Interval Scheduling

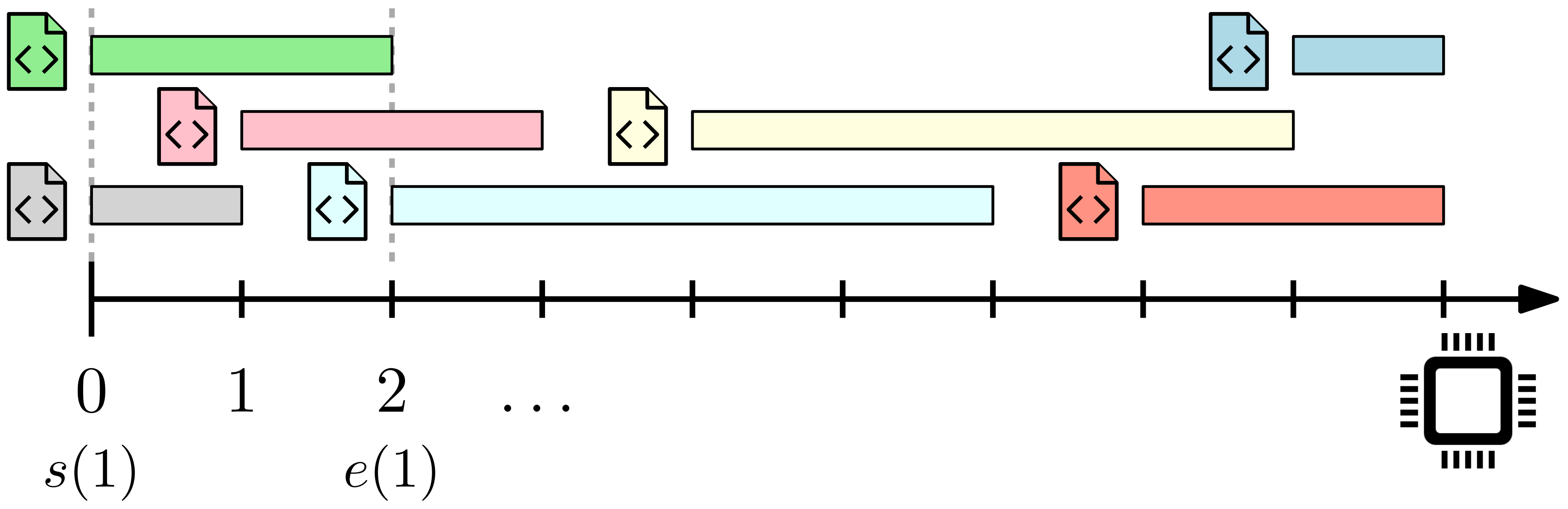
Interval Scheduling

You need to compute a non-preemptive schedule on a supercomputer.

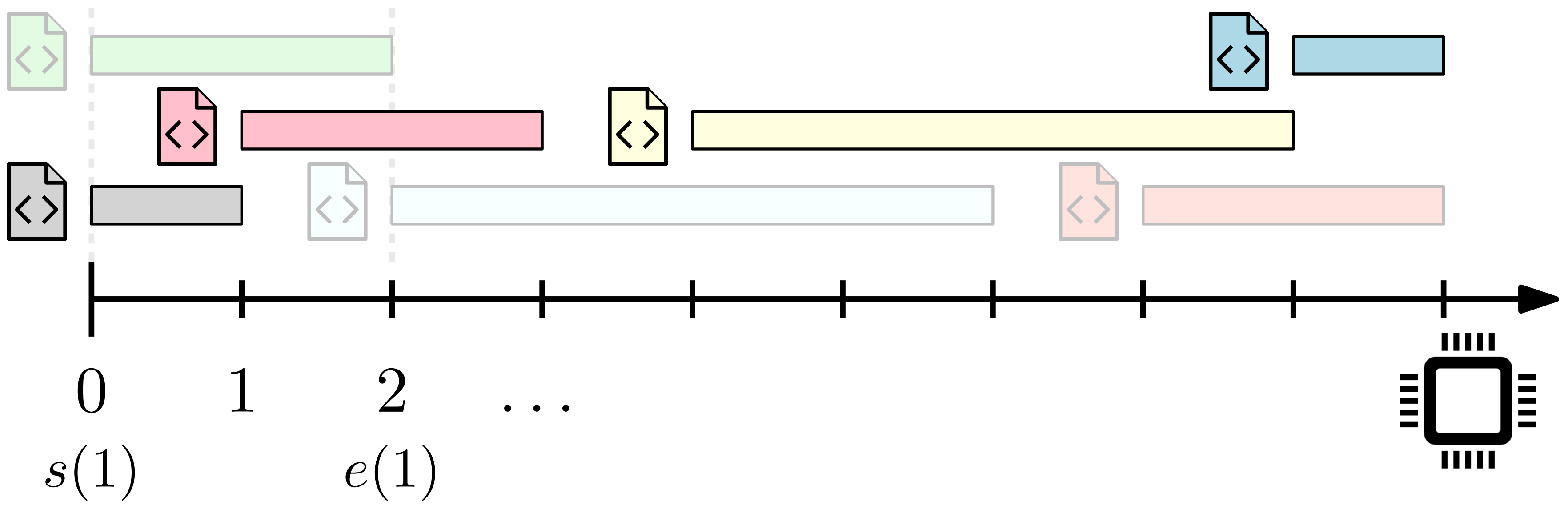
- There are n jobs indexed by $1, \ldots, n$ submitted for execution.
- Each job i has a desired start time $s(i)$ and a completion time $e(i) > s(i)$.
- Two jobs i and j are *compatible* if the intervals $[s(i), e(i))$ and $[s(j), e(j))$ are disjoint.

Goal: Find a subset of mutually compatible jobs of maximum cardinality.

Example



Example



Optimal solution: $\{$ 🗎 , 🗎 , 🗎 , 🗎 $\}$

Greedy template:

- Start with an empty set of jobs $R = \emptyset$.
- Examine jobs in some order.
    - When job i is examined: add i to R if it is compatible with all jobs j already in R.
- Finally, return R.

Greedy template:

- Start with an empty set of jobs $R = \emptyset$.
- Examine jobs in some order.
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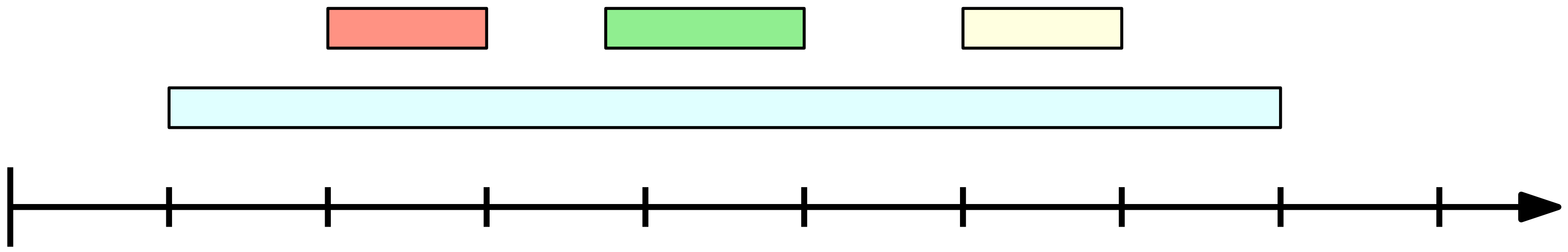
- Finally, return R.

Key question:
In what order should we process the jobs?

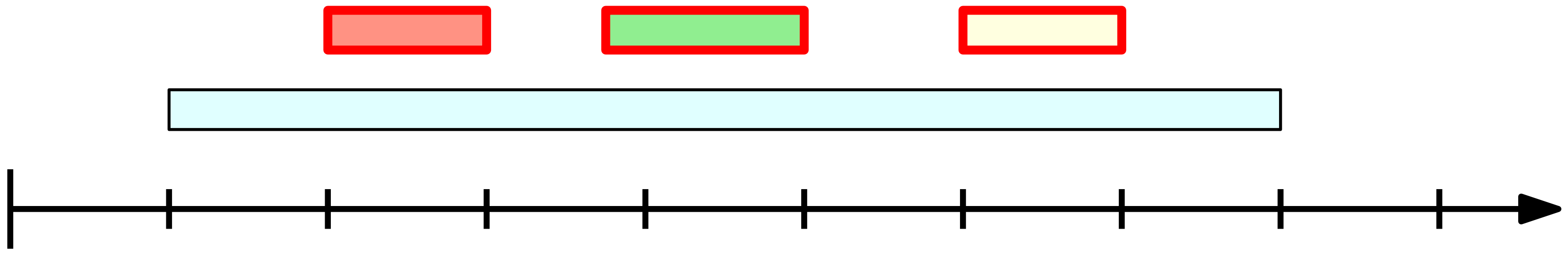
Some Possibilities:

- **Earliest Start Time**: Increasing order of $s(i)$.
- **Earliest Finish Time**: Increasing order of $e(i)$.
- **Shortest Interval**: Increasing order of $e(i) - s(i)$.
- **Fewest Conflicts**: Increasing order w.r.t. the number of conflicting jobs.

Earliest Start Time



Earliest Start Time



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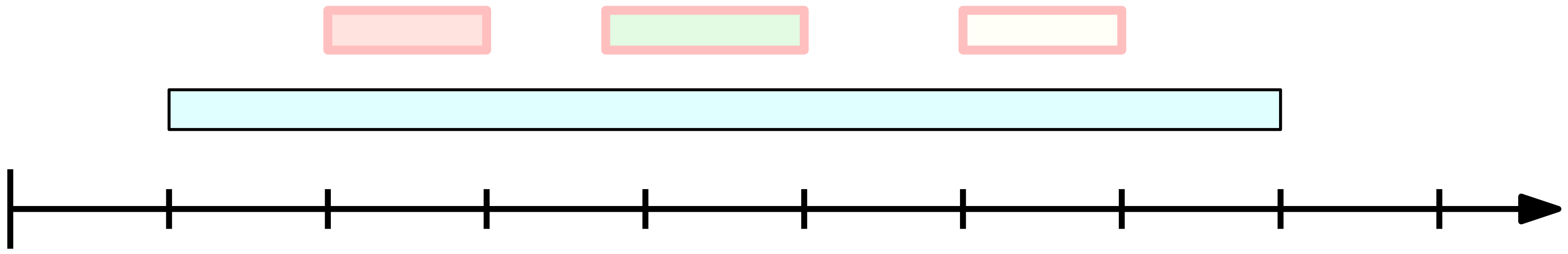
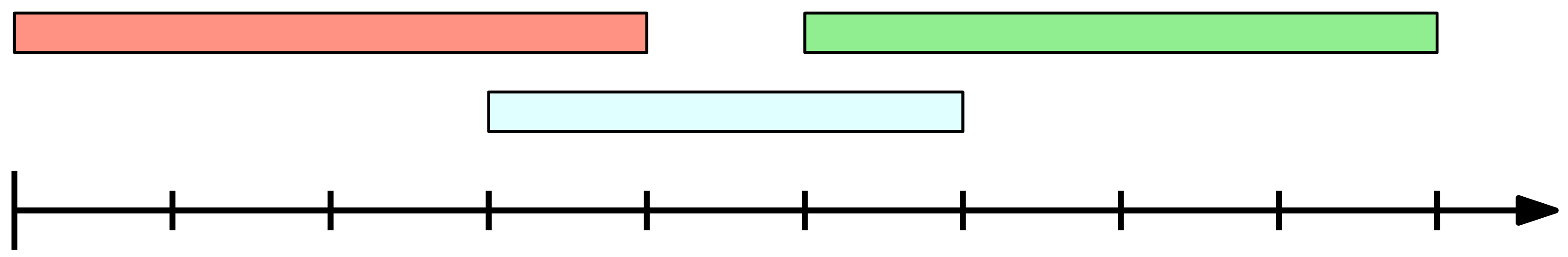


Shortest Interval



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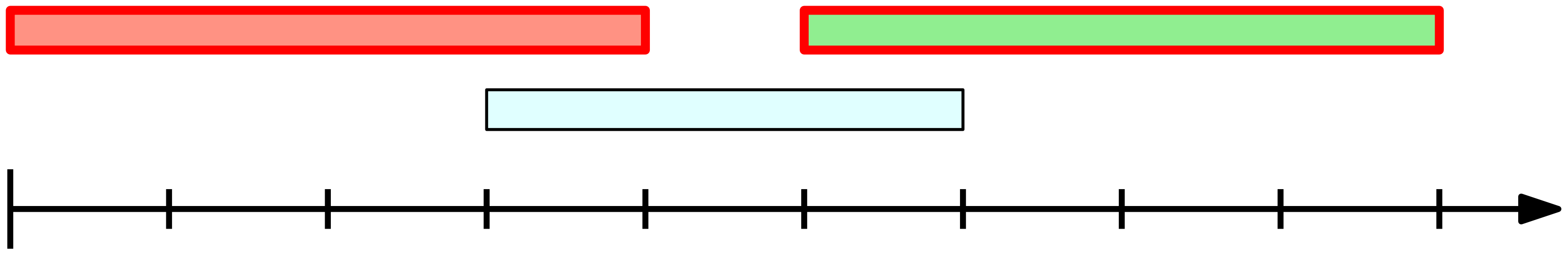

