# Subset Sum

# Subset Sum

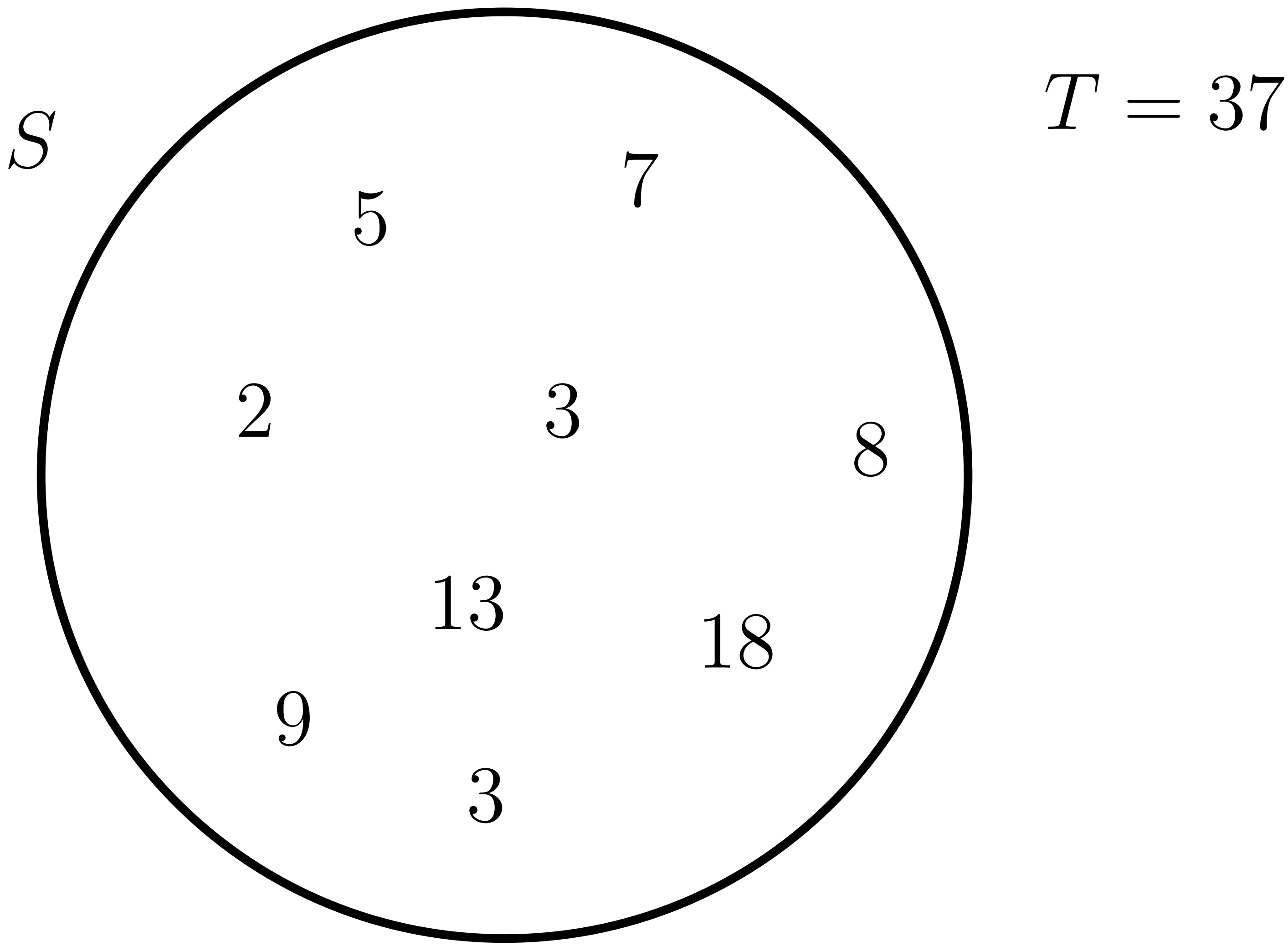
## Input

- A (multi)-set $S \subseteq \mathbb{N}^+$ of $n$ positive integers $s_1, \ldots, s_n$.
- A *target value* $T \in \mathbb{N}^+$.

