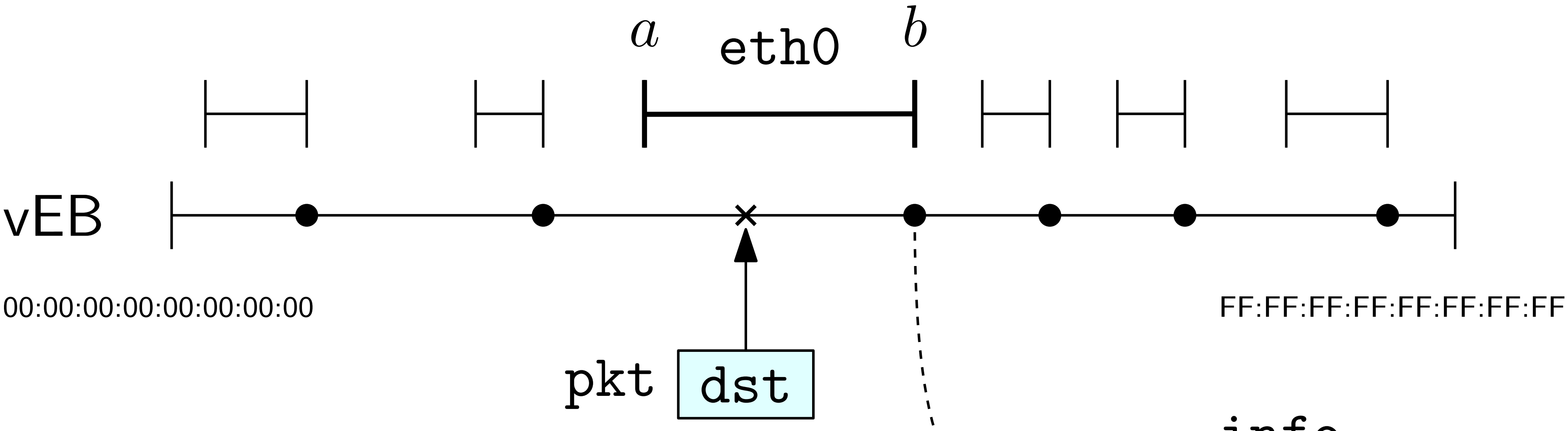
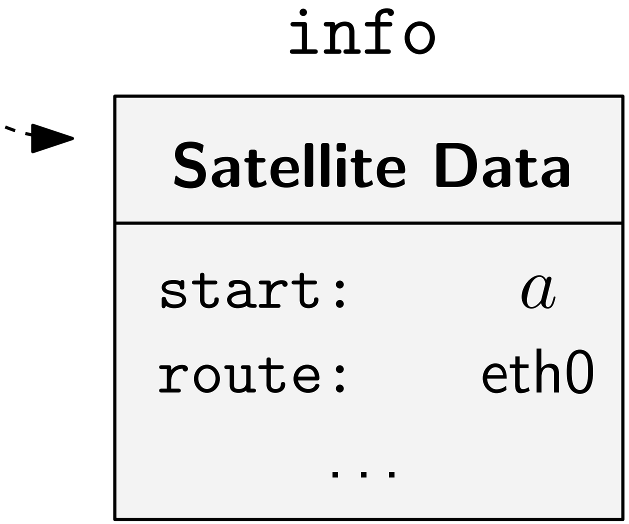


$\texttt{info} \leftarrow \text{vEB.successor}(\texttt{dst})$

dst $\geq$ $\texttt{info.start}$?

- **Yes**: send $\texttt{pkt}$ via $\texttt{info.route}$



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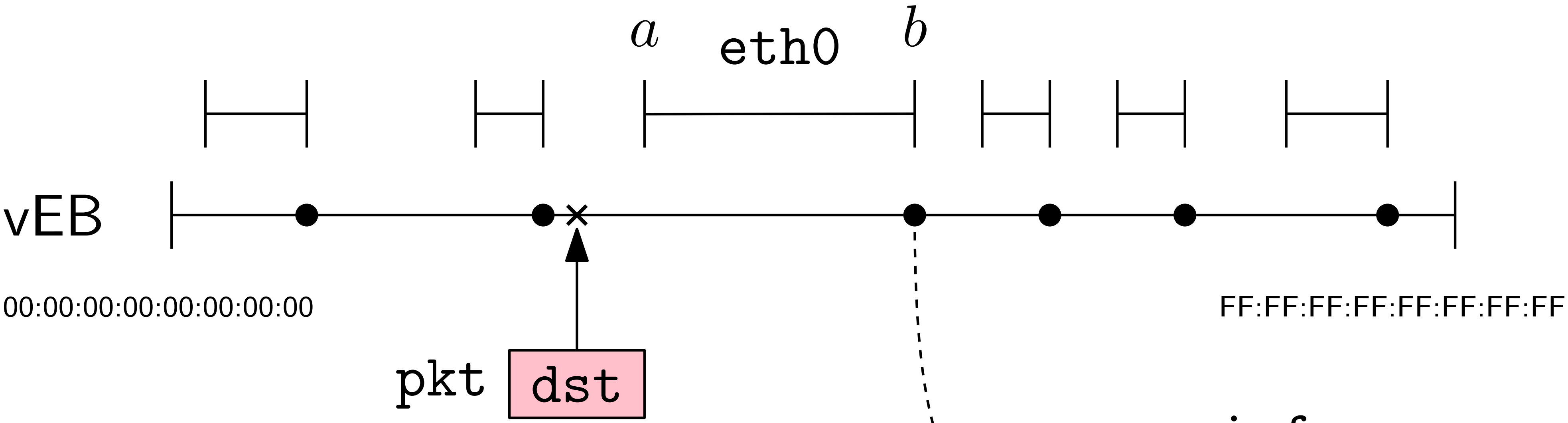