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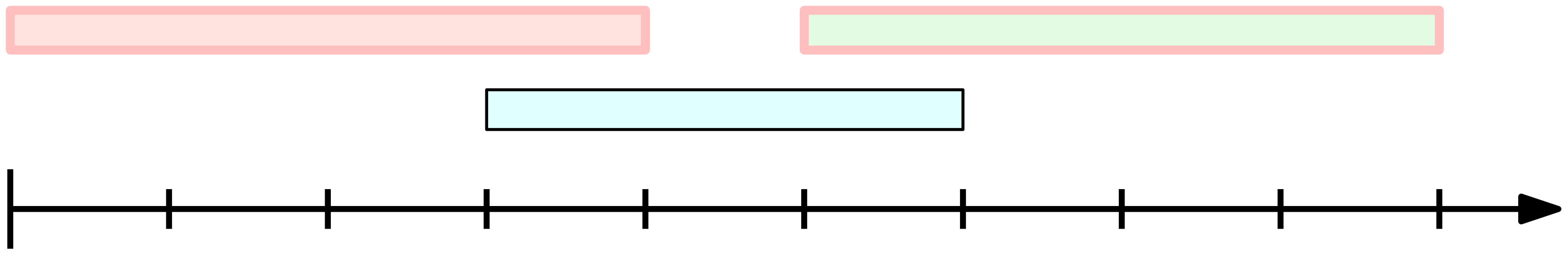
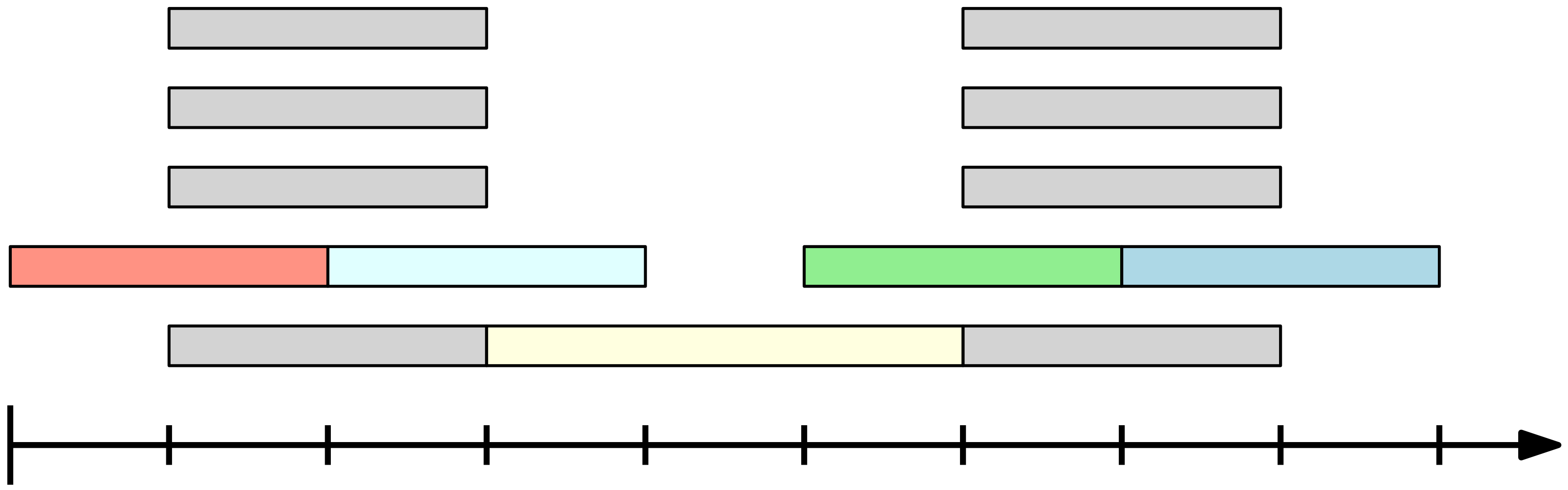


Fewest Conflicts



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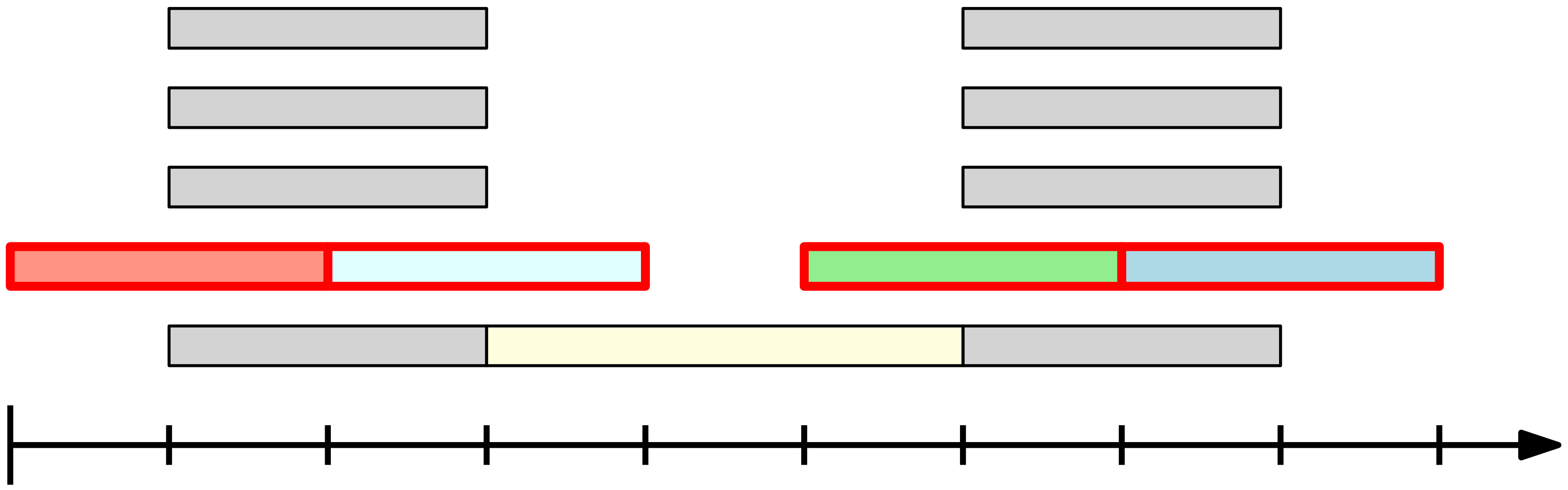

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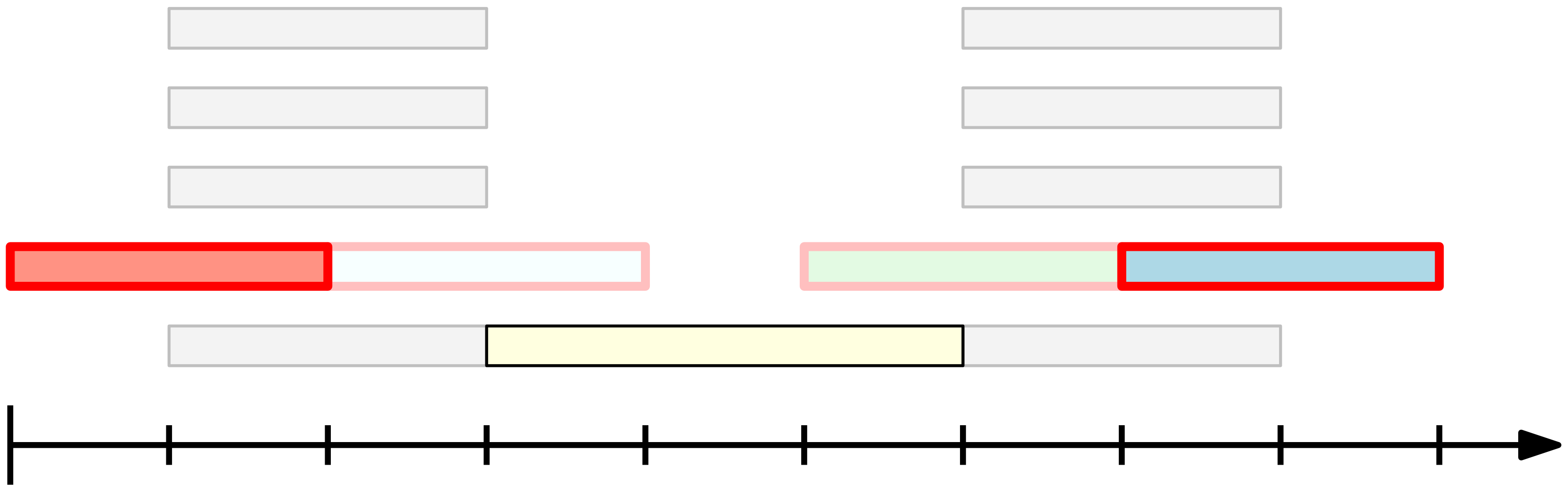

Some Possibilities:

- ~~**Earliest Start Time**: Increasing order of $s(i)$.~~

- **Earliest Finish Time**: Increasing order of $e(i)$.

- ~~**Shortest Interval**: Increasing order of $e(i) - s(i)$.~~

- ~~**Fewest Conflicts**: Increasing order w.r.t. the number of conflicting jobs.~~

Earliest Finish Time

- Let $\mathcal{J} = \{1 \ldots, n\}$ be the set of jobs in input.
- $R \leftarrow \emptyset$
- While $\mathcal{J}$ is not empty:
    - Find a job $i \in \mathcal{J}$ minimizing $e(i)$.
    - Add i to R
    - Remove from $\mathcal{J}$ all jobs $j \in \mathcal{J}$ that are not compatible with i (including i itself).
- Return R

Observation: R is always a set of mutually compatible jobs.

EFT: Proof of Correctness

Let R^* be an optimal set of jobs.

Let $i_1, i_2, \ldots, i_m$ (resp. $i_1^*, i_2^*, \ldots, i_\ell^*$) be the indices of the jobs in R (resp. R^*), sorted w.r.t. $e(\cdot)$.

We want to prove $m = |R| \geq |R^*| = \ell$.

Claim: For $k = 1, \ldots, \ell$, index i_k exists and $e(i_k) \leq e(i_k^*)$.

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Claim: For $k = 1, \ldots, \ell$, index i_k exists and $e(i_k) \leq e(i_k^*)$.

Base case ($k = 1$):

- Since $n \geq 1$, $\mathcal{J}$ is not empty before the first iteration, and i_1 exists.
- By the choice of i_1: $e(i_1) \leq \min_{j=1,\ldots,n} e(j) \leq e(i_1^*)$

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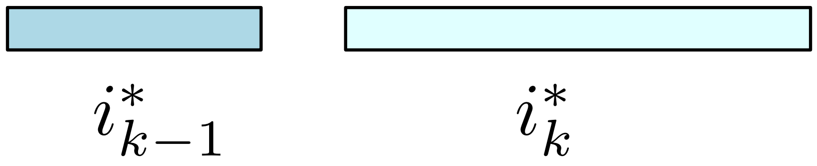
Claim: For $k = 1, \dots, \ell$, index i_k exists and $e(i_k) \leq e(i_k^*)$.

Induction step $(k > 1)$:

- i_k^* is compatible with i_{k-1}^*, thus $e(i_{k-1}^*) \leq s(i_k^*)$



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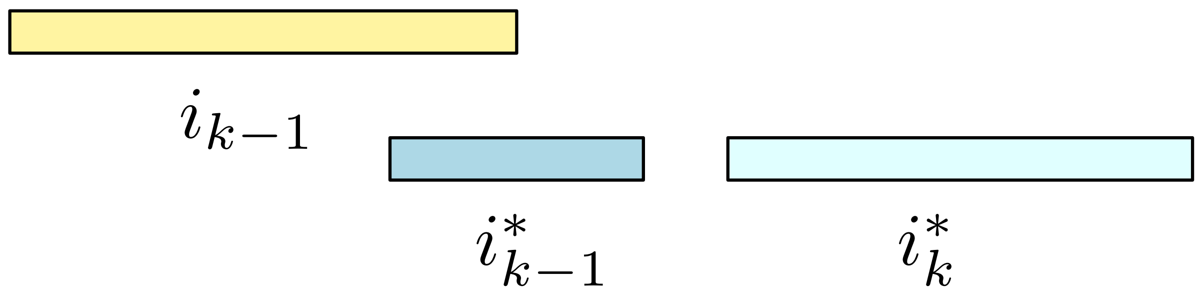

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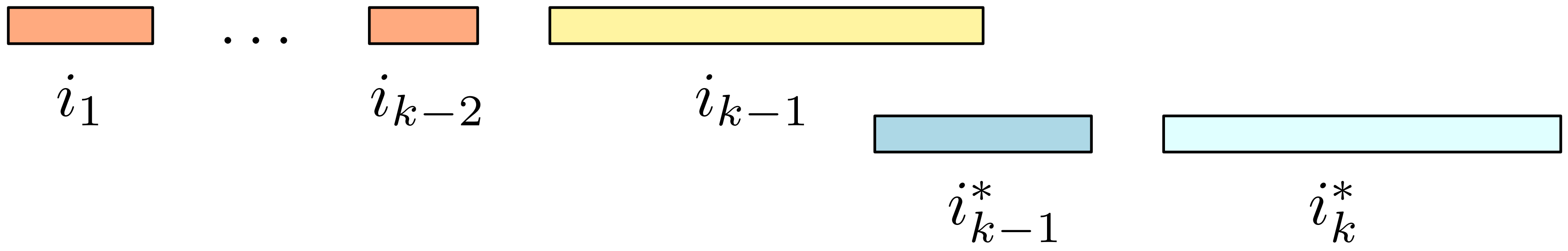
Induction step $(k > 1)$:

- i^*_k is compatible with i^*_{k-1}, thus $e(i^*_{k-1}) \leq s(i^*_k)$
- by induction hypothesis $e(i_{k-1}) \leq e(i^*_{k-1})$
- Thefore, at the beginning ot the k-th iteration, $i^*_k \in \mathcal{J}$ since it is compatible with $i_1, \ldots, i_{k-1}$



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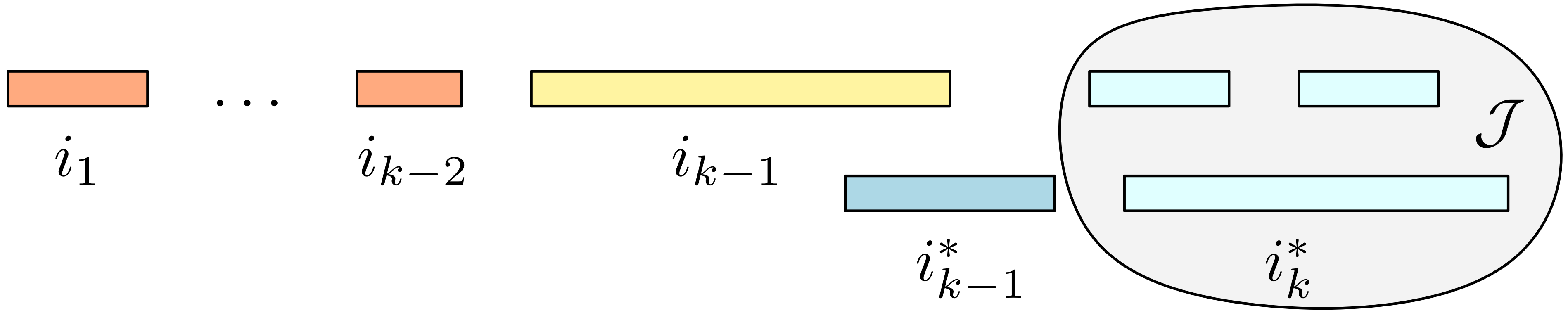
- Thefore, at the beginning ot the k-th iteration, $i^*_k \in \mathcal{J}$ since it is compatible with $i_1, \dots, i_{k-1}$
- $\mathcal{J} \neq \emptyset \implies \exists i_k$
- By the greedy choice: $e(i_k) = \min_{j \in \mathcal{J}} e(j) \leq e(i^*_k)$. $\square$



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Claim: For $k = 1, \ldots, \ell$, index i_k exists and $e(i_k) \leq e(i_k^*)$.

