Range Trees

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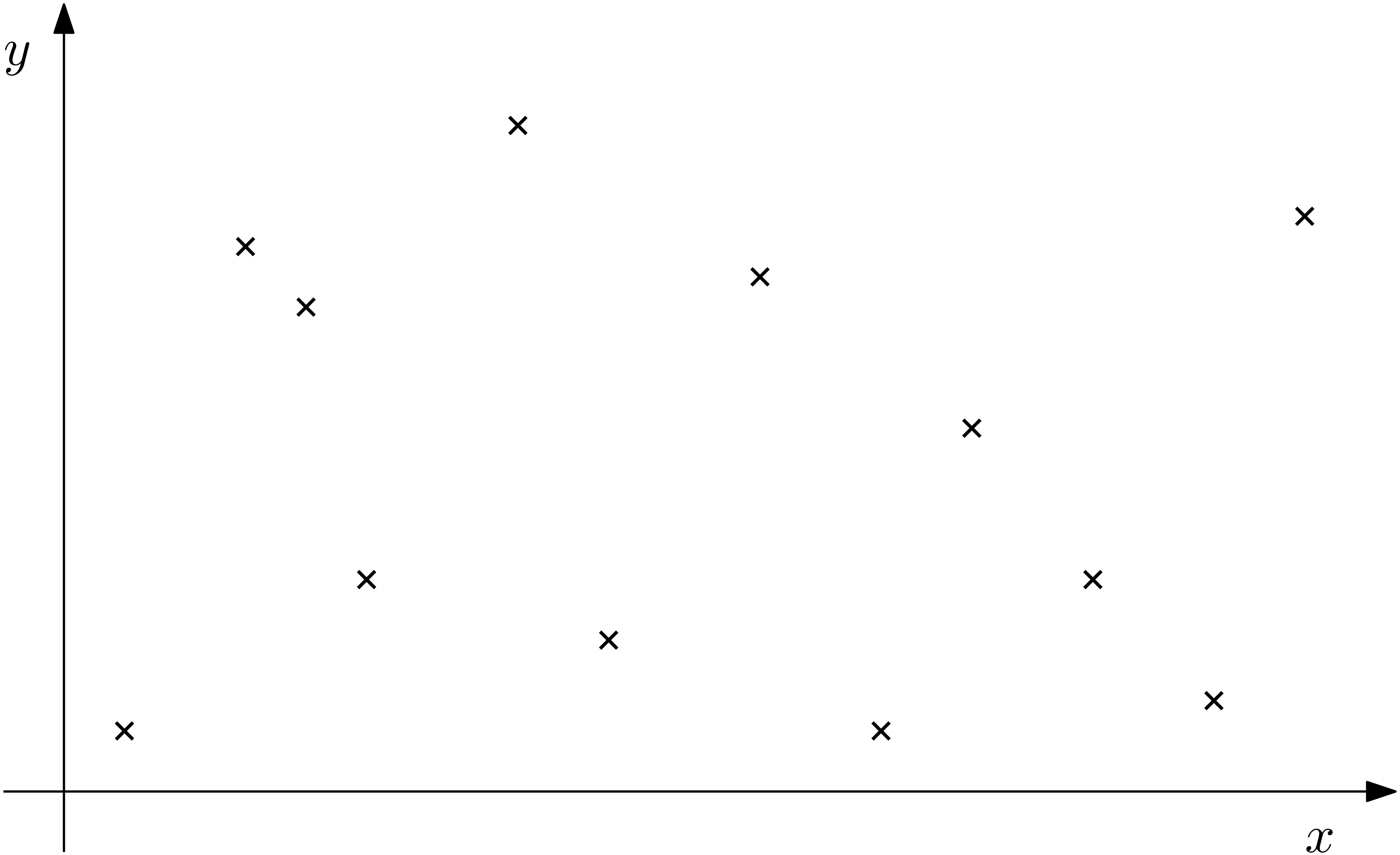

