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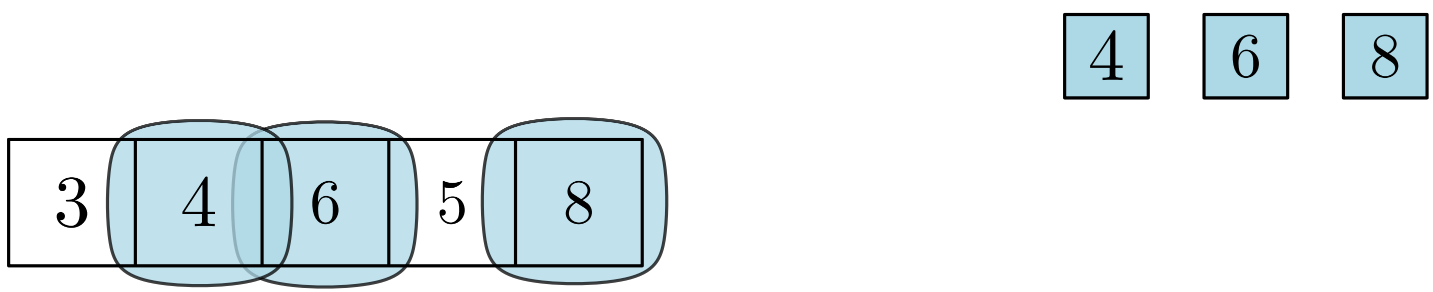

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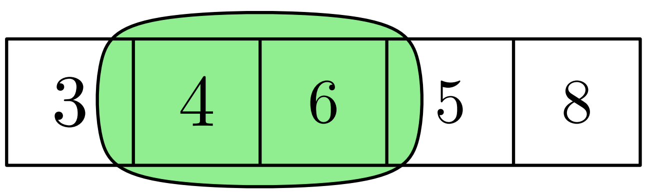

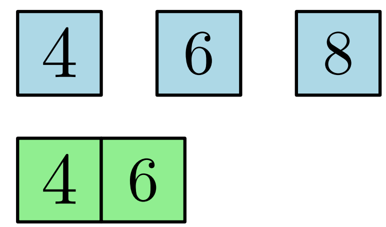

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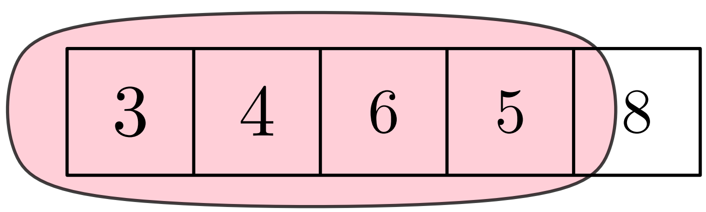

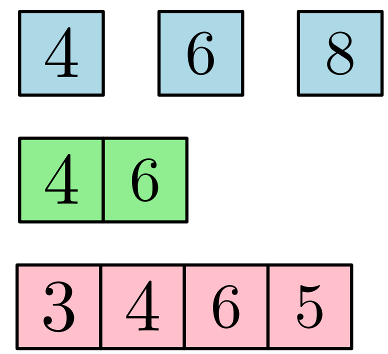

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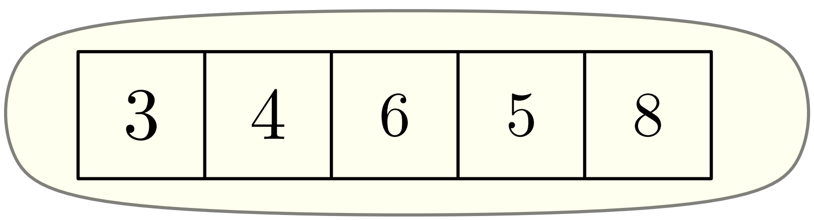