Trick/Technique: Greedy Stays Ahead

At each step, the solution produced by greedy is not worse than the one produced by any other algorithm.

Implementing EFT

- Naive implementation: $O(n^2)$ time.

A better implementation:

```
• $\langle i_1, \ldots, i_n \rangle \leftarrow$ sort $\{1, \ldots, n\}$ w.r.t. $e(\cdot)$.
• Let $R = \emptyset$ be the current (partial) solution.
• Let $f = 0$ be the current finish time.
• For $j = 1, \ldots, n$:
    • If $s(i_j) \geq f$:
        • $R \leftarrow R \cup \{i_j\}$
        • $f \leftarrow e(i_j)$
• Return $R$
```

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A better implementation:

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• $\langle i_1, \ldots, i_n \rangle \leftarrow$ sort $\{1, \ldots, n\}$ w.r.t. $e(\cdot)$.        $O(n \log n)$
• Let $R = \emptyset$ be the current (partial) solution.
• Let $f = 0$ be the current finish time.
• For $j = 1, \ldots, n$:                  ⎫
    • If $s(i_j) \geq f$:                  ⎪
        • $R \leftarrow R \cup \{i_j\}$    ⎬ $O(n)$
        • $f \leftarrow e(i_j)$            ⎭
• Return $R$
```

Time complexity: $O(n \log n)$

Implementing EFT

```
struct job { int id; int start; int end; };
std::vector<job> jobs;

//[...] Read jobs

std::sort(jobs.begin(), jobs.end(), [](const job &j1, const job &j2)
                                    { return j1.end < j2.end; })

int f = 0;
std::vector<int> schedule;
for(const job &j : jobs)
{
    if(j.start >= f)
    {
        schedule.push_back(j.id);
        f = j.end;
    }
}