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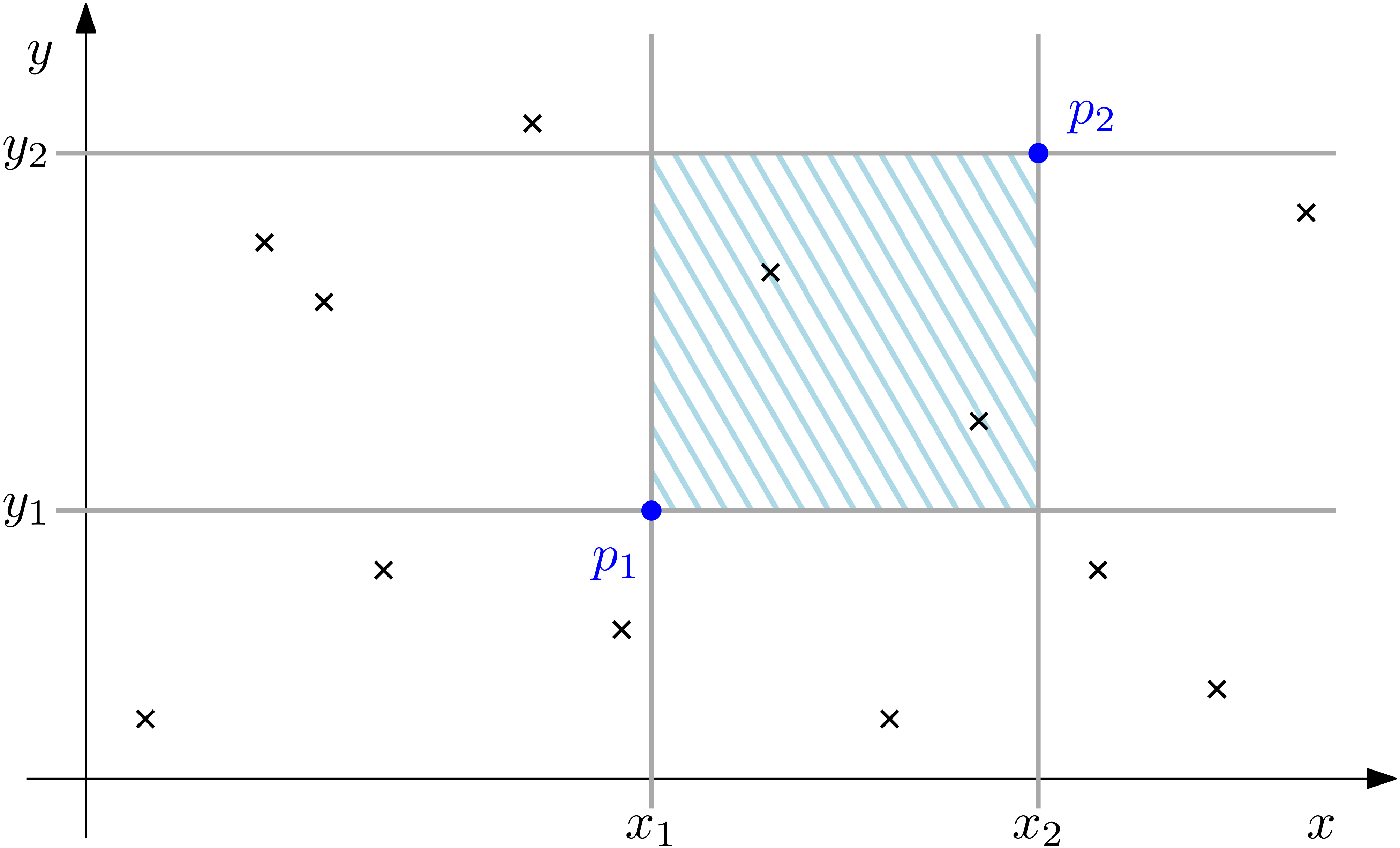