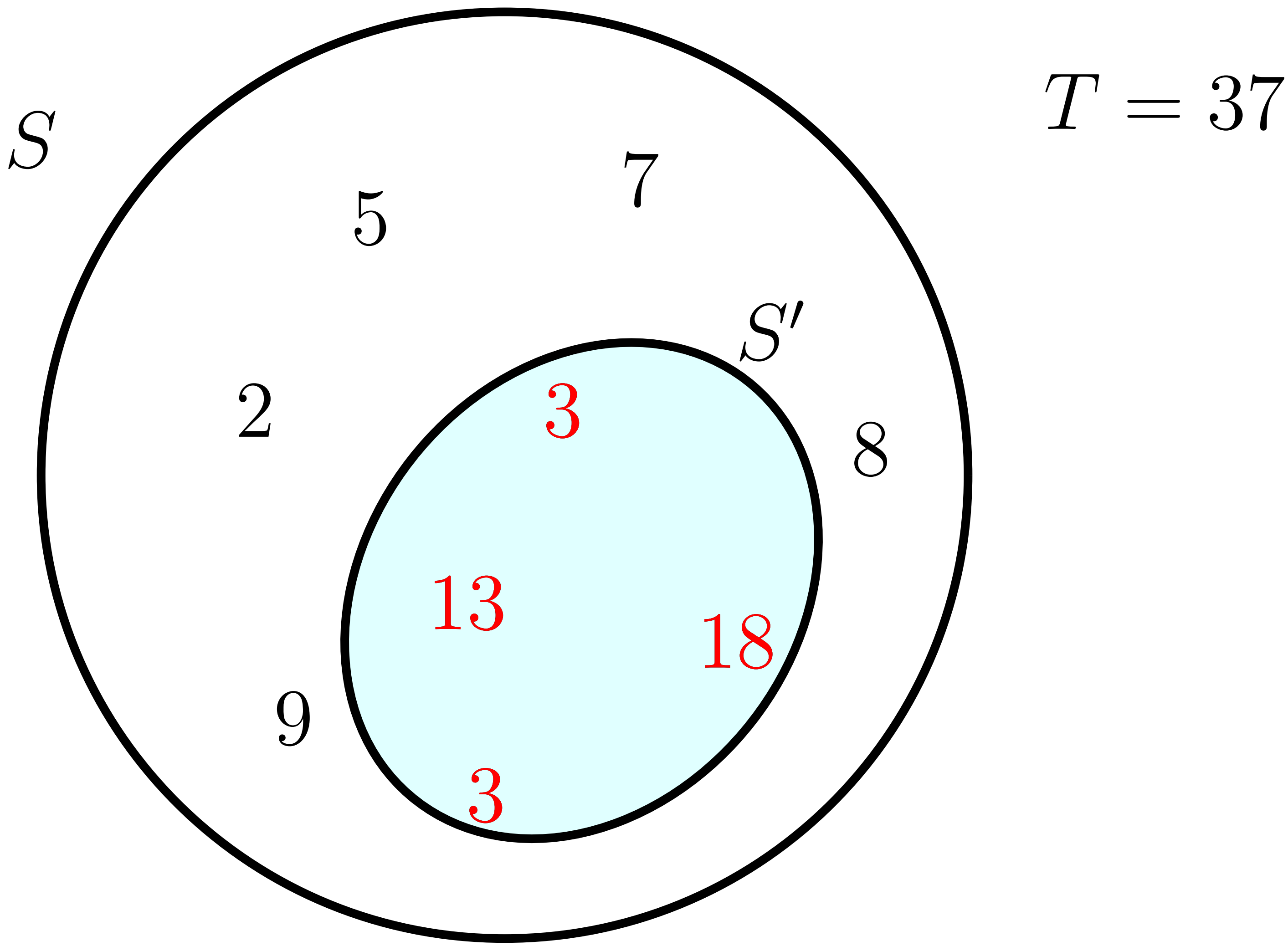
## Question

Is there a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = T$?

# Example



# Example




**Answer:** YES!

# A Dynamic Programming Algorithm

**Subproblem definition:**

$OPT[i,t] = \texttt{true}$ iff $\exists S'' \subseteq \{s_1, \ldots, s_i\}$ such that $\displaystyle\sum_{x \in S''} x = t$.