//schedule contains an optimal set of jobs
```

Interval Partitioning

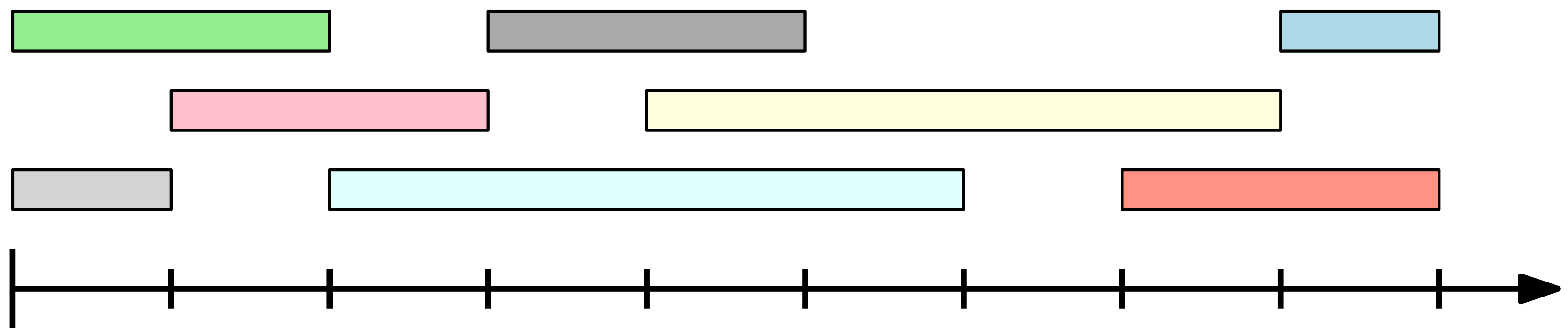
Interval Partitioning

- There are n jobs indexed by $1, \dots, n$.
- Each job i has a start time $s(i)$ and a completion time $e(i) > s(i)$.
- Two jobs i and j are *compatible* if the intervals $[s(i), e(i))$ and $[s(j), e(j))$ are disjoint.
- **All jobs must be executed**, but you can use k processors.
- Jobs scheduled on the same processor must be mutually compatible.

Goal: Minimize k.

(and return the k corresponding schedules)

Example



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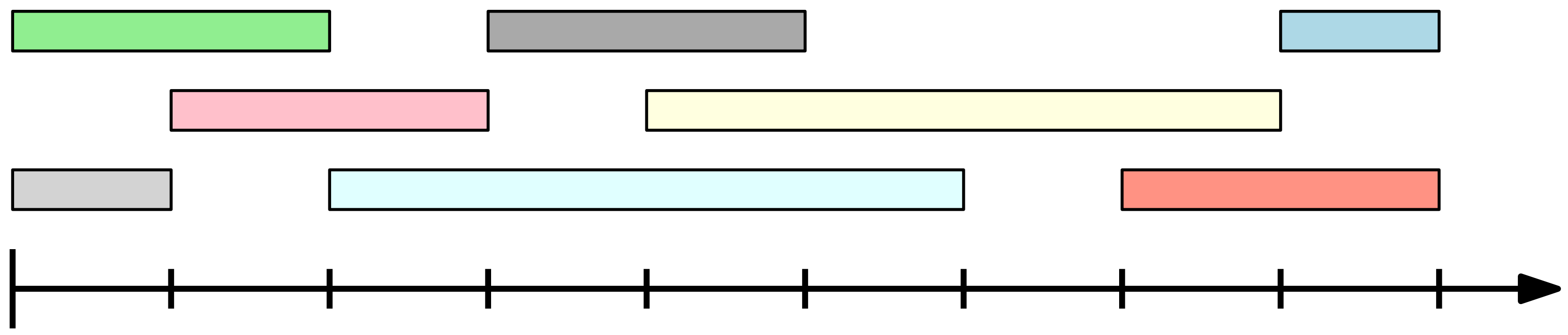
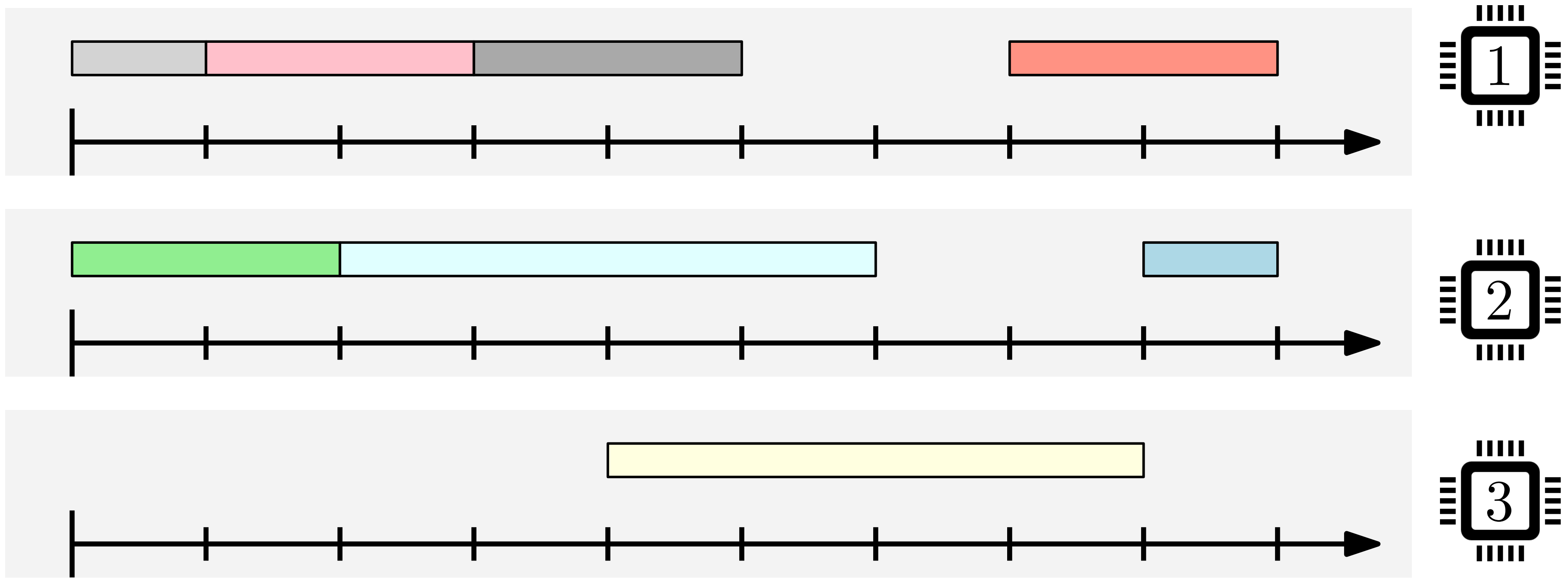


$k = 3$



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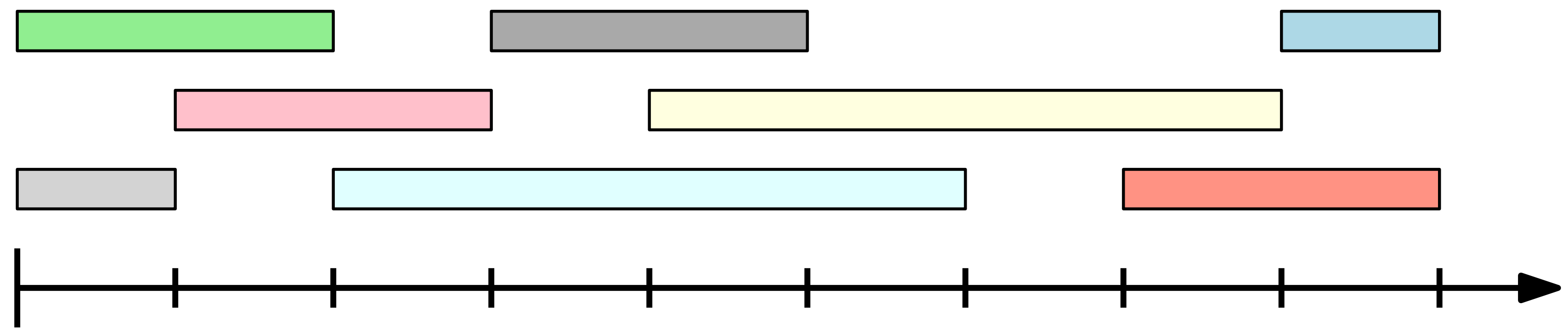

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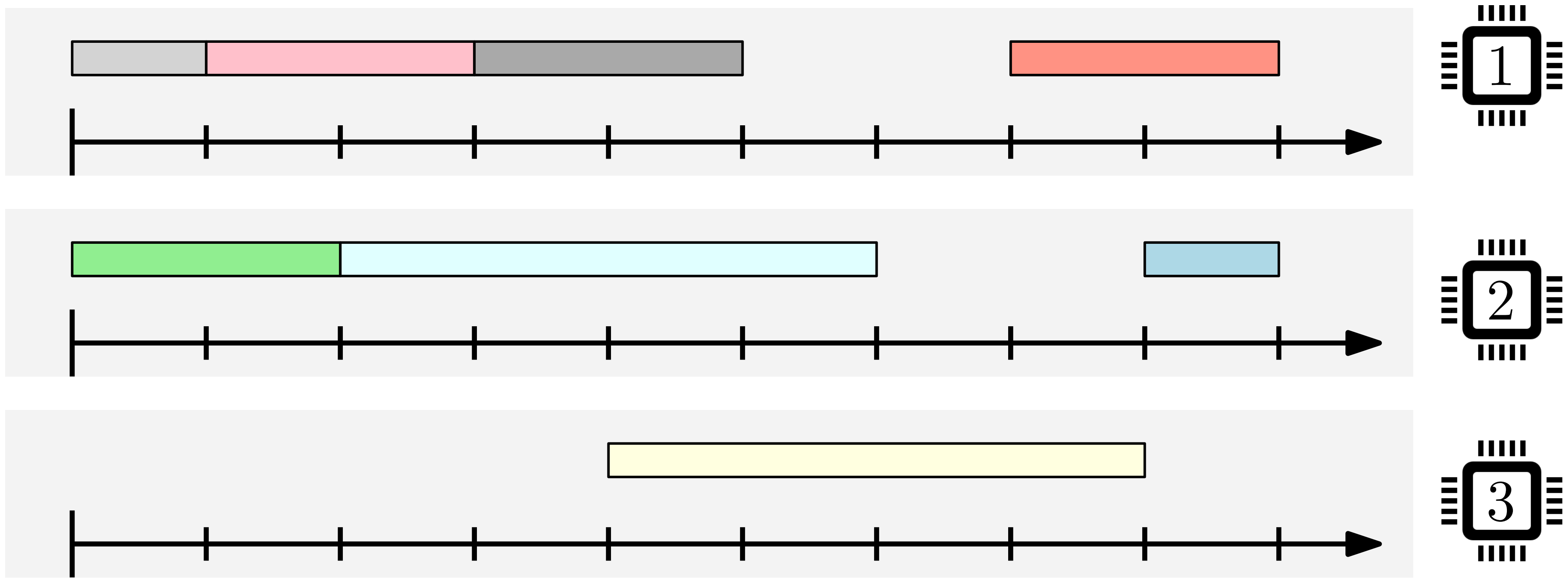
Is $k = 3$ optimal?



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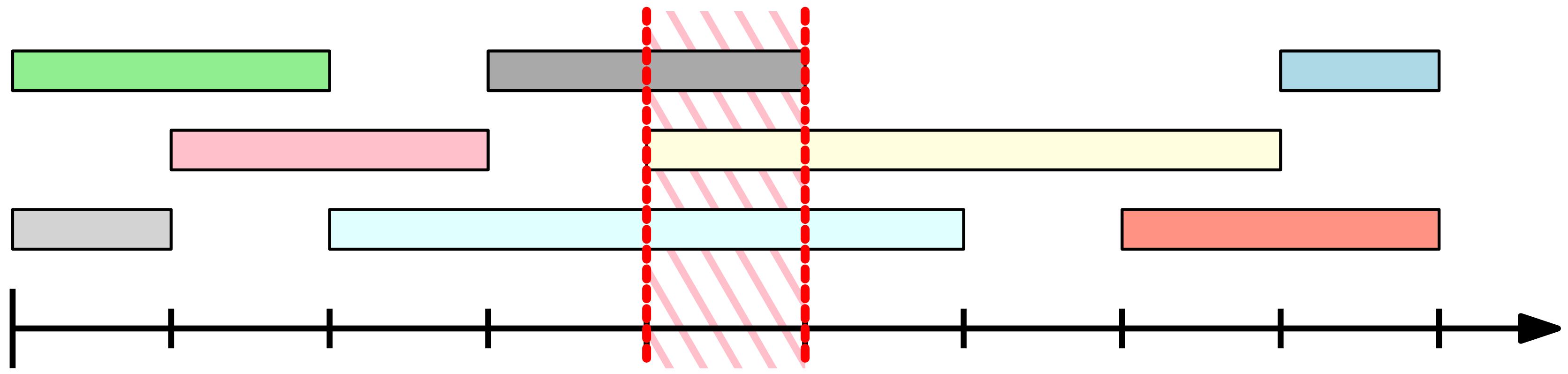
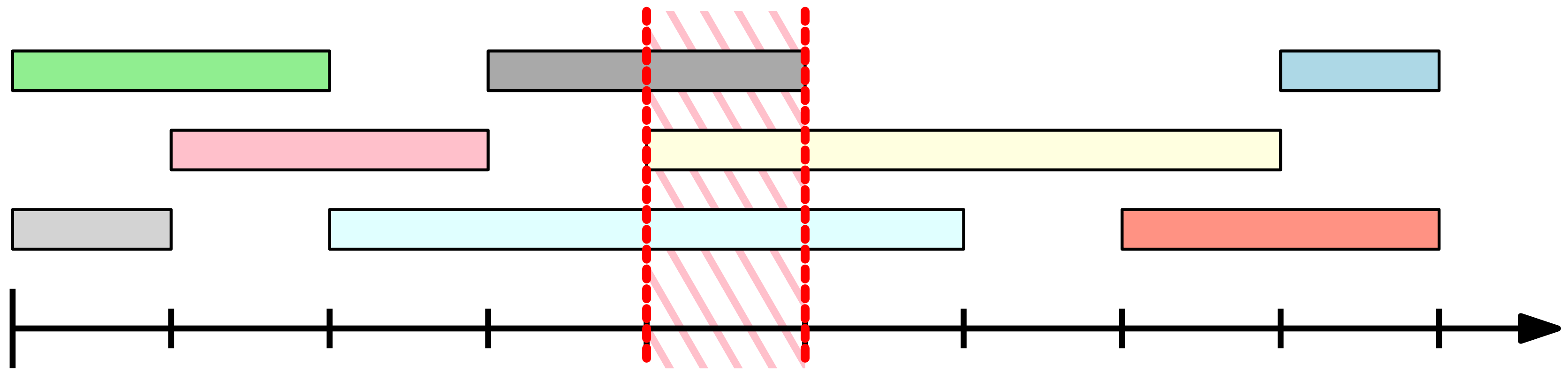


- **Observation:** There are 3 jobs that must be executed simultaneously.
- 3 is a lower bound to the optimal solution k^*.

Is $k = 3$ optimal?



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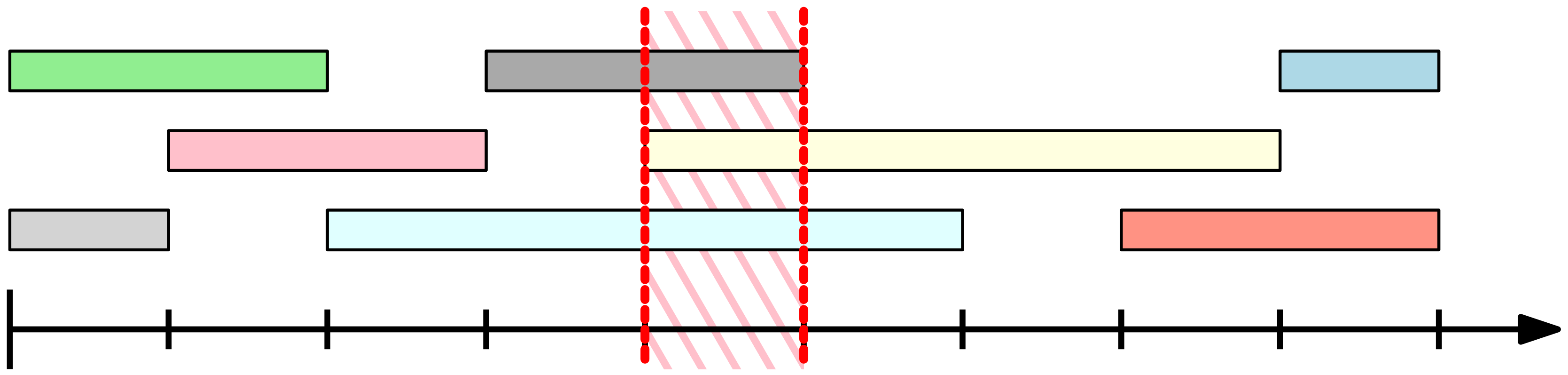

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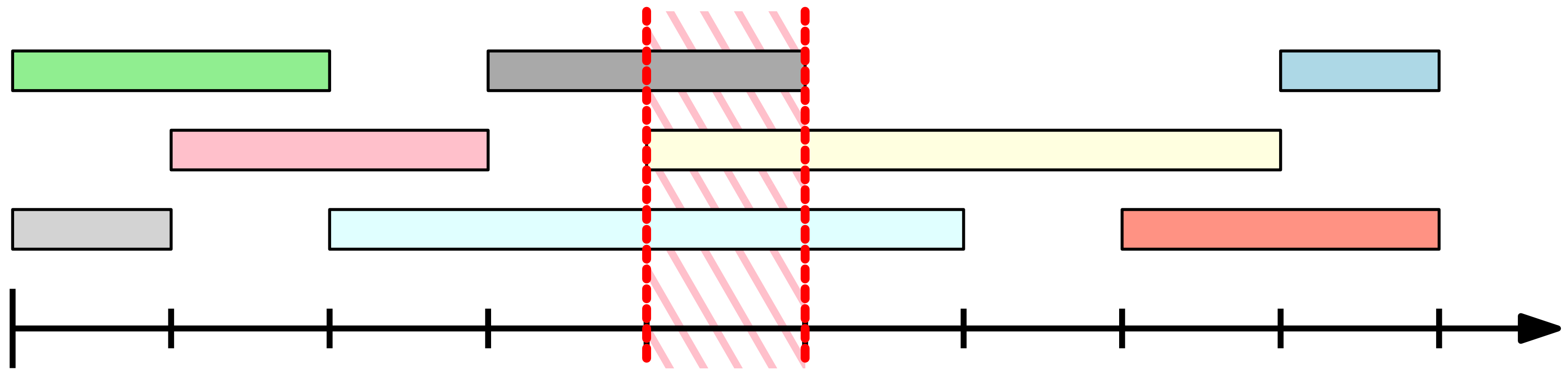
Is $k = 3$ optimal?



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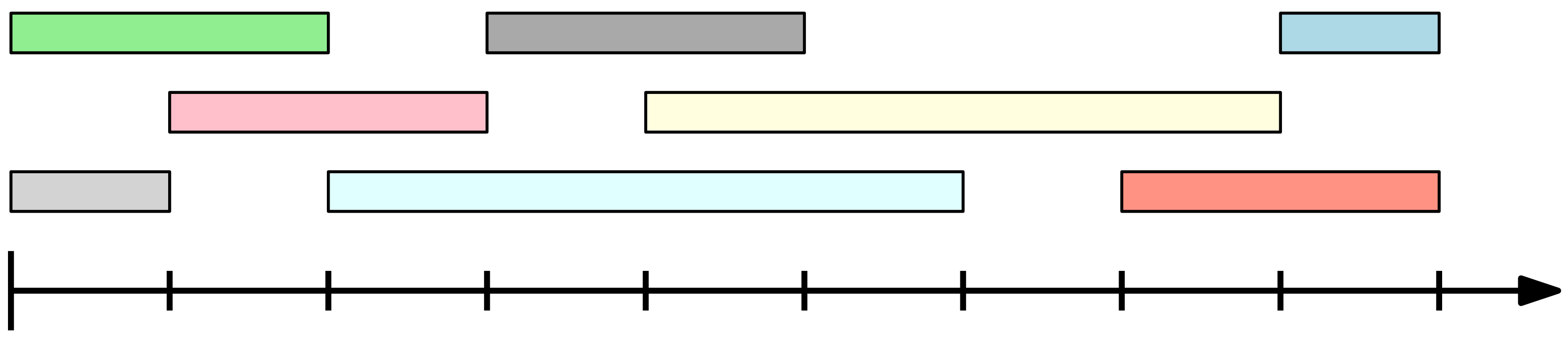
- **Definition:** The *depth* D of a set of intervals is the maximum number of intervals $[s(i), e(i))$ that contain any single point.
- **Observation:** $k^* \geq D$. Is $k^* \leq D$?

A greedy algorithm

- Assume that $\mathcal{J} = \{1, \ldots, n\}$ is sorted w.r.t. $s(\cdot)$.
- Each job $j \in \mathcal{J}$ will get a label $\ell(j) \in \mathbb{N}^+$.
- For $j = 1 \ldots, n$:
    - $C_j \leftarrow$ set of jobs in $1, \ldots, j-1$ that conflict with j.
    - $\ell(j) \leftarrow$ smallest positive integer not in $\{\ell(i) : i \in C_j\}$
- $k \leftarrow \max_{j=1,\ldots,n} \ell(j)$.
- Return a solution on k processors. The jobs assigned to the h-th processor are those in $\{i : \ell(i) = h\}$.

A greedy algorithm



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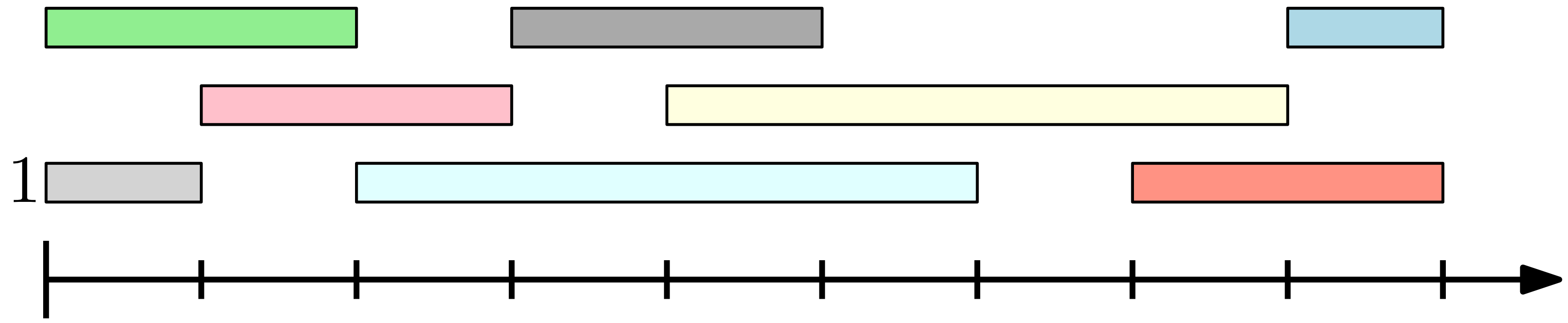

