Binary Knapsack

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Input

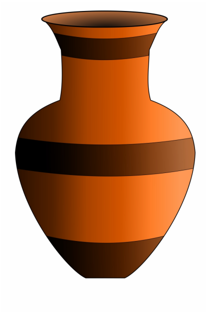
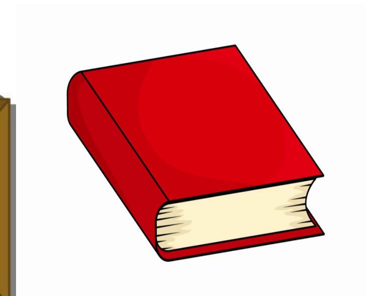
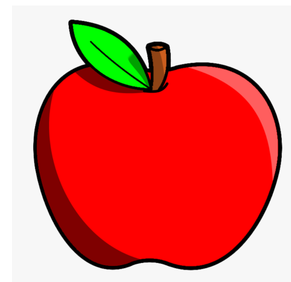
- You are given a collection $\mathcal{I}$ of n items indexed from 1 to n.
- Item i has a weight $w_i : \mathbb{N}^+$ and a value $v_i \in \mathbb{N}^+$.
- You can carry an overall weight of at most $W \in \mathbb{N}$.

Goal

Find a subset of $S \subset \mathcal{I}$ such that:

- Its overall weight $w(S) = \sum_{i \in \mathcal{I}} w_i$ is at most W; and
- Its overall value $v(S) = \sum_{i \in \mathcal{I}} v_i$ is maximized.

Example

1	2	3	4	5	6
					
$w_1 = 12$	$w_2 = 1$	$w_3 = 19$	$w_4 = 14$	$w_5 = 4$	$w_6 = 3$
$v_1 = 15$	$v_2 = 1$	$v_3 = 8$	$v_4 = 20$	$v_5 = 10$	$v_6 = 4$

Maximum Weight: 20

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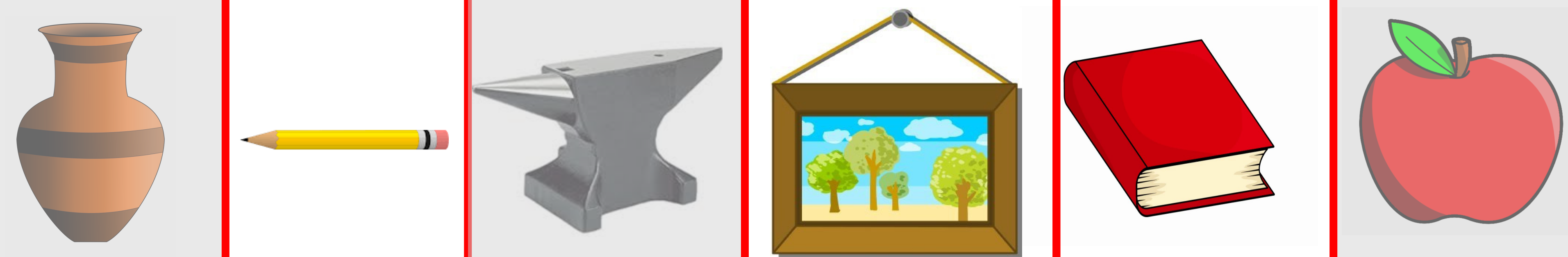

Maximum Weight: 20

$$w(S) = 19$$

$$v(S) = 31$$

A Dynamic Programming Algorithm

Subproblem definition:

$OPT[i, x]$ = Maximum overall value $v(S)$ among all subsets S of $\{1, \ldots, i\}$ such that $w(S) \leq x$.

Base case:

For any $x \geq 0$, $OPT[0, x] = 0$.

A Dynamic Programming Algorithm

Recursive Formula

- Either we ignore item i...

$$OPT[i, x] = OPT[i-1, x]$$

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- Or we select item i and we can still carry a weight of $x - w_i$

$$OPT[i, x] = v_i + OPT[i-1, x - w_i]$$

This is only viable if $x \geq w_i$!

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$$OPT[i, x] = \begin{cases} OPT[i-1, x] & \text{if } x < w_i \\ \max \begin{cases} OPT[i-1, x] \\ v_i + OPT[i-1, x-w_i] \end{cases} & \text{if } x \geq w_i \end{cases}$$