**Base cases:**

$OPT[0,0] = \texttt{true}$.

$OPT[0,t] = \texttt{false}$, for $t > 0$.

**Recursive formula:**

# A Dynamic Programming Algorithm

**Subproblem definition:**

$OPT[i,t] =$ `true` iff $\exists S'' \subseteq \{s_1, \ldots, s_i\}$ such that $\displaystyle\sum_{x \in S''} x = t$.

**Base cases:**

$OPT[0,0] =$ `true`. $OPT[0,t] =$ `false`, for $t > 0$.

**Recursive formula:**

- Either we ignore $s_i$ $\quad OPT[i,t] = OPT[i-1,t]$
- Or we include $s_i$ in $S''$... $\quad OPT[i,t] = OPT[i-1,t-s_i]$

(as long as $t \geq s_i$)

# A Dynamic Programming Algorithm

**Subproblem definition:**

$OPT[i,t] =$ `true` iff $\exists S'' \subseteq \{s_1, \ldots, s_i\}$ such that $\sum_{x \in S''} x = t$.

**Base cases:**

$OPT[0,0] =$ `true`. $OPT[0,t] =$ `false`, for $t > 0$.

**Recursive formula:**

$$OPT[i,t] = \begin{cases} OPT[i-1,t] & \text{if } t < s_i \\ OPT[i-1,t] \vee OPT[i-1,t-s_i] & \text{if } t \geq s_i \end{cases}$$

# Time Complexity

- $\Theta(n \cdot T)$ subproblems
- Each problem can be solved in constant time
- **Overall time:** $\Theta(n \cdot T)$

Is this a polynomial-time algorithm?

# Time Complexity

- $\Theta(n \cdot T)$ subproblems
- Each problem can be solved in constant time
- **Overall time:** $\Theta(n \cdot T)$

Is this a polynomial-time algorithm?

NO!

The input size is $O(n \log T)$ (Under reasonable assumptions)

Choose, e.g., $T = 2^n$.

This is called a *pseudo*-polynomial-time algorithm.

# Can we do better?

- Subset Sum is a well-known NP-complete problem.
- A polynomial-time algorithm for Subset Sum would imply P$=$NP. ☹
- Let's give up on polynomial-time algorithms and look at exponential algorithms.
- **Easy exercise:** come up with an algorithm with time complexity $O^*(2^n)$.

  (OK for $n \approx 25$)
- Can the exponent be improved?

# Can we do better?

- Subset Sum is a well-known NP-complete problem.
- A polynomial-time algorithm for Subset Sum would imply P=NP. ☹
- Let's give up on polynomial-time algorithms and look at exponential algorithms.
- **Easy exercise:** come up with an algorithm with time complexity $O^*(2^n)$.

(OK for $n \approx 25$)

- C

$O^*(2^n)$ is a shorthand for $O(2^n \cdot \text{poly}(n))$.

# Split & List

# Split & List

Partition $S$ into $S_1$ and $S_2$.

**Observation:** The following two statements are equivalent:

- $\exists S' \subseteq S$ such that $\sum_{x \in S'} x = T$; and
- $\exists S'_1 \subseteq S_1, S'_2 \subseteq S_2$ such that $\sum_{x \in S'_1} x + \sum_{x \in S'_2} x = T$.

**Idea:** Check whether the second statement hold.

How does this help?

# The Algorithm

- Partition $S$ into $S_1$ and $S_2$.

- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of $S_1$.
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of $S_2$.
- $T_2 \leftarrow$ Sort $T_2$.

- For each $t \in T_1$
  - Check whether $T - t \in T_2$

# The Algorithm

- Partition $S$ into $S_1$ and $S_2$. $O(n)$

- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of $S_1$. $O(2^{|S_1|})$
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of $S_2$. $O(2^{|S_2|})$
- $T_2 \leftarrow$ Sort $T_2$. $O(|S_2| \cdot 2^{|S_2|})$

- For each $t \in T_1$ $|T_1| = O(2^{|S_1|})$
  - Check whether $T - t \in T_2$ $O(\log |T_2|) = O(|S_2|)$

$$O\left(|S_2| \cdot 2^{|S_1|} + |S_2| \cdot 2^{|S_2|}\right) = O^*\left(2^{|S_1|} + 2^{|S_2|}\right)$$