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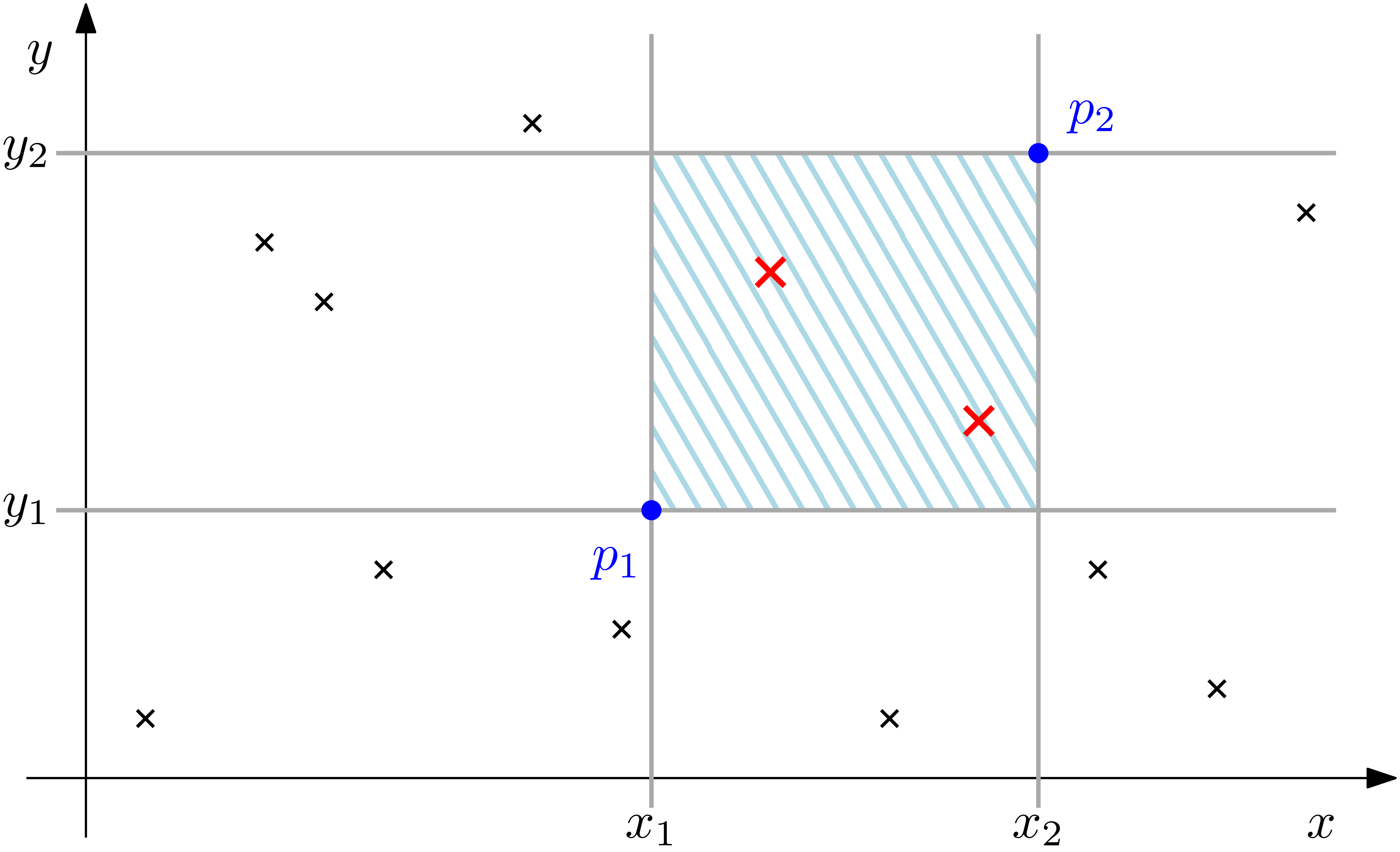


Range Trees

Input:

A set S of n D-dimensional points.

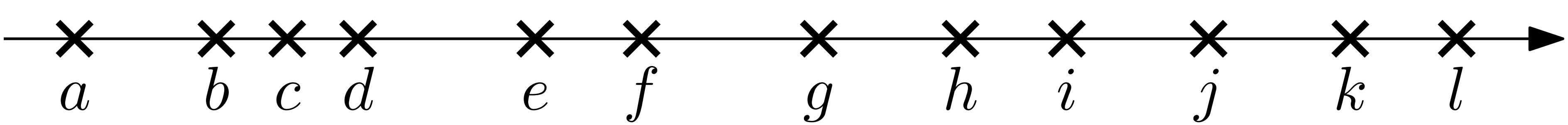
Goal:

Design a data stucture that, given $p_1 \in \mathbb{Z}^D, p_2 \in \mathbb{Z}^D$ can:

- Report *the number* of points $q \in S$ such that $p_1 \leq q \leq p_2$.
- Report *the set* of points $q \in S$ such that $p_1 \leq q \leq p_2$.
- Report the point $q \in S$, $p_1 \leq q \leq p_2$, with *smallest* D-th coordinate.
- $\ldots$

An easy case: $D = 1$

- Points are integers
- Store points in a sorted array (in time $O(n \log n)$).
- Perform queries by binary searching for p_1 and p_2



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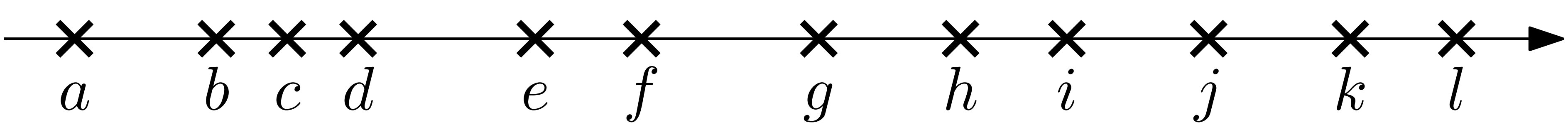
Query time: $O(\log n + k)$ $k =$ "size" of the output.

- $k = \#$ reported points.
- $k = \Theta(1)$ if we only care about the *number* of points.