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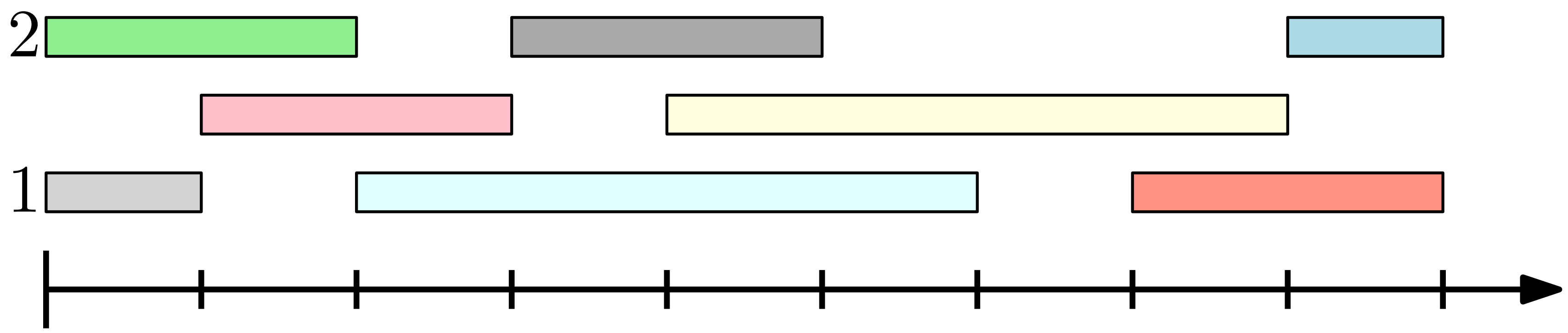

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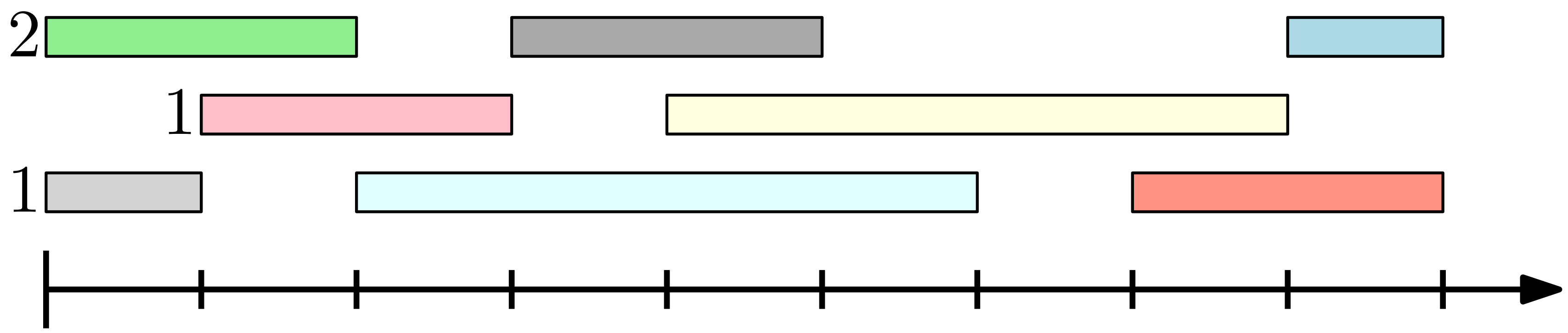

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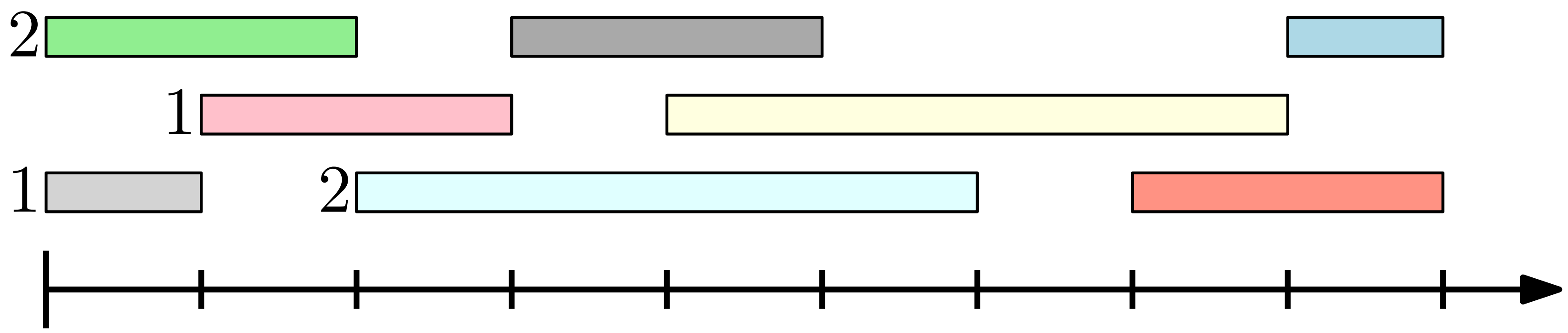

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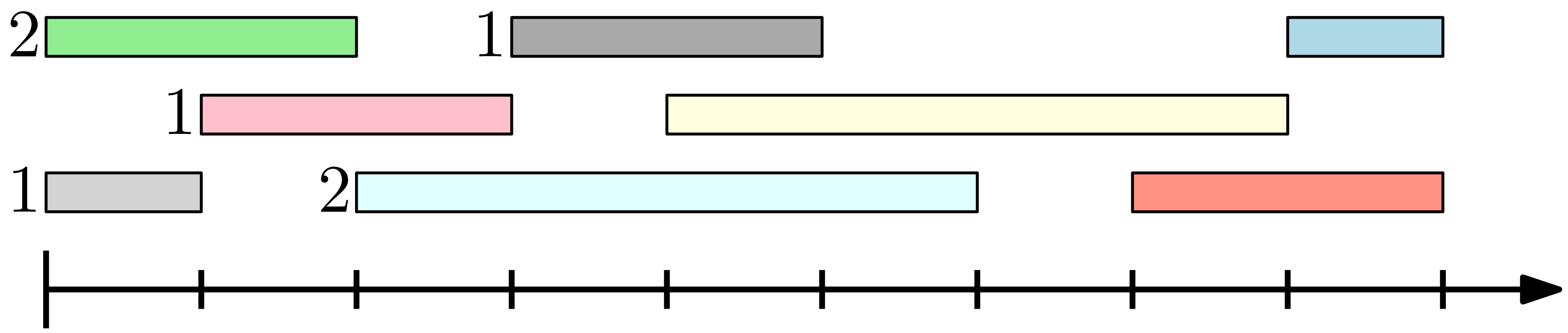

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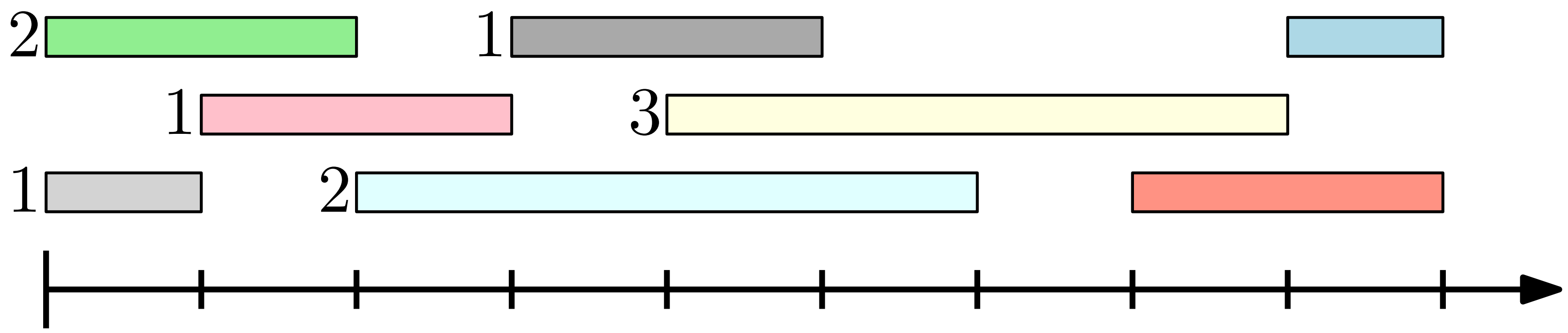

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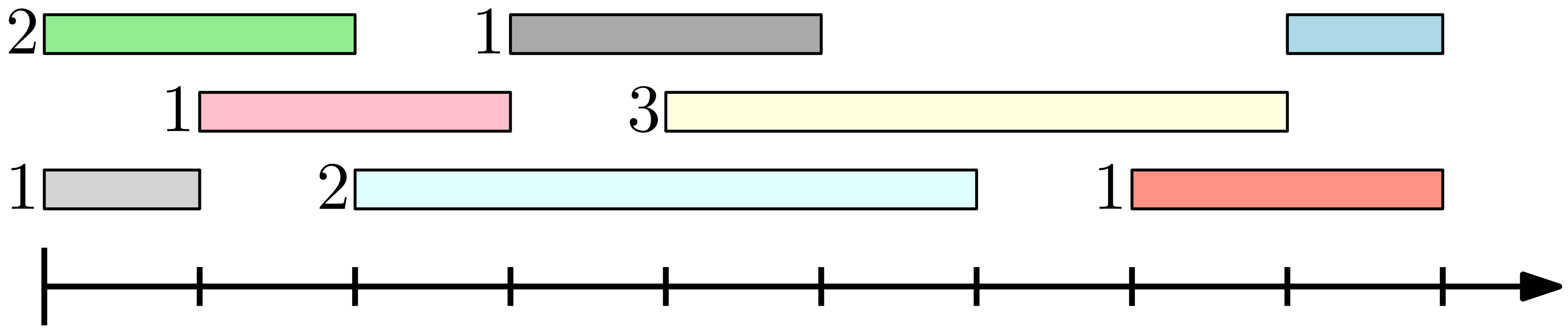

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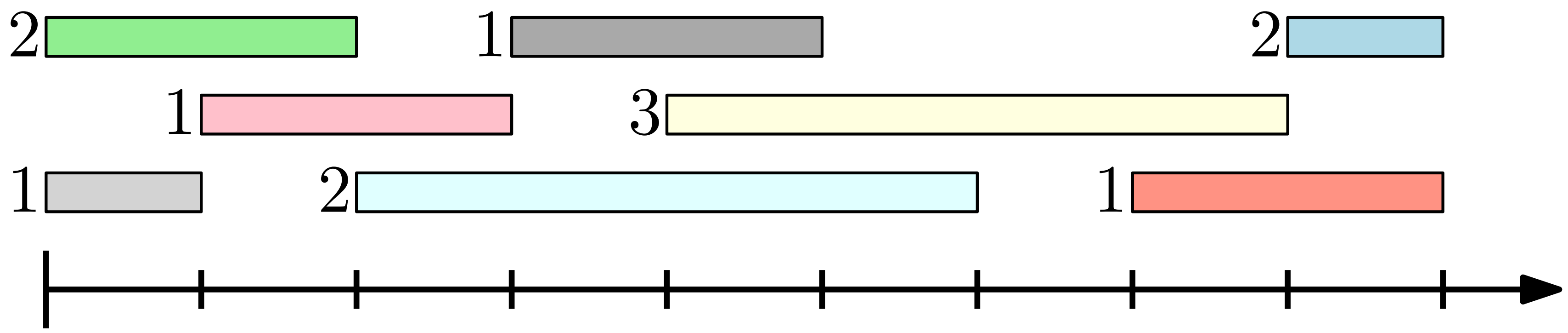

Analysis

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- **Claim:** $k \leq D$.
  - Let j be a job for which $\ell(j) = k$.
  - By the choice of $\ell(j)$: $1, \ldots, k-1 \in \{\ell(i) : i \in C_j\}$
  - For all $i \in C_j$, $e(i) > s(j)$, i.e., $s(j) \in [s(i), e(i))$.
  - $s(j)$ belongs to at least k intervals $\implies D \geq k$ $\square$

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$$k^* \leq k \leq D$$

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$$\left.\begin{array}{r} k^* \leq k \leq D \\ D \leq k^* \end{array}\right\} \implies k = k^* = D$$

Analysis

- **Observation:** $k^* \geq D$.

- **Claim:** $k \leq D$.

Trick/Technique: Finding Structural Properties

Find a structural property that implies optimality. (e.g., a lower bound to the measure of an optimal solution). Prove that greedy returns a solution with that property.

A possible implementation

- Every starting time $s(j)$ or finish time $e(j)$ of a job j is an *event* $\langle s(j), j \rangle$ or $\langle e(j), j \rangle$.

$O(n \log n)$

- Create a sorted list of events. (break ties in favor of ending events)

- $k \leftarrow 0$ (number distinct labels)

- Mantain a min-heap H. (stores unused labels in $\{1, \ldots, k\}$)

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- Create a sorted list of events. (break ties in favor of ending events)
- $k \leftarrow 0$ (number distinct labels)
- Mantain a min-heap H. (stores unused labels in $\{1, \dots, k\}$)
- For each event $\langle t, j \rangle$: $O(n)$
  - If $t = s(j)$
    - If H is empty, increment k and set $\ell(j) \leftarrow k$
    - Otherwise $\ell(j) \leftarrow$ pop from H $O(\log k)$
  - Otherwise $(t = e(j))$:
    - Push $\ell(j)$ into H. $O(\log k)$

A possible implementation