# The Algorithm

- Partition $S$ into $S_1$ and $S_2$. $O(n)$
- $T_1 \leftarrow$ Set of the sums of *all possible* subsets of $S_1$. $O(2^{|S_1|})$
- $T_2 \leftarrow$ Set of the sums of *all possible* subsets of $S_2$. $O(2^{|S_2|})$
- $T_2 \leftarrow$ Sort $T_2$. $O(|S_2| \cdot 2^{|S_2|})$
- For each $t \in T_1$ $|T_1| = O(2^{|S_1|})$
  - Check whether $T - t \in T_2$ $O(\log |T_2|) = O(|S_2|)$

Choosing $|S_1| = \lfloor \frac{n}{2} \rfloor$ and $|S_2| = \lceil \frac{n}{2} \rceil$:

$$O^* \left( 2^{|S_1|} + 2^{|S_2|} \right) = O^* \left( 2^{n/2} + 2^{n/2} \right) = O^*(2^{\frac{n}{2}})$$

# Intermission: Generating All Subsets

- Let $S$ be a set of $n$ elements, where $n$ is *small*.
- **Option 1:** use integers to encode the characteristic vectors of all subsets $S' \subseteq S$

$$
\begin{array}{rccccccccccc}
S = \{ & 2, & 5, & 3, & 13, & 7, & 8, & 9, & 18, & 3 & \} \\
x = 0b & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \\
S' = \{ & & 5, & 3, & & & 8, & & 18 & & \}
\end{array}
$$

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> sums(nsums, 0);
for(uint64_t x=0; x<nsums; x++)
    for(unsigned int i=0; i<S.size(); i++)
        sums[x] += ( (x>>i) & 1u )?S[i]:0;
```

Time: $O(n \cdot 2^n)$ $\approx$ 2s for $n = 25$

# Intermission: Generating All Subsets

- Let $S$ be a set of $n$ elements, where $n$ is *small*.
- **Option 1:** use integers to encode the characteristic vectors of all subsets $S' \subseteq S$

$$
\begin{array}{lccccccccccc}
S = \{ & 2, & 5, & 3, & 13, & 7, & 8, & 9, & 18, & 3 & \} \\
x = 0b & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \\
S' = \{ & & 5, & 3, & & & 8, & & 18 & & \}
\end{array}
$$

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> sums(nsums, 0);
for(uint64_t x=0; x<nsums; x++)
    for(unsigned int i=0; i<S.size(); i++)
        sums[x] += ( (x>>i) & 1u ) * S[i];
```

Time: $O(n \cdot 2^n)$ $\approx$ 0.75s for $n = 25$

# Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0$$

sum $= 34$

# Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1$$

sum $= 37$

# Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ \ 5\ ,\ \ 3\ ,\ \ 13\ ,\ \ 7\ ,\ \ 8\ ,\ \ 9\ ,\ \ 18\ ,\ \ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad \color{red}{1} \quad \color{red}{0} \quad \color{red}{0}$$

sum $= \color{red}{25}$

# Intermission: Generating All Subsets

- **Option 2:** explicitly maintain the characteristic vector.
- *Update* the previous sum when the characteristic vector changes.

$$S = \{\ 2\ ,\ 5\ ,\ 3\ ,\ 13\ ,\ 7\ ,\ 8\ ,\ 9\ ,\ 18\ ,\ 3\ \}$$

$$0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0$$

sum $= 25$

Time complexity?

# Intermission: Generating All Subsets

| $b_{n-1}$ | | | $\ldots$ | | | $b_2$ | $b_1$ | $b_0$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |

- $b_0$ flips at every iteration
- $b_1$ flips every 2 iterations
- $b_2$ flips every 4 iterations
- $\ldots$
- $b_i$ flips every $2^i$ iterations

# Intermission: Generating All Subsets

| $b_{n-1}$ | | | $\ldots$ | | | $b_2$ | $b_1$ | $b_0$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |

- $b_0$ flips at every iteration
- $b_1$ flips every 2 iterations
- $b_2$ flips every 4 iterations
- $\ldots$
- $b_i$ flips every $2^i$ iterations