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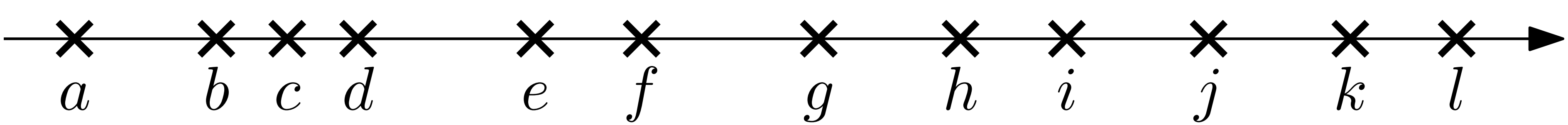
Space complexity: $O(n)$



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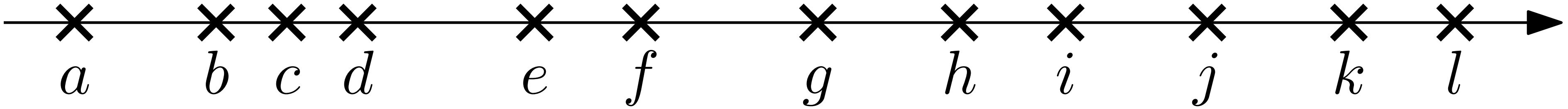

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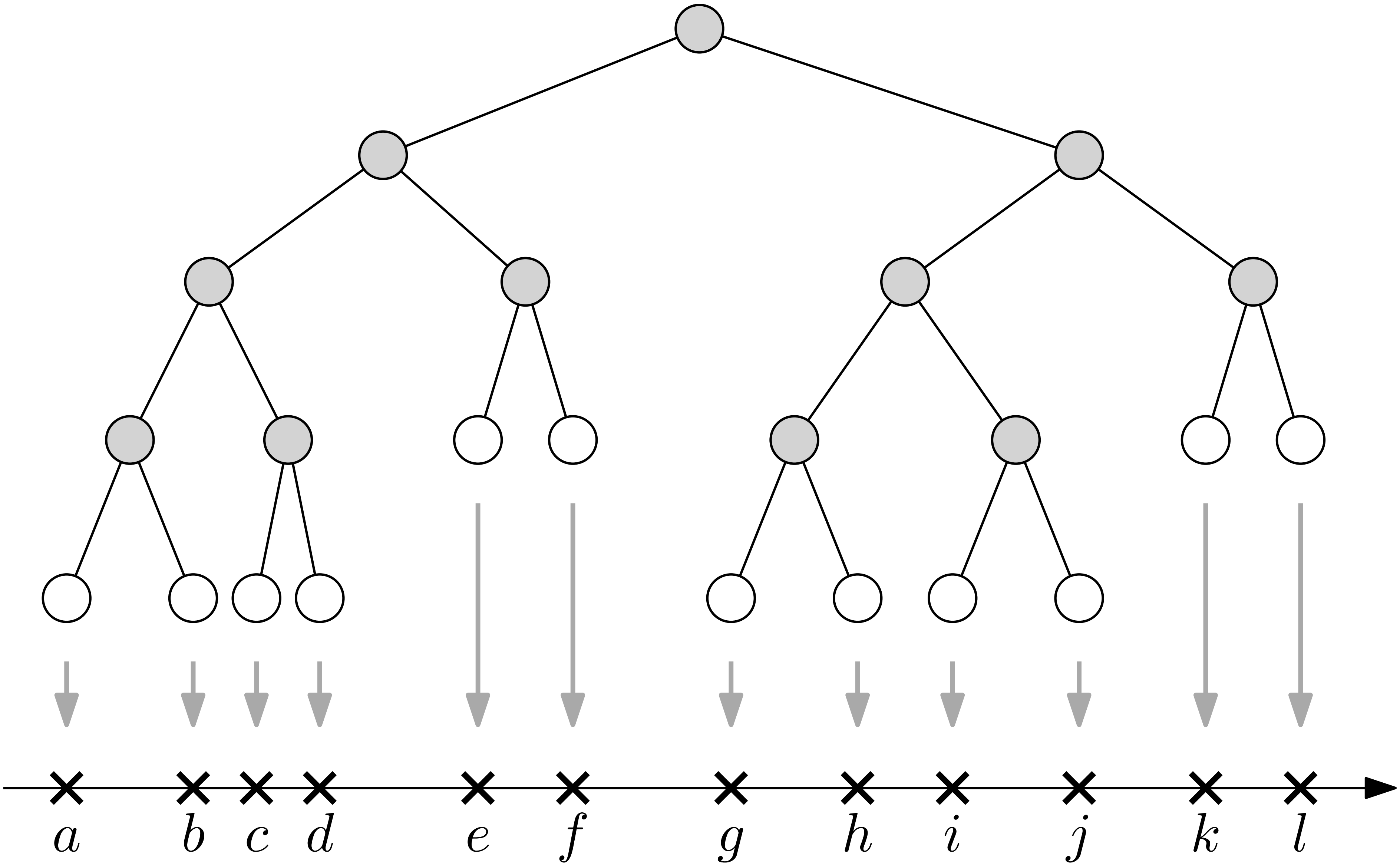