```cpp
struct job { int id; int start; int end; };
std::vector<job> jobs;

//[...] Read jobs

std::vector<std::tuple<int, bool, int>> events;
for(const job &j : jobs)
{
    //Use second entry for tie breaking (false<true)
    events.push_back( std::make_tuple(j.start, true, j.id) );
    events.push_back( std::make_tuple(j.end, false, j.id) );
}

std::sort(events.begin(), events.end());
```

A possible implementation

```cpp
int k=0;
std::vector<int> H; //A min-heap of available labels
std::vector<int> labels(jobs.size()); //Labels assigned to jobs
for(const auto &event : events)
{
    if(std::get<1>(event)) //Start event
    {
        if(H.empty())
            labels[std::get<2>(event)] = ++k;
        else
        {
            std::pop_heap(H.begin(), H.end(), std::greater<int>());
            labels[std::get<2>(event)] = H.back();
            H.pop_back();
        }
    }
    else //End event
    {
        H.push_back(labels[std::get<2>(event)]);
        std::push_heap(H.begin(),  H.end(), std::greater<int>());
    }
}
//labels[i] contains the label of job i
```

Minimizing Lateness

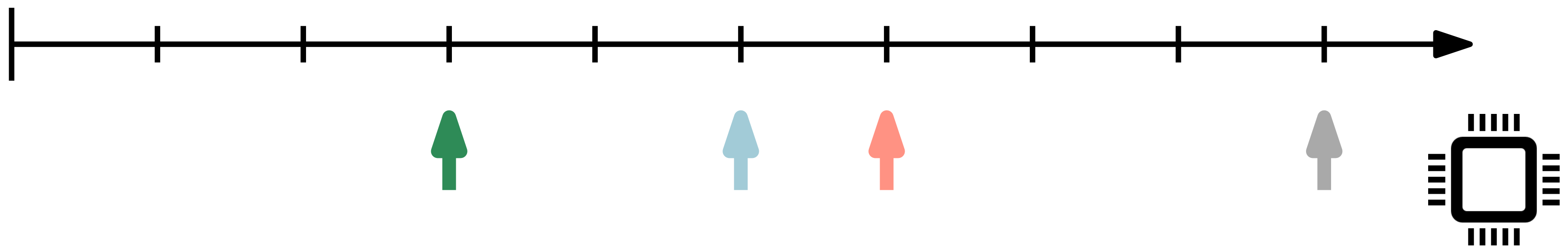
Minimizing Lateness

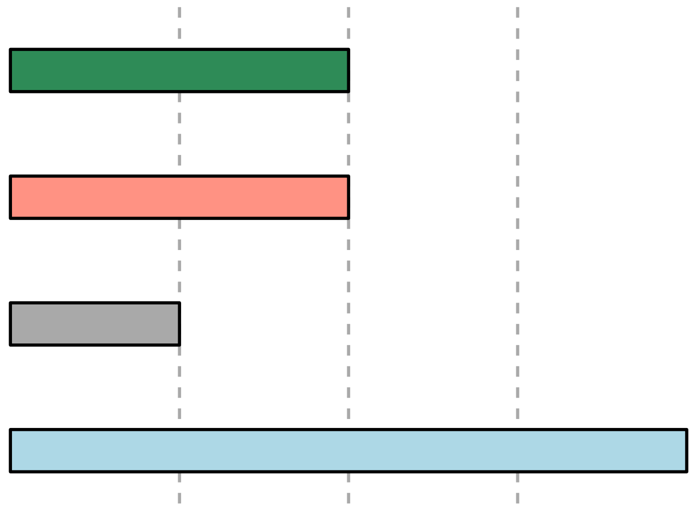
- There are n jobs indexed by $1, \ldots, n$.
- Each job i has a length $t(i)$ and a distinct deadline $d(i)$.
- All jobs have to be scheduled on a single processor (one at a time).
- If job i completes by time $f_i \leq d(i)$ its *lateness* ℓ_i is 0. Otherwise $\ell_i = f_i - d(i)$.

Goal: Find a schedule S minimizing the maximum lateness

$$L(S) = \max_{i=1,\ldots,n} \max\{0, f_i - d(i)\}.$$

Example





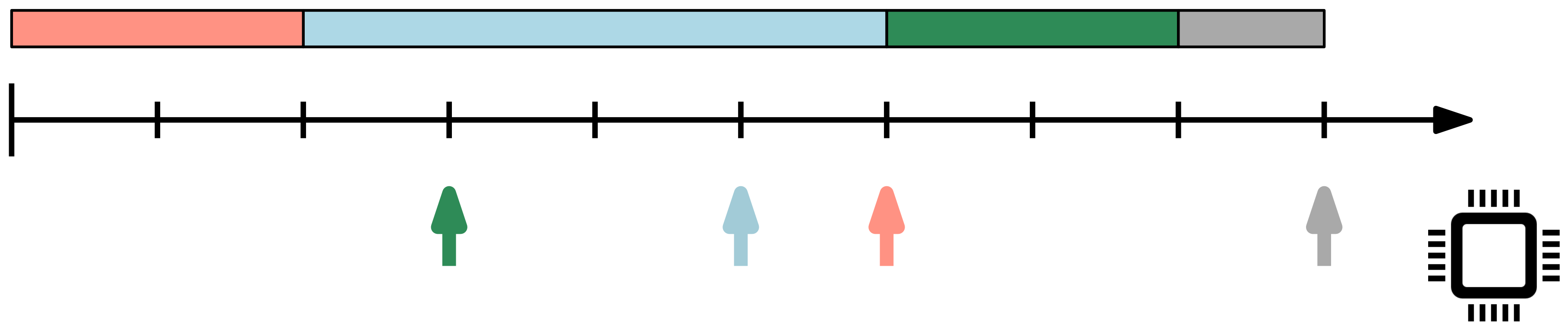
Job 1: $d(1) = 3$

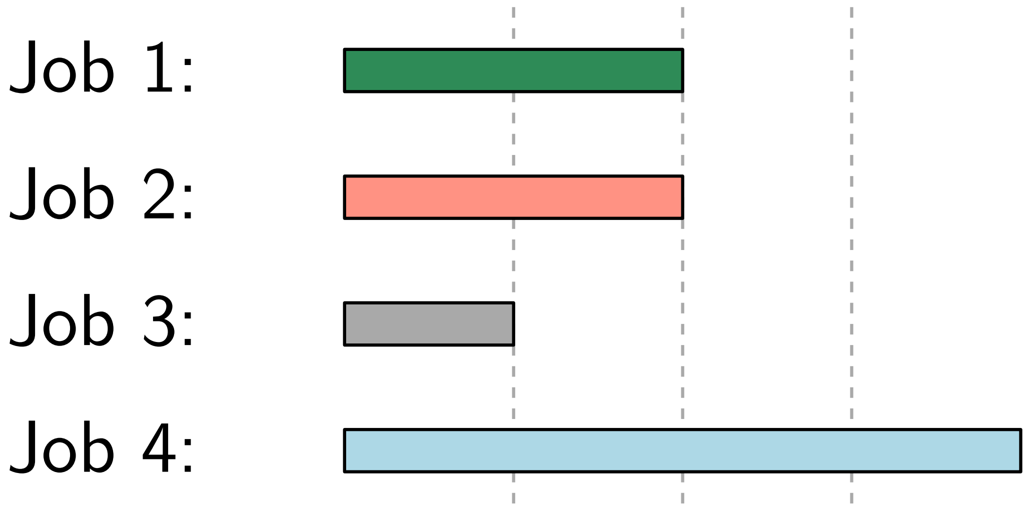
Job 2: $d(2) = 6$

Job 3: $d(3) = 9$

Job 4: $d(4) = 5$

Example





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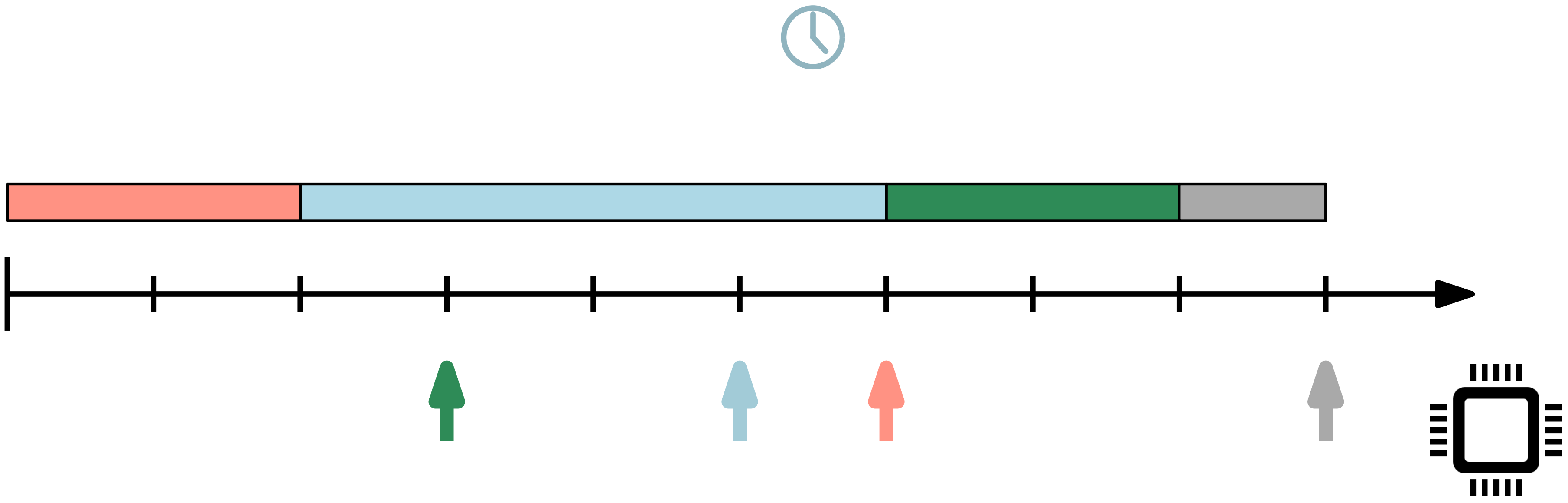

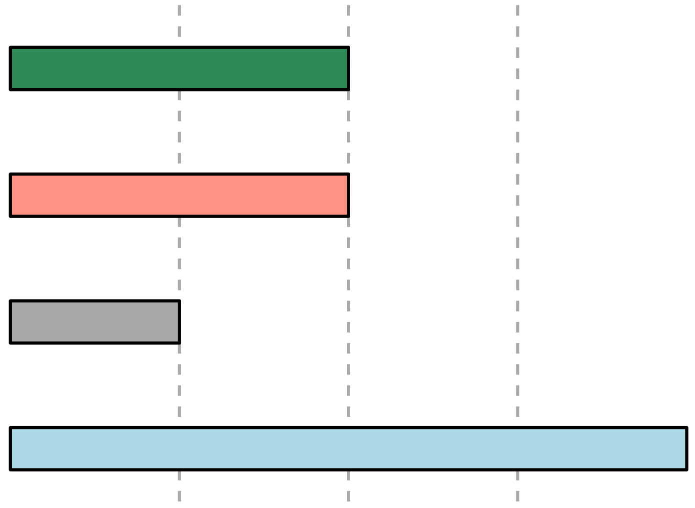

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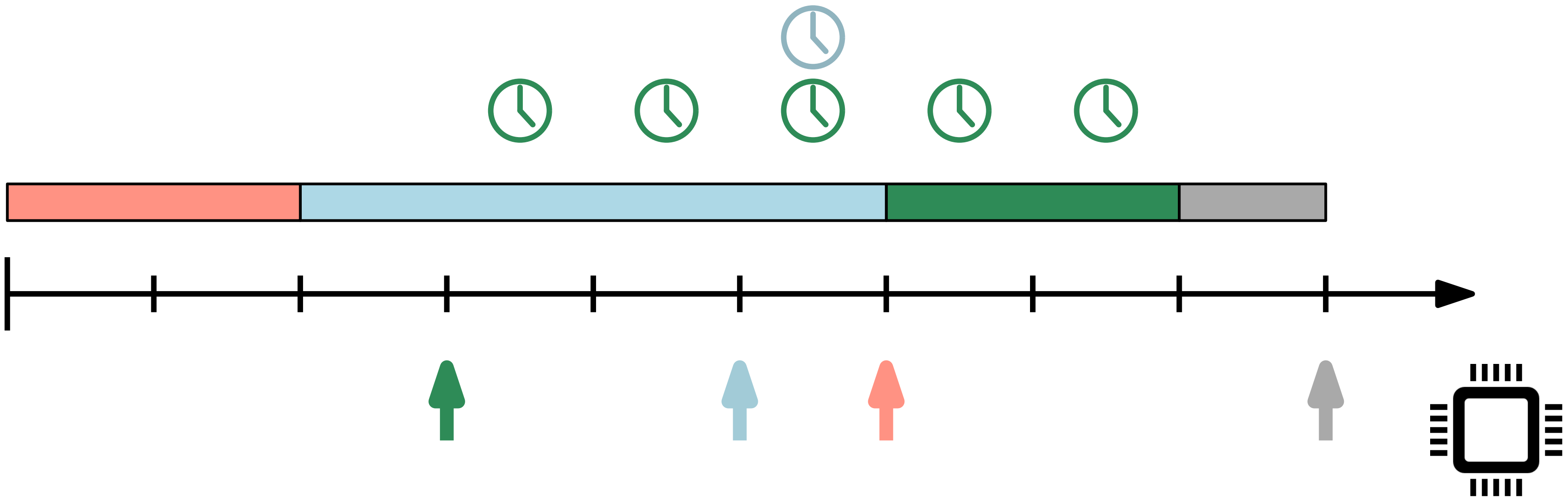