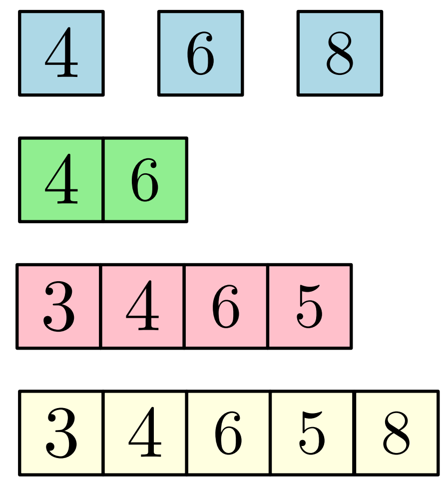

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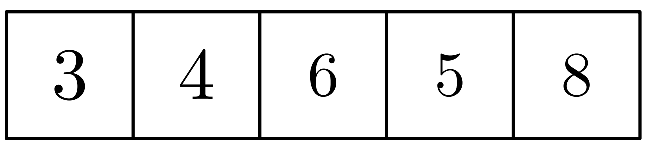
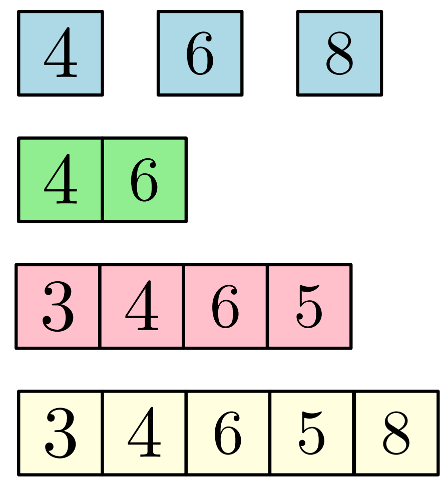
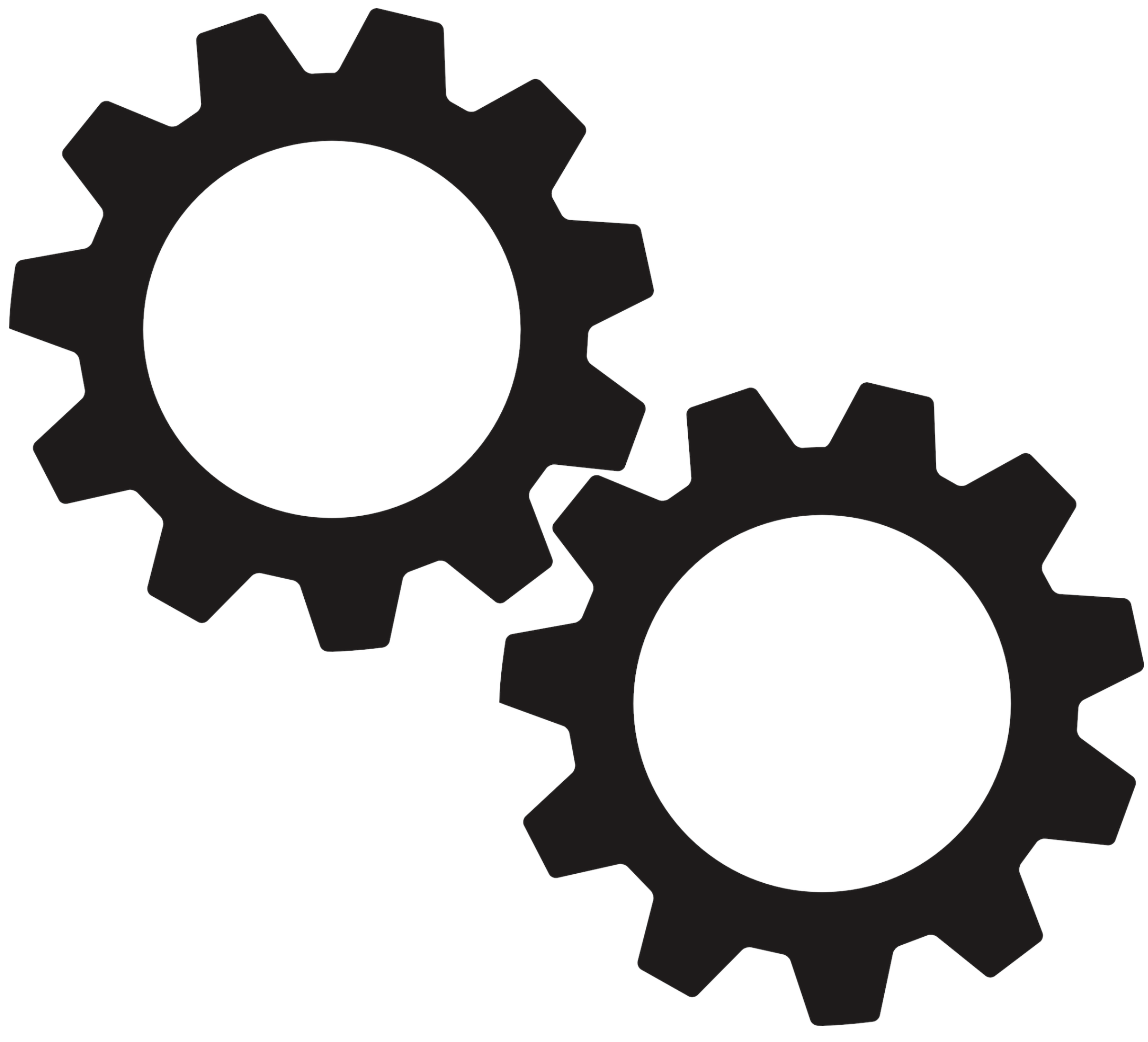