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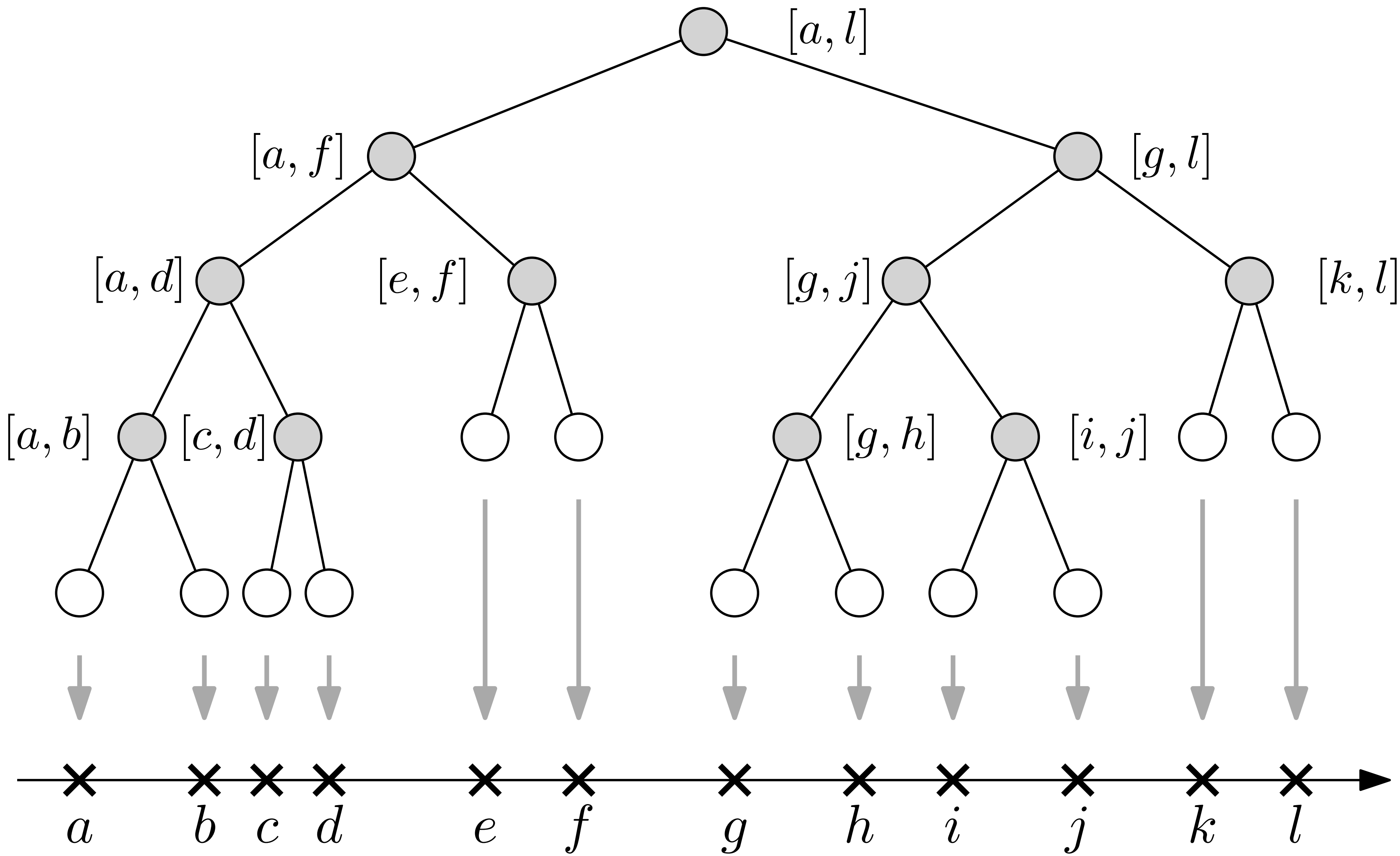