Total # of operations (including updates to sum) $\propto$ # bit flips

$$\sum_{i=0}^{n-1} \frac{2^n}{2^i} = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2 = \Theta(2^n).$$

# Generating All Subsets

```
uint64_t nsums = static_cast<uint64_t>(1)<<S.size(); //2^n
std::vector<int> sums(nsums);
std::vector<bool> bits(S.size());

for(uint64_t i=1; i<nsums; i++)
{
    sums[i] = sums[i-1];

    int j=0;
    while(bits[j])
    {
        bits[j] = 0;
        sums[i] -= S[j];
        j++;
    }

    bits[j]=1;
    sums[i] += S[j];
}
```

$\approx$ 0.2s for $n = 25$

# Back to Subset Sum: Which Algorithm?

| Dynamic Programming | Split and List |
| --- | --- |
| $O(n \cdot T)$ | $O(n \cdot 2^{\frac{n}{2}})$ |
| $T \leq 2^{\frac{n}{2}}$ | $T \geq 2^{\frac{n}{2}}$ |
| OK for "small" $T$ | OK for $n \leq 50$, regardless of $T$ |

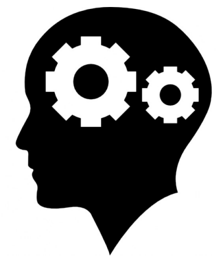
# Split & List

- Split input into two sets $S_1$, $S_2$
- Explicitly compute all possible (partial) solutions w.r.t. $S_1$ and $S_2$

**Brute force!**

- Combine the solutions of $S_1$ with those of $S_2$

**Quicker than brute force**



Can we split into $3$ sets?

# Binary Knapsack

# Binary Knapsack

## Input

- You are given a collection $\mathcal{I}$ of $n$ items indexed from $1$ to $n$.
- Item $i$ has a weight $w_i : \mathbb{N}^+$ and a value $v_i \in \mathbb{N}^+$.
- You can carry an overall weight of at most $W \in \mathbb{N}$.

## Goal

Find a subset of $S \subset \mathcal{I}$ such that:

- Its overall weight $w(S) = \sum_{i \in \mathcal{I}} w_i$ is at most $W$; and
- Its overall value $v(S) = \sum_{i \in \mathcal{I}} v_i$ is maximized.

# Two Algorithms

- **Dynamic programming:** parameterize weights, store values.

$$O(nW)$$ Good for $W = O(nV)$

- **Dynamic programming:** parameterize values, store weights.

$$O(nV^*) = O(n^2V)$$ Good for $W = \Omega(nV)$

Neither algorithm runs in polynomial-time!

**What if $V$ and $W$ are large but there are few items ($n$ is small)?**

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$.

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$.

- **List:** Let $L_1$ (resp. $L_2$) be the list of all the pairs $(w(X), v(X))$ for each $X \subseteq S_1$ (resp. $X \subseteq S_2$).

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$.
- **List:** Let $L_1$ (resp. $L_2$) be the list of all the pairs $(w(X), v(X))$ for each $X \subseteq S_1$ (resp. $X \subseteq S_2$).
- Sort $L_2$ in lexicographic order.
- For each $(w, \cdot) \in L_2$, add a pair $(w, v)$ in $L_2'$, where $v = \max\{v' \, : \, (w', v') \in L_2, w' \leq w\}$

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$.
- **List:** Let $L_1$ (resp. $L_2$) be the list of all the pairs $(w(X), v(X))$ for each $X \subseteq S_1$ (resp. $X \subseteq S_2$).
- Sort $L_2$ in lexicographic order.
- For each $(w, \cdot) \in L_2$, add a pair $(w, v)$ in $L'_2$, where $v = \max\{v' : (w', v') \in L_2, w' \leq w\}$
- For each pair $(w, v) \in L_1$ such that $w \leq W$:
  - Binary search for the last pair $(w', v') \in L'_2$ for which $w' \leq W - w$, if any.
  - If $w'$ exists, $v + v'$ is the value of a candidate solution.