Answer: 6



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                                                      %2 == 0)
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Naive Solution

- For each pair of integers (i, j) with $i \leq j$ $O(n^2)$
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Partial Sums Technique

$O(n)$

Compute the partial sums $P_k = \sum_{i=1}^{k} a_k$ in a single sweep

$$P_0 = 0 \qquad\qquad P_k = P_{k-1} + a_k$$

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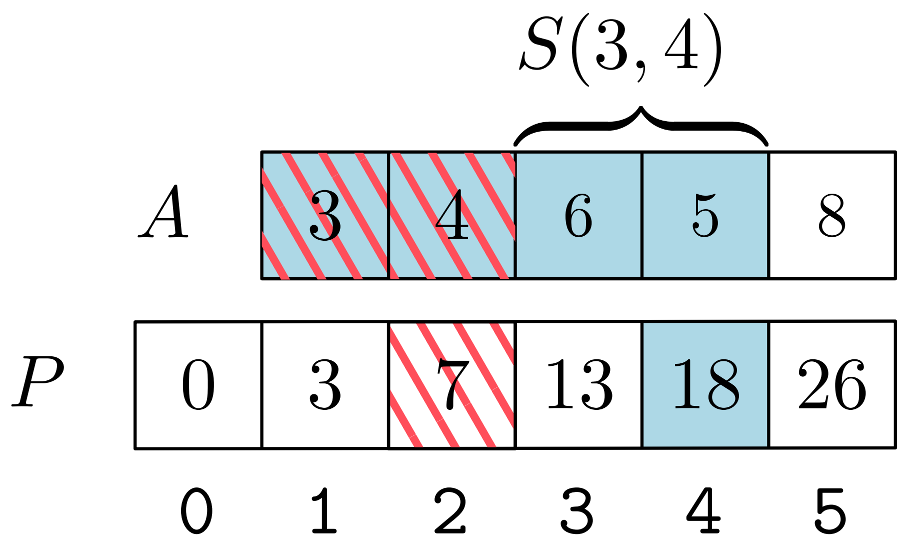

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A $O(n^2)$ algorithm

- Compute the vector P of prefix sums.
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```
std::vector<int> P(A.size()+1);
std::partial_sum(A.begin(), A.end(), P.begin()+1);