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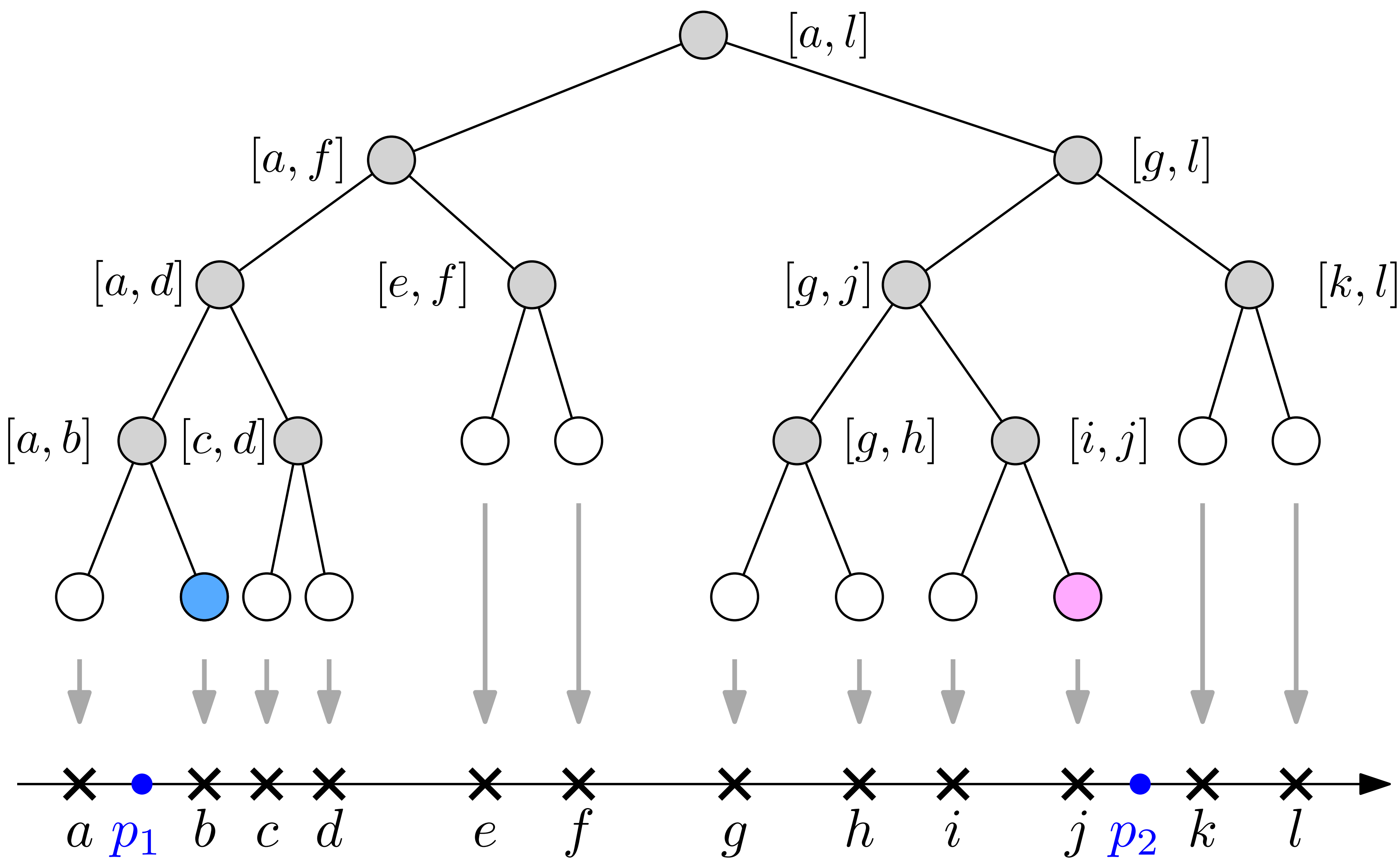