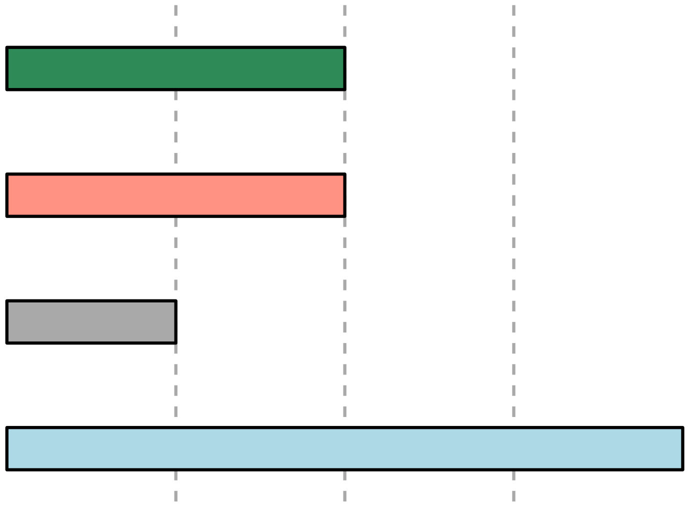

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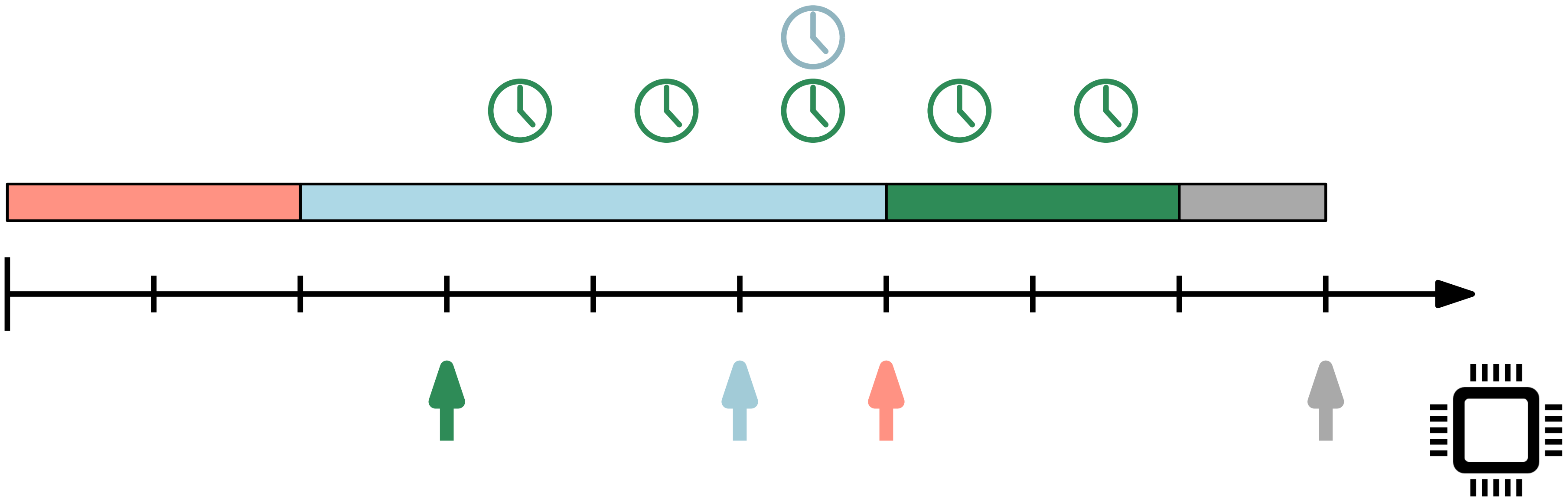
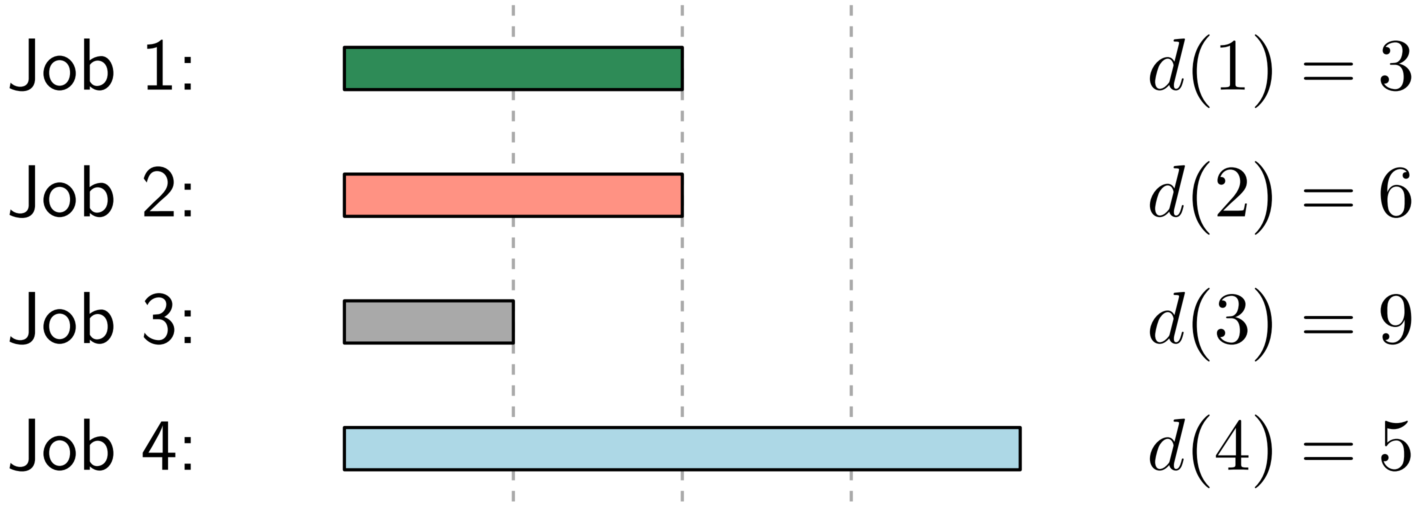


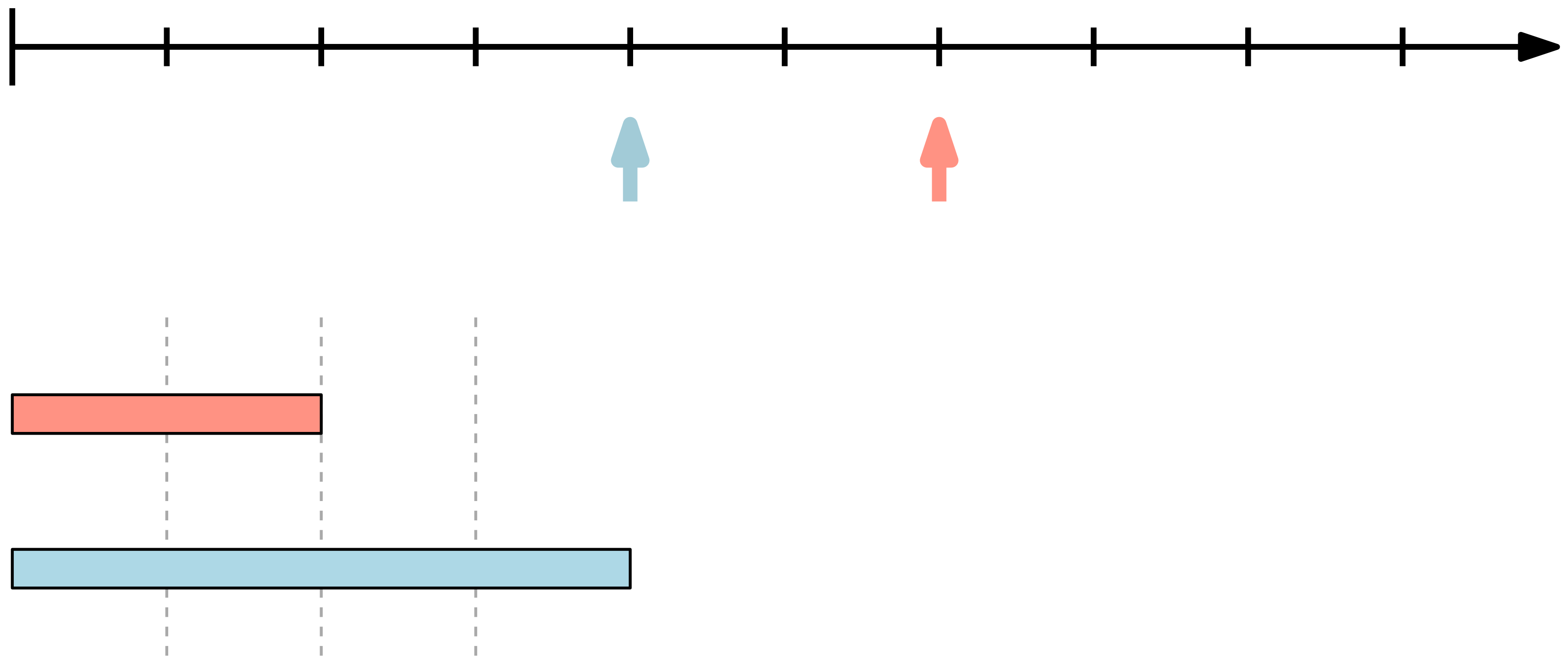
Maximum Lateness: 5



Which order for the jobs?

- **Shortest Job First**: Increasing order of $t(i)$.
- **Shortest Slack Time First:** Increasing order of $d(i) - t(i)$.
- **Earliest Deadline First:** Increasing order of $d(i)$.

Shortest Job First



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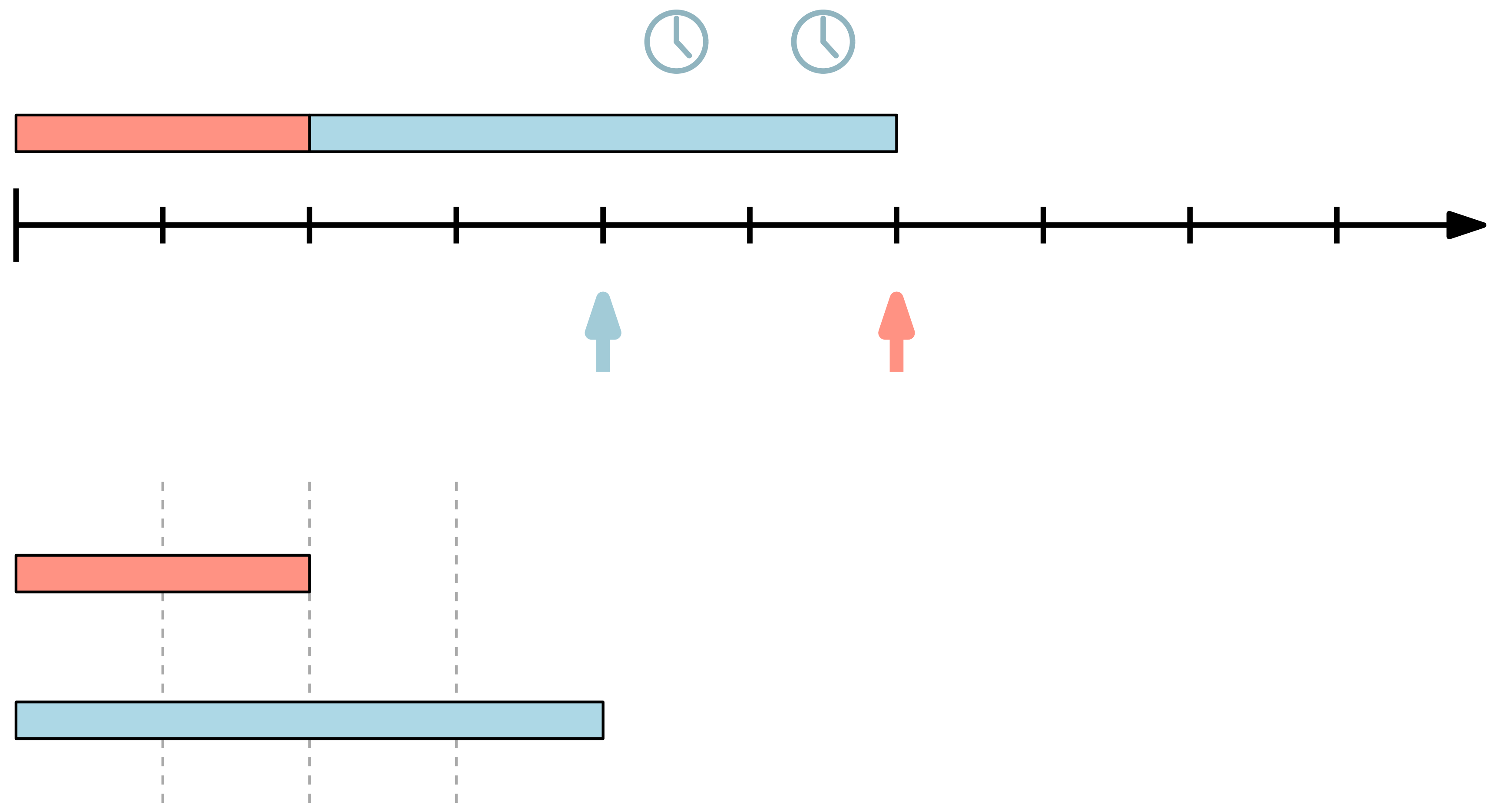

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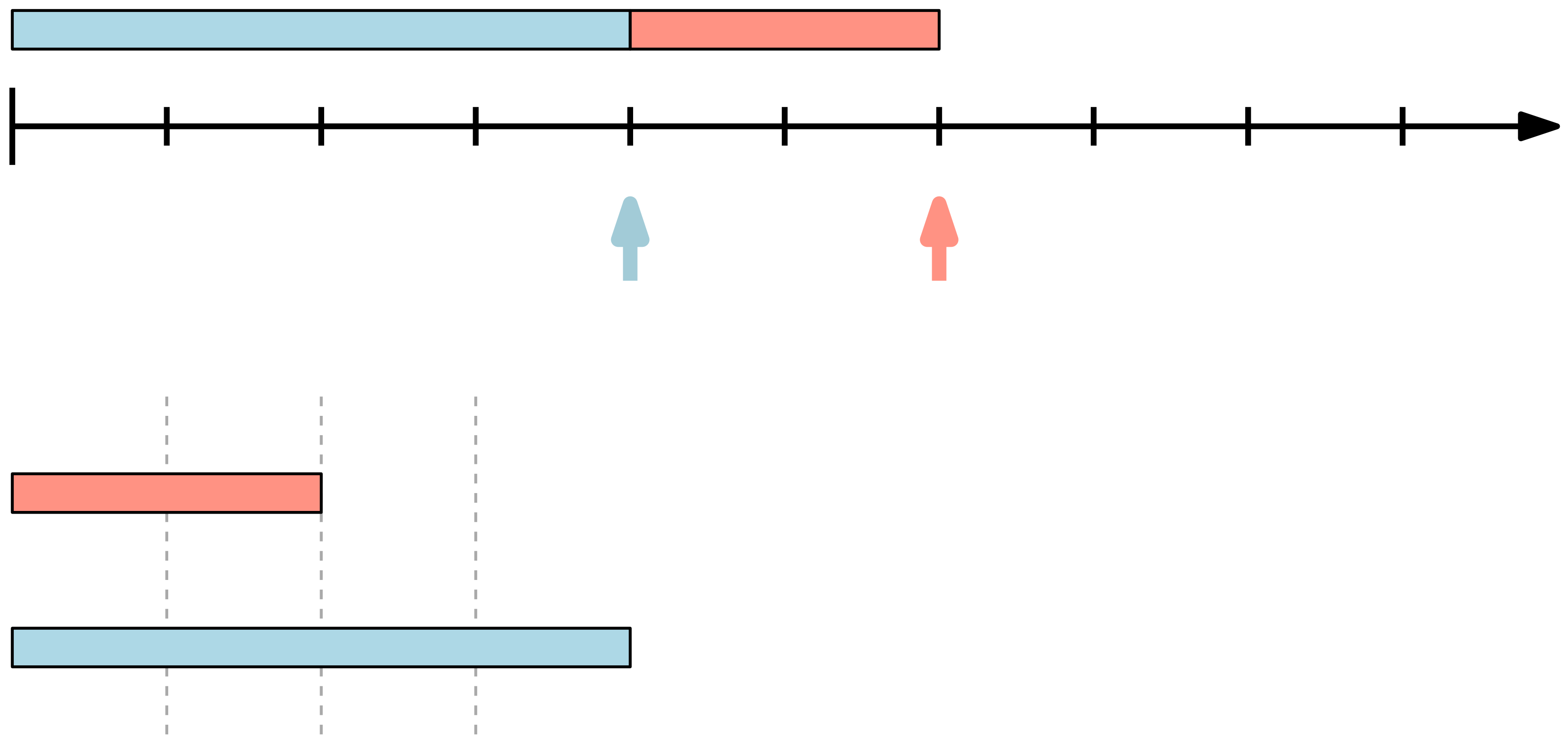
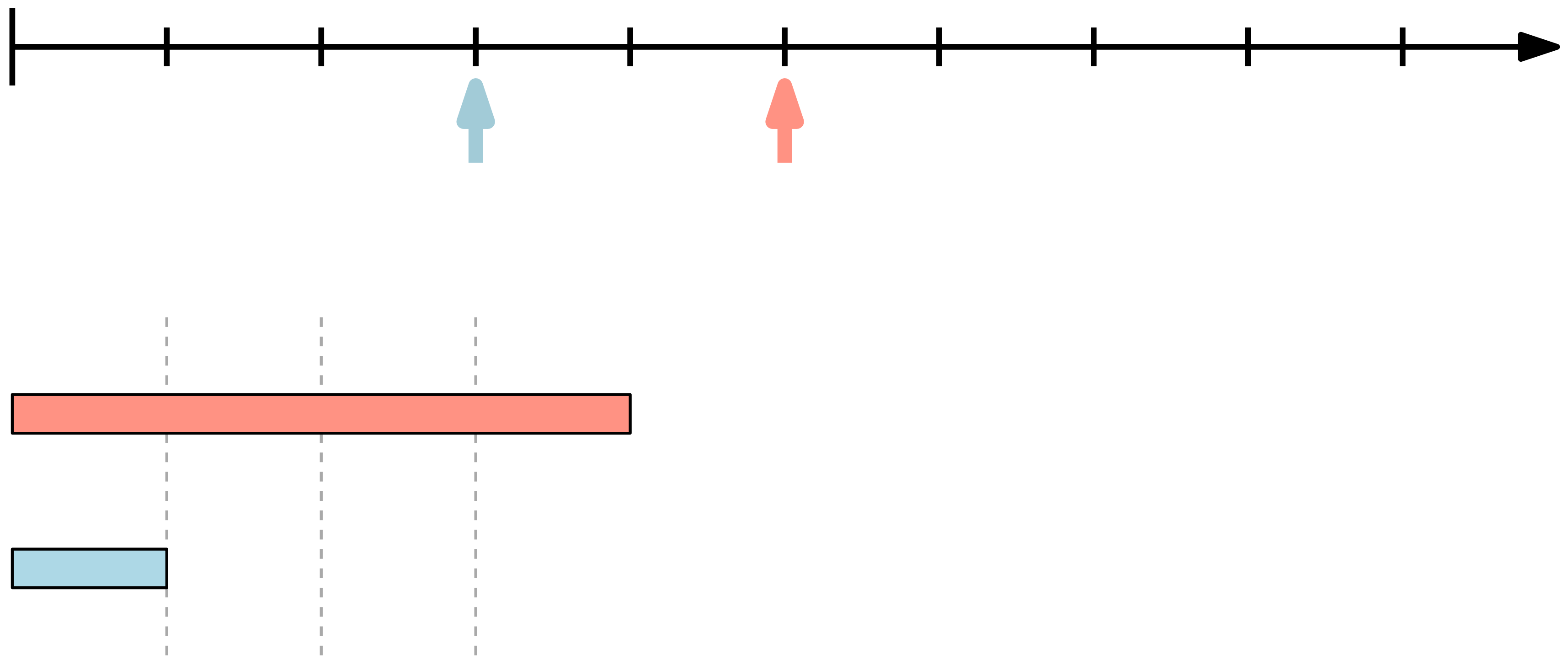
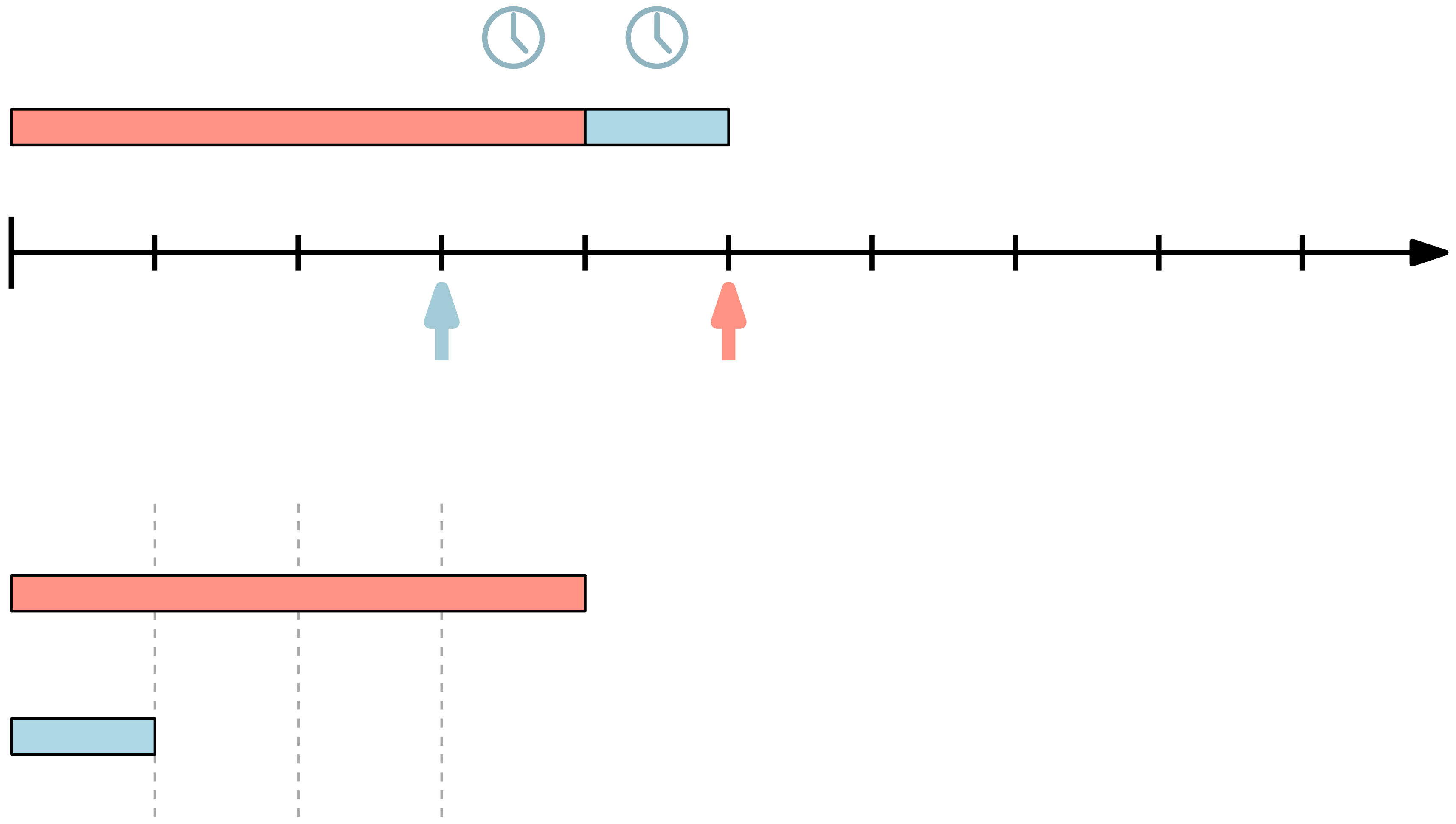


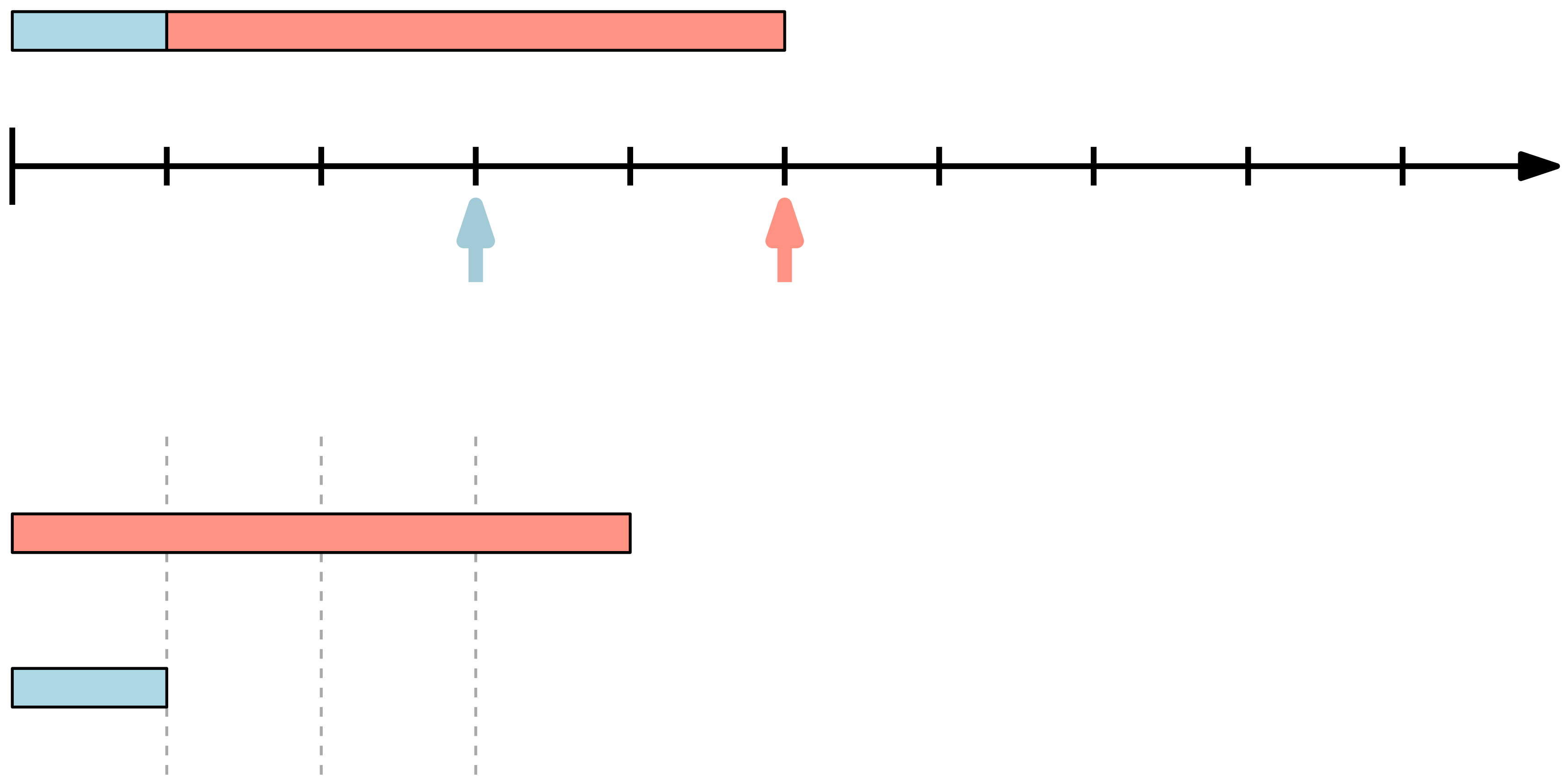
Shortest Slack Time First



Shortest Slack Time First



Shortest Slack Time First



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Earliest Deadline First

The algorithm:

- $\langle j_1, \ldots, j_n \rangle \leftarrow$ sort jobs w.r.t. $d(\cdot)$.
- For $i = 1 \ldots, n$
    - Schedule j_i at time $\sum_{k=1}^{i-1} t(k)$

Earliest Deadline First

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- **Observation:** The greedy schedule has no idle time.

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Proof of correctness:

- **Observation:** The greedy schedule has no idle time.
- **Definition:** An inversion of a schedule S is a pair of jobs (i, j) such that job i is scheduled before job j but $d(i) > d(j)$.
- **Observation:** The greedy schedule has no inversion.

EDF - Proof of Correctness

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- **Claim:** All schedules with no idle time and no inversions are identical.

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Claim: For every optimal schedule S^* there is an optimal schedule S with no idle time and the same number of inversions as S^*.