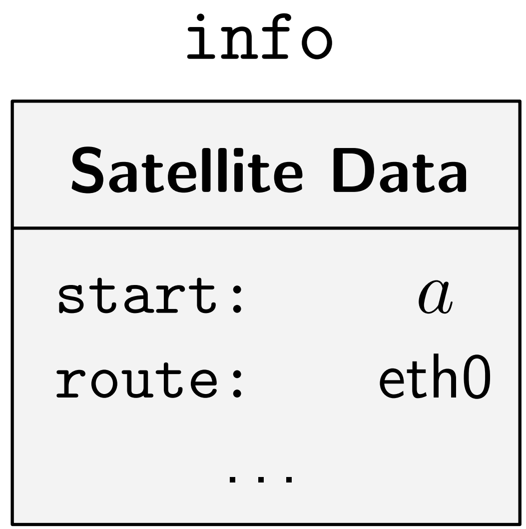

`info` $\leftarrow$ vEB.successor(`dst`)

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- **Yes**: send `pkt` via `info.route`
- **No**: drop `pkt`

An Application

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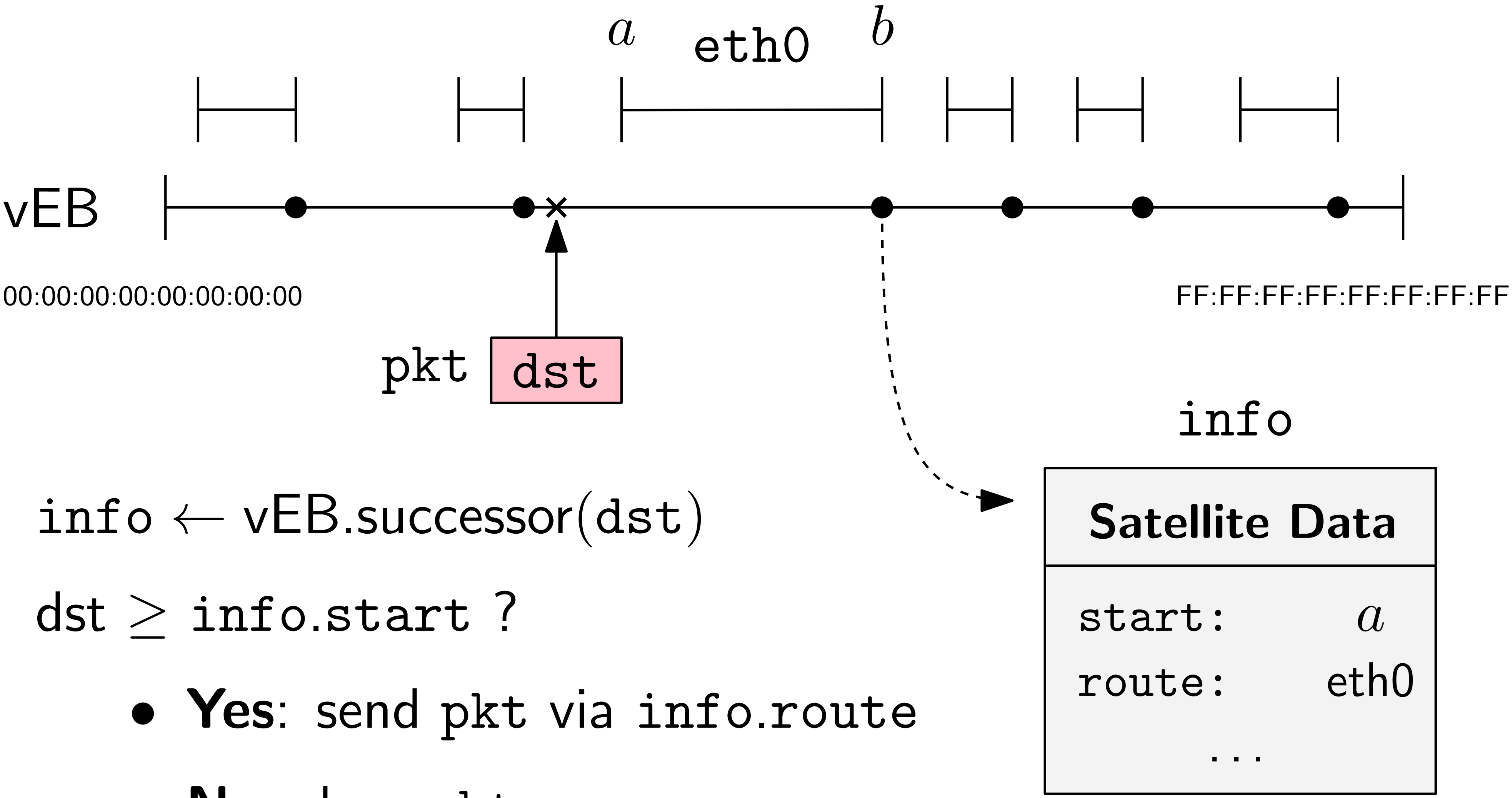

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Universe size: $u = 2^{128}$ $\log \log u = \log 128 = 7$