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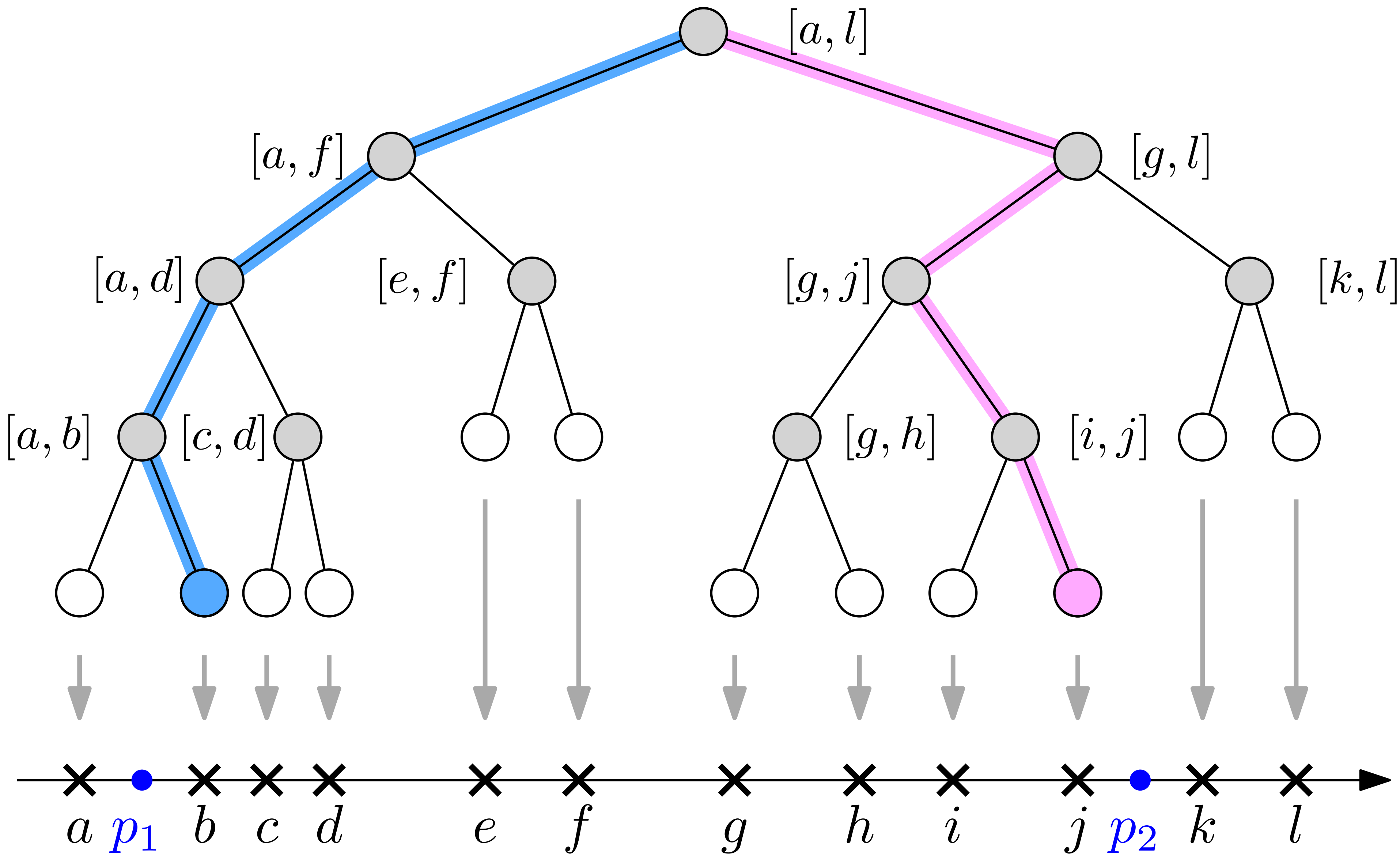

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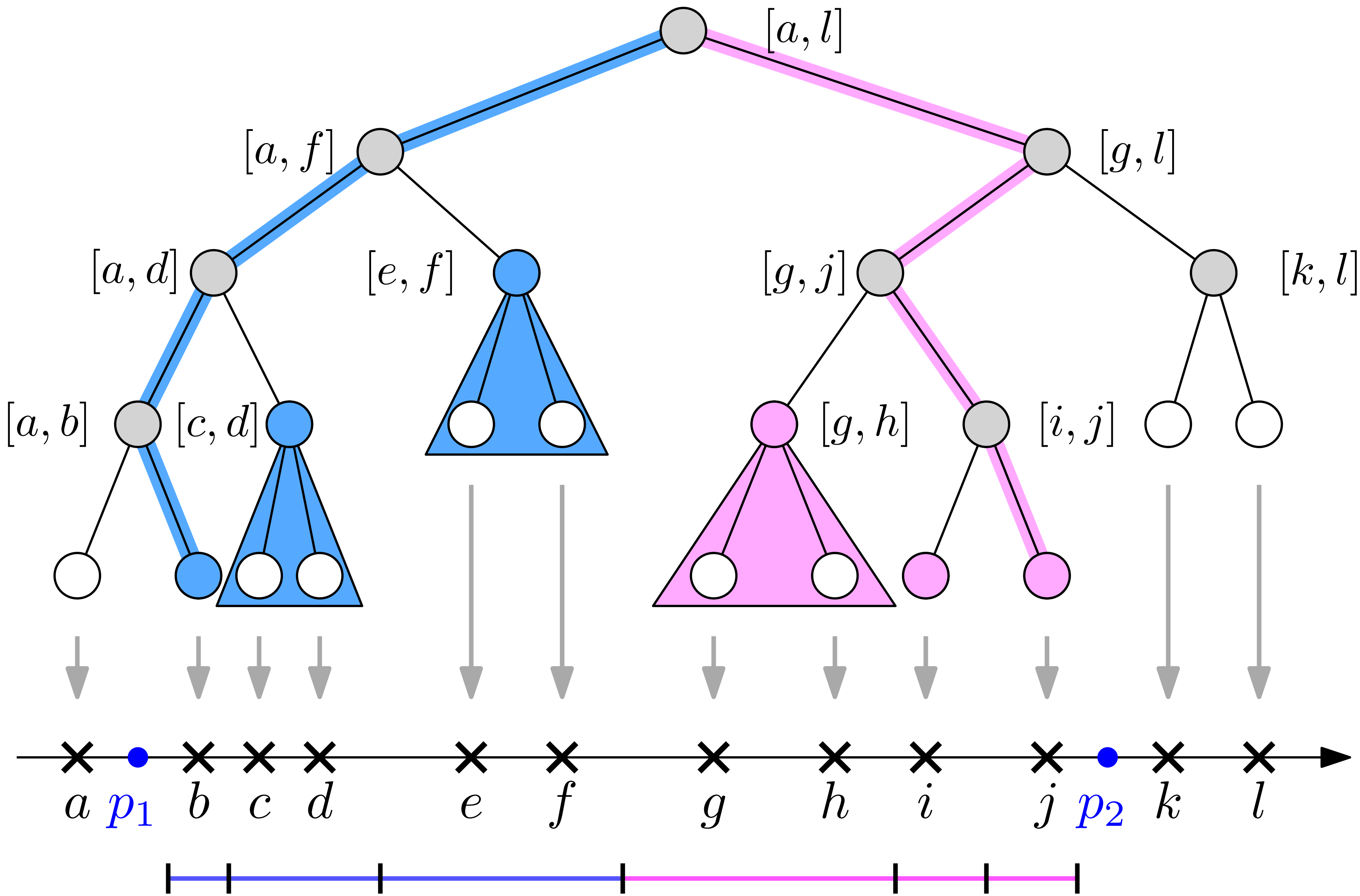