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$.
- **List:** Let $L_1$ (resp. $L_2$) be the list of all the pairs $(w(X), v(X))$ for each $X \subseteq S_1$ (resp. $X \subseteq S_2$).
- Sort $L_2$ in lexicographic order.
- For each $(w, \cdot) \in L_2$, add a pair $(w, v)$ in $L_2'$, where $v = \max\{v' \,:\, (w', v') \in L_2, w' \leq w\}$
- For each pair $(w, v) \in L_1$ such that $w \leq W$:
  - Binary search for the last pair $(w', v') \in L_2'$ for which $w' \leq W - w$, if any.
  - If $w'$ exists, $v + v'$ is the value of a candidate solution.
- **Return:** Best candidate solution, if any.

# A Split & List Algorithm

- **Split:** let $S_1 = \{1, \ldots, \lceil n/2 \rceil\}$ and $S_2 = S \setminus S_1$. $O(n)$
- **List:** Let $L_1$ (resp. $L_2$) be the list of all the pairs $(w(X), v(X))$ for each $X \subseteq S_1$ (resp. $X \subseteq S_2$). $O(2^{\frac{n}{2}})$
- Sort $L_2$ in lexicographic order. $O(n \cdot 2^{\frac{n}{2}})$
- For each $(w, \cdot) \in L_2$, add a pair $(w, v)$ in $L_2'$, where $v = \max\{v' : (w', v') \in L_2, w' \leq w\}$ $O(2^{\frac{n}{2}})$
- For each pair $(w, v) \in L_1$ such that $w \leq W$: $O(2^{\frac{n}{2}})$
  - Binary search for the last pair $(w', v') \in L_2'$ for which $w' \leq W - w$, if any. $O(n)$
  - If $w'$ exists, $v + v'$ is the value of a candidate solution.
- **Return:** Best candidate solution, if any.

# Three Algorithms

- **Dynamic programming:** parameterize weights, store values.

$$O(nW)$$

- **Dynamic programming:** parameterize values, store weights.

$$O(nV^*) = O(n^2V)$$

- **Split & List:**

$$O(n2^{\frac{n}{2}})$$

# $1$-in-$3$ positive SAT

# $1$-in-$3$ positive SAT

**Input:** A formula $\phi$ consisting of

- A set of $n$ boolean variables $x_1, \ldots, x_n$
- A collection of $m$ clauses $C_1, \ldots, C_m$, i.e., triples of variables $C_j = (c_j^{(1)}, c_j^{(2)}, c_j^{(3)}) \in \{x_1, \ldots, x_n\}^3$

A truth assignment is a function $\tau : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$

- A clause $C_j = (c_j^{(1)}, c_j^{(2)}, c_j^{(3)})$ is *satisfied* by $\tau$ iff *exactly* one of $\tau(c_j^{(1)})$, $\tau(c_j^{(2)})$, and $\tau(c_j^{(3)})$ is $\top$.
- $\phi$ is satisfied iff all $m$ clauses $C_1, \ldots, C_m$ are satisfied.

**Question:** Is there a truth assigment that satisfies $\phi$?

# Example

**Formula**

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

# Example

**Formula**

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

**Satisfying assignment:**

$$x_1 = \bot \quad x_2 = \bot \quad x_3 = \top \quad x_4 = \top \quad x_5 = \bot$$