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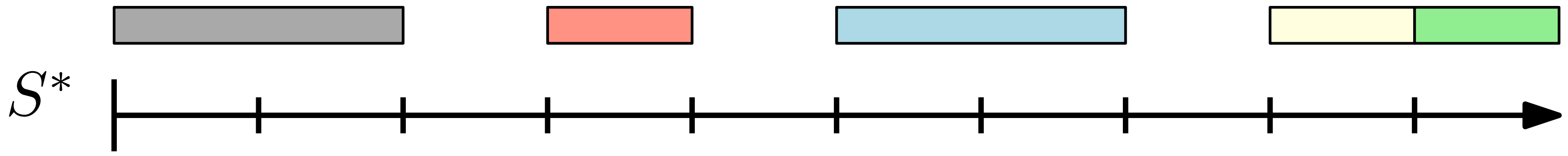
Proof: Let $j_1, \ldots, j_n$ be the sequence of jobs of S^*. Let f_k^* and ℓ_k^* be the finish time and lateness of job k according to S^*, respectively.

Consider the schedule S that excecutes $j_1, \ldots, j_n$ (in order) with no idle time.

Notice that $f_i = \sum_{k=1}^{i} t(j_k) \leq f_i^*$ and hence $\ell_i \leq \ell_i^*$.

S is feasible and has the same inversions as S^*. $\square$



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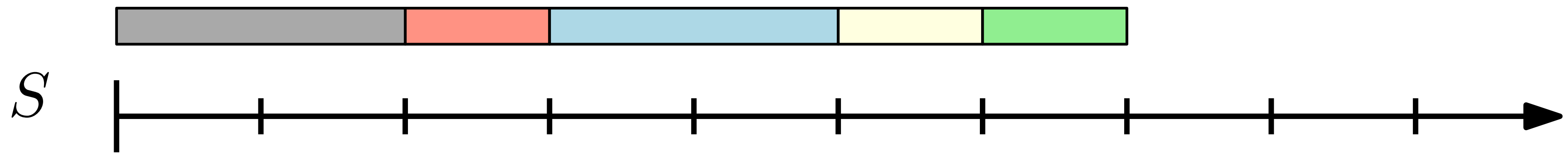

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DONE

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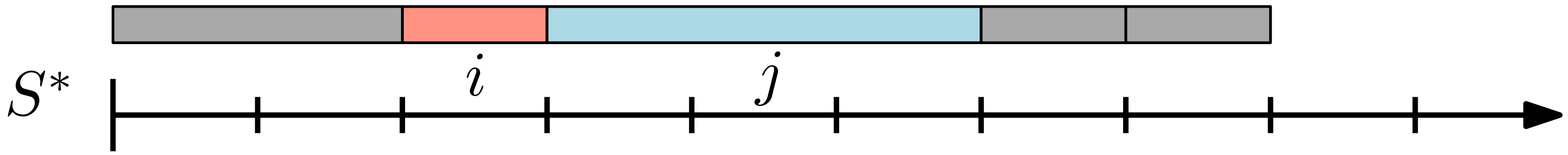

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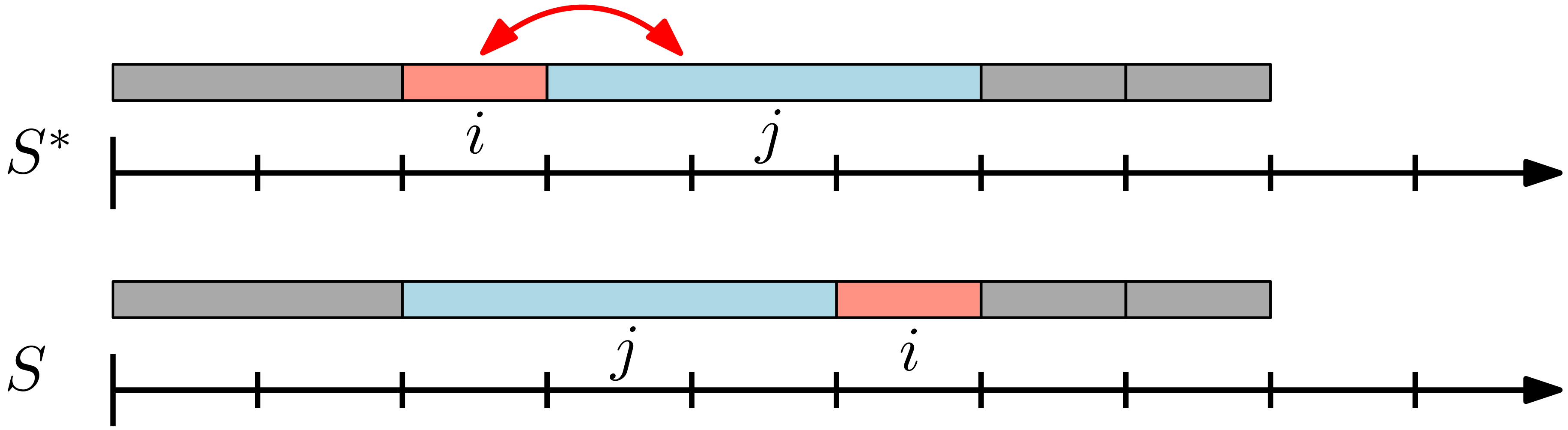
Proof (sketch): S^* must also contain an inversion (i, j) such that no job is scheduled between i and j.

Consider the schedule S obtained by swapping job i with job j.



$f_j < f_j^* \leq d(j) + \ell_j^*$

$f_i = f_j^* \leq d(j) + \ell_j^* < d(i) + \ell_j^*$

$\square$

EDF - Proof of Correctness

Claim: Let S^* be an optimal schedule with no idle time and at least 1 inversion. There is an optimal schedule S with no idle time and less inversions than S^*.

- Pick any optimal schedule S^*
- Initially S^* can have at most $\binom{n}{2}$ inversions.
- Iteratively apply the claim until no inversions are left.
- We have obtained an optimal schedule with no idle time and no inversions.

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This is exactly the greedy schedule!

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Trick/Technique: Exchange Argument

Iteratively transform the optimal solution into the greedy solution without worsening its quality.

Recap

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At each step, the solution produced by greedy is not worse than the one produced by any other algorithm.

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