## Gustavo's Pizza

Gustavo has very peculiar tastes when it comes to pizza al taglio: he wants his *slice* to have as many olives as possible, but never more than k.

The pizza can be cut into discrete positions  $t_1 < \cdots < t_n$ . A slice (i, j) with j > i represents the interval  $[t_i, t_j]$ .



The interval  $[t_i, t_{i+1}]$  contains  $\eta_i$  olives. Where to cut?

#### Example



#### Example



Solution: (3,6). Number of olives:  $\sum_{i=3}^{6-1} \eta_i = 8$ .



## A Naive Solution

- For i = 1, ..., n 1:
  - For j = i + 1, ..., n:
    - $\bullet \ \text{olives} \leftarrow 0$
    - For k = i, ..., j 1:
      - olives  $\leftarrow$  olives  $+ \eta_k$

### A Naive Solution

- For i = 1, ..., n 1:
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- O(n)

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Total time:  $O(n^3)$ 

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Can we do better?



• Compute partial sums vector S O(n)



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- Compute partial sums vector S O(n)
- For i = 1, ..., n 1: O(n)
  - Binary search S for the largest index  $j \ge i$  such that  $S[j] \le S[i-1] + k$ .
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Total time:  $O(n \log n)$ 

#### Recap



 $O(n^3)$ 

• Partial Sums

 $O(n^2)$ 

• Partial Sums + Binary Search

 $O(n\log n)$ 

Time

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 $O(n^3)$ 

• Partial Sums

 $O(n^2)$ 

Time

Partial Sums + Binary Search

 $O(n\log n)$ 

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- Return "best" feasible window among those considered.

(plus suitable handling of edge cases)

```
int left=0, right=-1, sum=0;
int best_left=-1, best_right=-1, best_sum=-1;
do
{
   if(sum<=k && right<n-1)</pre>
       sum += A[++right];
   else
       sum -= A[left++];
   if(sum<=k && sum>best_sum)
   {
       best_sum = sum; best_left = left; best_right = right;
    }
} while(left<n-1 || right<n-1);</pre>
std::cout << "Cut from position " << best_left+1</pre>
          << " to position " << best_right+2 << "\n";
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                                                O(n)
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#### **Trick/Technique: Sliding Window**

Some problems in which you need to find an interval can be solved in linear time using a sliding window approach, if you can ensure that an optimal interval will be considered.

Gustavo is in a Sushi-belt restaurant: n small plates are lined up on a conveyor belt and will soon reach him

- Gustavo can eat an unlimited amount of food, as long as he never stops eating
- Gustavo does not want to eat any repeated dish
- Gustavo wants to eat as much as possible



What is the maximum number of dishes that Gustavo can eat?

Given an array A of n integers in  $\{1, \ldots, n\}$ , find the longest contiguous subarray of A that contains only distinct elements.



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**Solution:** A[3...,6], length: 4

Start eating from the 3rd plate, eat up to (and including) the 6-th plate



- For i = 1, ..., n:
  - For j = i, ..., n:
    - If  $A[i \dots j]$  contains no duplicates:
      - $A[i \dots j]$  is a candidate solution
      - . . .
- Return longest candidate solution found

O(n)

O(n)

 $O(n^2)$ 

• For 
$$i = 1, ..., n$$
:

• For 
$$j = i, ..., n$$
:

• If  $A[i \dots j]$  contains no duplicates:

- $A[i \dots j]$  is a candidate solution
- Return longest candidate solution found

#### Total time: $O(n^4)$

O(n)

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• For 
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#### Total time: $O(n^4)$

#### Total time: $O(n^3)$ via counting sort

# Sushi Belt: Checking for Duplicates

- Do not run counting sort each time we need to check for duplicates
- Keep the number of occurrences of each type updated
- Keep track of the number of duplicates, i.e., counts  $\geq 2$



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## Sushi Belt: (A Less) Naive Solution

O(n)

O(n)

 $Q(n^2) O(n)$ 

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:

• For 
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- If  $A[i \dots j]$  contains no duplicates:
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#### Total time: $O(n^4)$

Total time:  $O(n^3)$  via counting sort

# Sushi Belt: (A Less) Naive Solution

O(n)

O(n)

 $Q(n^2) Q(n) O(1)$ 

• For 
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• For 
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:

- If  $A[i \dots j]$  contains no duplicates:
  - $A[i \dots j]$  is a candidate solution
- Return longest candidate solution found

Total time: 
$$O(n^4)$$
  
Total time:  $O(n^3)$  via counting sort  
Total time:  $O(n^2)$  by updating counts in  $O(1)$  time

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![](_page_46_Figure_3.jpeg)

• If  $A[p_1 \dots p_2]$  contains no duplicates and  $p_2 < n$ : increase  $p_2$ 

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![](_page_47_Figure_3.jpeg)

- If  $A[p_1 \dots p_2]$  contains no duplicates and  $p_2 < n$ : increase  $p_2$
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![](_page_48_Figure_3.jpeg)

- If  $A[p_1 \dots p_2]$  contains no duplicates and  $p_2 < n$ : increase  $p_2$
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- If  $p_1 = p_1^*$  then  $p_2 < p_2^*$  and  $A[p_1 \dots p]$  contains no duplicates for all  $p = p_2, \dots, p_2^* 1$

![](_page_50_Figure_4.jpeg)

- Let  $W^* = [p_1^*, p_2^*]$  be an optimal interval
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- Therefore,  $p_2$  will be incremented until it reaches  $p_2^*$  while  $p_1 = p_1^*$  remains constant  $\implies$  the algorithm considers  $W^*$ .

![](_page_51_Figure_5.jpeg)

- Let  $W^* = [p_1^*, p_2^*]$  be an optimal interval
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![](_page_52_Figure_4.jpeg)

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Some problems in which you need to find an interval can be solved in linear time using a sliding window approach, if you can ensure that an optimal interval will be considered.

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int best_len = -1;
std::vector<int> counts(n);
do
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   if(duplicates==0 && right<n-1)</pre>
       duplicates += ( ++counts[A[++right]] >= 2 );
   else
       duplicates -= ( counts[A[left++]]-- >= 2 );
   if(duplicates==0)
       best_len = std::max(best_len, right-left+1);
} while(left<n-1 || right<n-1);</pre>
std::cout << "Gustavo can eat " << best_len << " dishes\n";</pre>
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