# Example

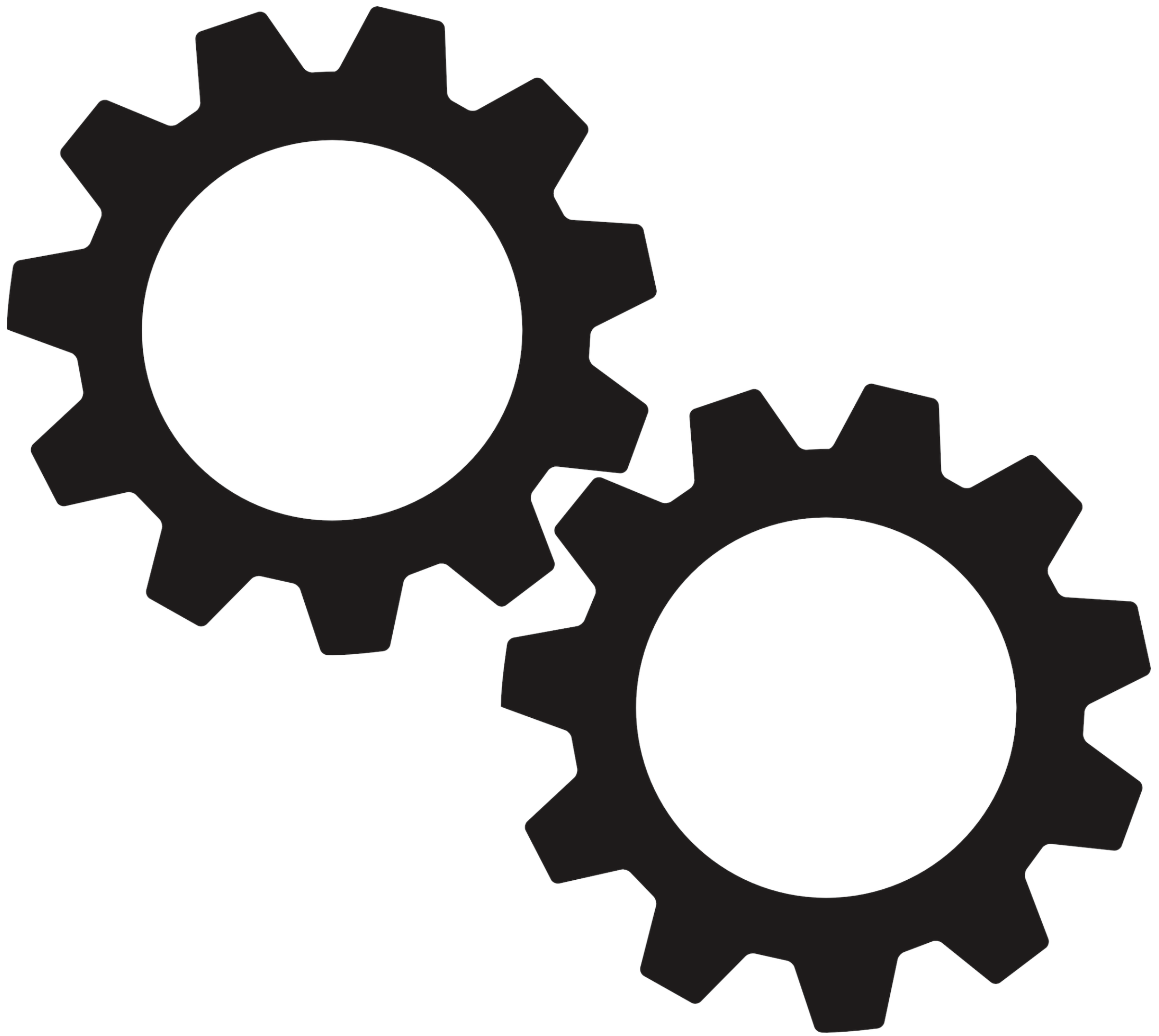
**Formula**

$$\phi = (x_1, x_2, x_4) \wedge (x_2, x_4, x_5) \wedge (x_1, x_3, x_5) \wedge (x_2, x_3, x_1)$$

**Satisfying assignment:**

$$x_1 = \bot \quad x_2 = \bot \quad x_3 = \top \quad x_4 = \top \quad x_5 = \bot$$

Trivial solution $O^*(2^n)$



# An Algorithm Based on Split & List

- Split the $n$ boolean variables into two sets $S_1, S_2$ of size $\approx \frac{n}{2}$
- For each possible truth assigment $\tau_1$ of the variables in $S_1$
  - If $\tau_1$ sets $\geq 2$ variables in the same clause to $\top$, discard it.
  - Otherwise, store in $X_1$ the characteristic vector $\chi(\tau_1) = (\chi_1, \chi_2, \ldots, \chi_m) \in \{\top, \bot\}^m$ of the satisfied clauses, where $\chi_j = \top$ iff $\tau_1$ satisfies $C_j$.
- Compute $X_2$ in a similar way.
- Sort $X_2$ (e.g., w.r.t. the lexicographic order)
- For each vector $\chi = (\chi_1, \ldots, \chi_m) \in X_1$
  - $\overline{\chi} \leftarrow (\overline{\chi_1}, \ldots, \overline{\chi_m})$
  - Binary search for $\overline{X}$ in $X_2$