unsigned int result = 0;
for(unsigned int i=1; i<=A.size(); i++)
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        if((P[j] - P[i-1])%2 == 0)
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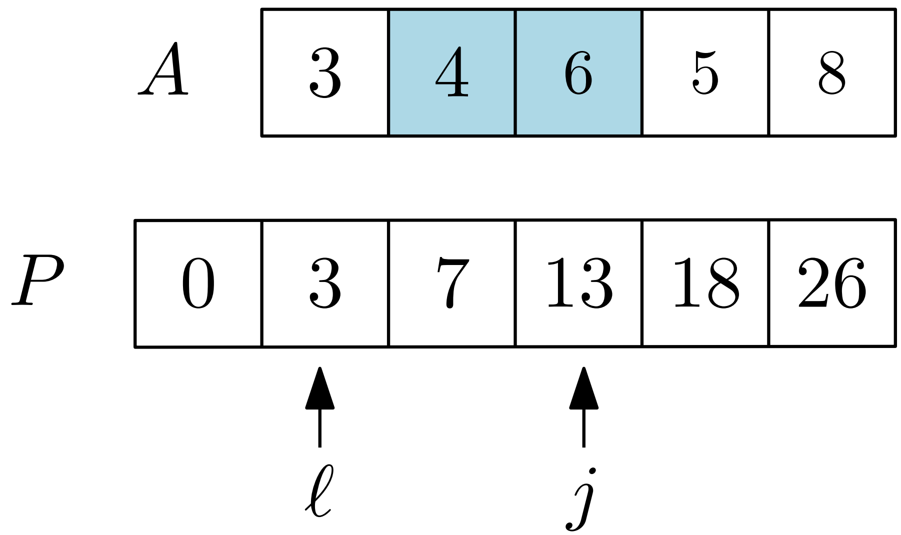
A Linear-Time Solution



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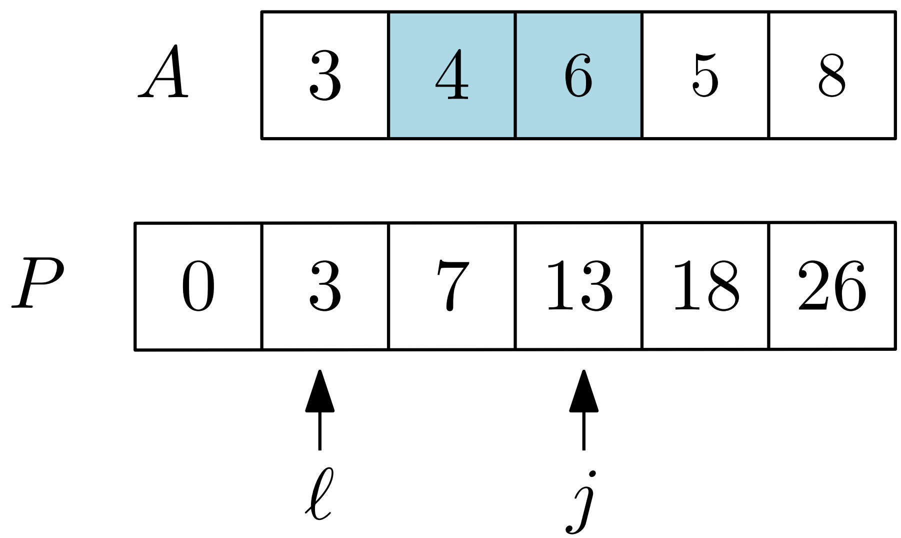

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Let E and F be the number of even and odd values is P, respectively.

Number of even pairs in P: $\binom{E}{2} = E(E-1)/2$.

Number of odd pairs in P: $\binom{F}{2} = F(F-1)/2$.

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- Compute the vector P of prefix sums $O(n)$
- Compute F from P $O(n)$
- Return $F(F-1)/2 + \overbrace{(n+1-F)}^{E}(n-F)/2$ $O(1)$

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std::partial_sum(A.begin(), A.end(), A.begin());
int odd = std::count_if(A.begin(), A.end(),
                                        [](int x) {return x%2;});

return odd*(odd-1)/2 + (A.size()-odd)*(A.size()+1-odd)/2;
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Can we do better?

Observation: Each partial sum is used only once.

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- Scan the input and keep track of the current sum $O(n)$
- Keep track of F $O(n)$
- Return $F(F-1)/2 + \overbrace{(n+1-F)}^{E}(n-F)/2$ $O(1)$

```
unsigned int n;
std::cin >> n;

int x, odd=0, prefix_sum=0;
for(unsigned int i=0; i<n; i++)
{
    std::cin >> x;
    prefix_sum += x;
    odd += prefix_sum%2;
}

std::cout << odd*(odd-1)/2 + (n-odd)*(n+1-odd)/2 << "\n";
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Streaming algorithm. $O(n)$ time $O(1)$ space