Time Complexity

- $\Theta(n \cdot W)$ subproblems
- Optimal solution in $OPT[n, W]$
- Each problem can be solved in constant time
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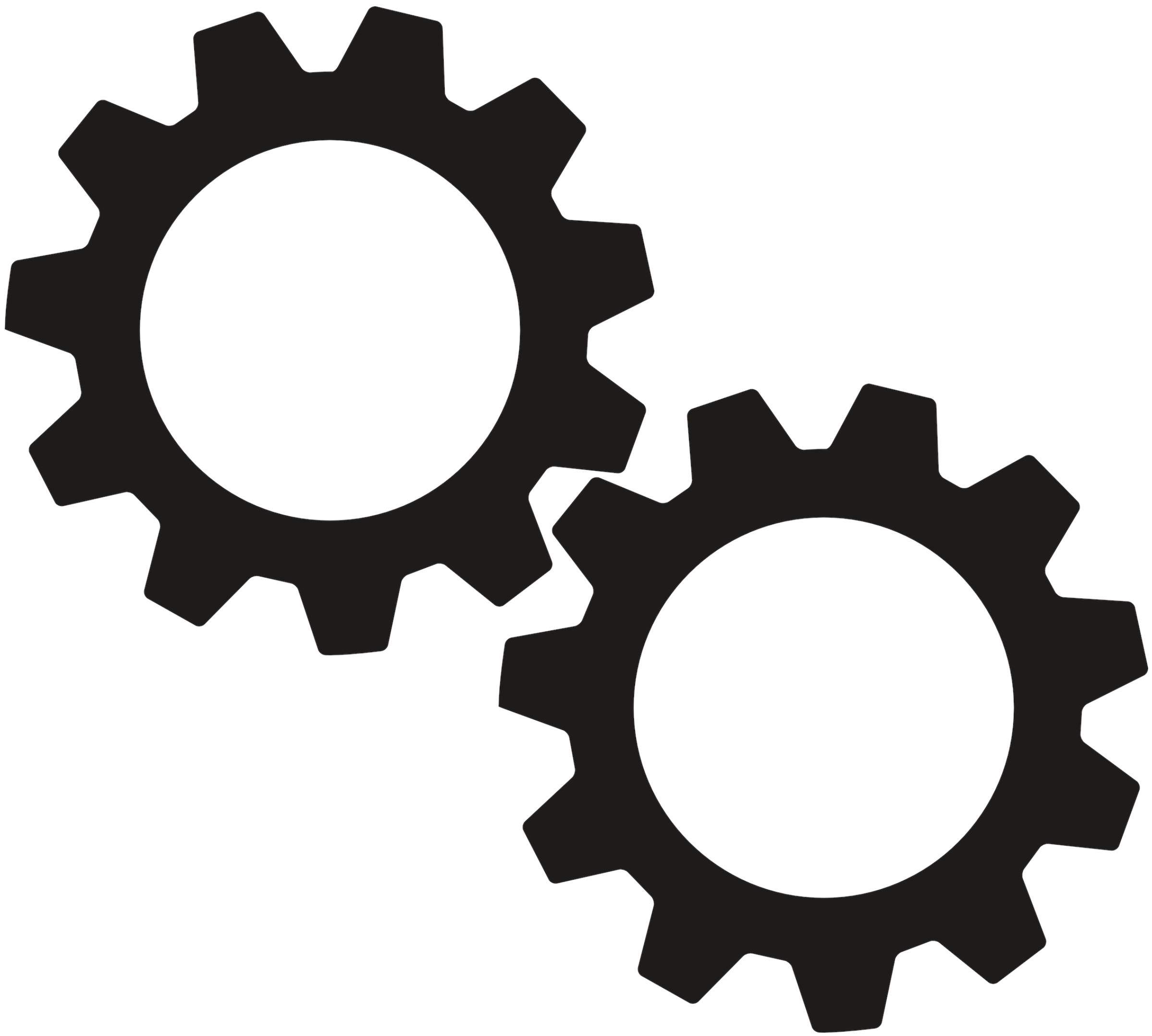
NO!

The input size is $O(n(\log W + \log V))$ where $V = \max_i v_i$

Choose, e.g., $W = 2^n$.

Small maximum value

Can we do better if W is large (e.g., 2^n) and $V = \max_i v_i$ is small?



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Subproblem definition (sketch):

$OPT[i, x] =$ Minimim overall weight $w(S)$ among all subsets S of $\{1, \ldots, i\}$ such that $v(S) \geq x$.

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$OPT[0, 0] = 0$.

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Use "$+\infty$" to encode "not feasible"

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$$OPT[i, x] = w_i + OPT[i-1, \max\{x - v_i, 0\}]$$

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$$OPT[i, x] = w_i + OPT[i-1, \max\{x - v_i, 0\}]$$

$$OPT[i, x] = \min \begin{cases} OPT[i-1, x] \\ w_i + OPT[i-1, \max\{x - v_i, 0\}] \end{cases}$$

Optimal Solution: $V^* = \max_{x\,:\,OPT[n,x] \leq W} x$

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Order of subproblems:

For each $x = 1, 2, \ldots$

Compute $OPT[1, x], OPT[2, x], \ldots, OPT[n, x]$

Stop computing subproblems as a soon as $OPT[n, x] > W$.

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Optimal Solution: $V^* = \max_{x : OPT[n,x] \leq W} x$

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Compute $OPT[1, x], OPT[2, x], \ldots, OPT[n, x]$

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Time complexity

- $\Theta(n \cdot V^*)$ subproblems
- Each problem can be solved in constant time
- **Overall time:** $\Theta(n \cdot V^*) = O(n^2 V)$ where $V = \max_i v_i$

Two Algorithms

- **Dynamic programming:** parameterize weights, store values.

$$O(nW)$$

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$$O(nV^*) = O(n^2V)$$ Good for $W = \Omega(nV)$

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What if V and W are large but there are few items (n is small)?