Range Trees: $D = 1$

Construction:

- **Preliminarily** sort S (only once!)
- Split S into S_1 and S_2 of $\approx \frac{n}{2}$ elements each. $O(1)$
- Recursively build T_1 and T_2 from S_1 and S_2, respectively.
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What if S is already sorted?

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$$O(n \log n)$$

What if S is already sorted? $O(n)$ (we will need this later)

Range Trees: $D = 1$

Preprocessing time: $O(n \log n)$

Query time: $O(\log n + k)$

- $k = \#$ reported points.
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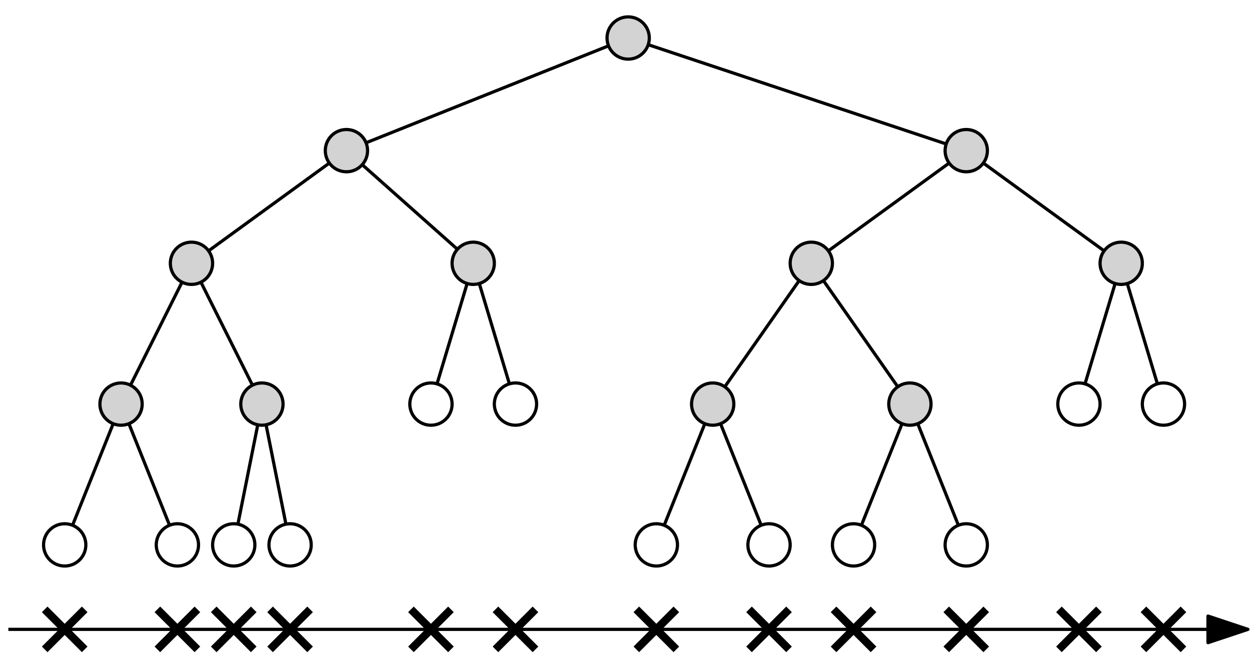
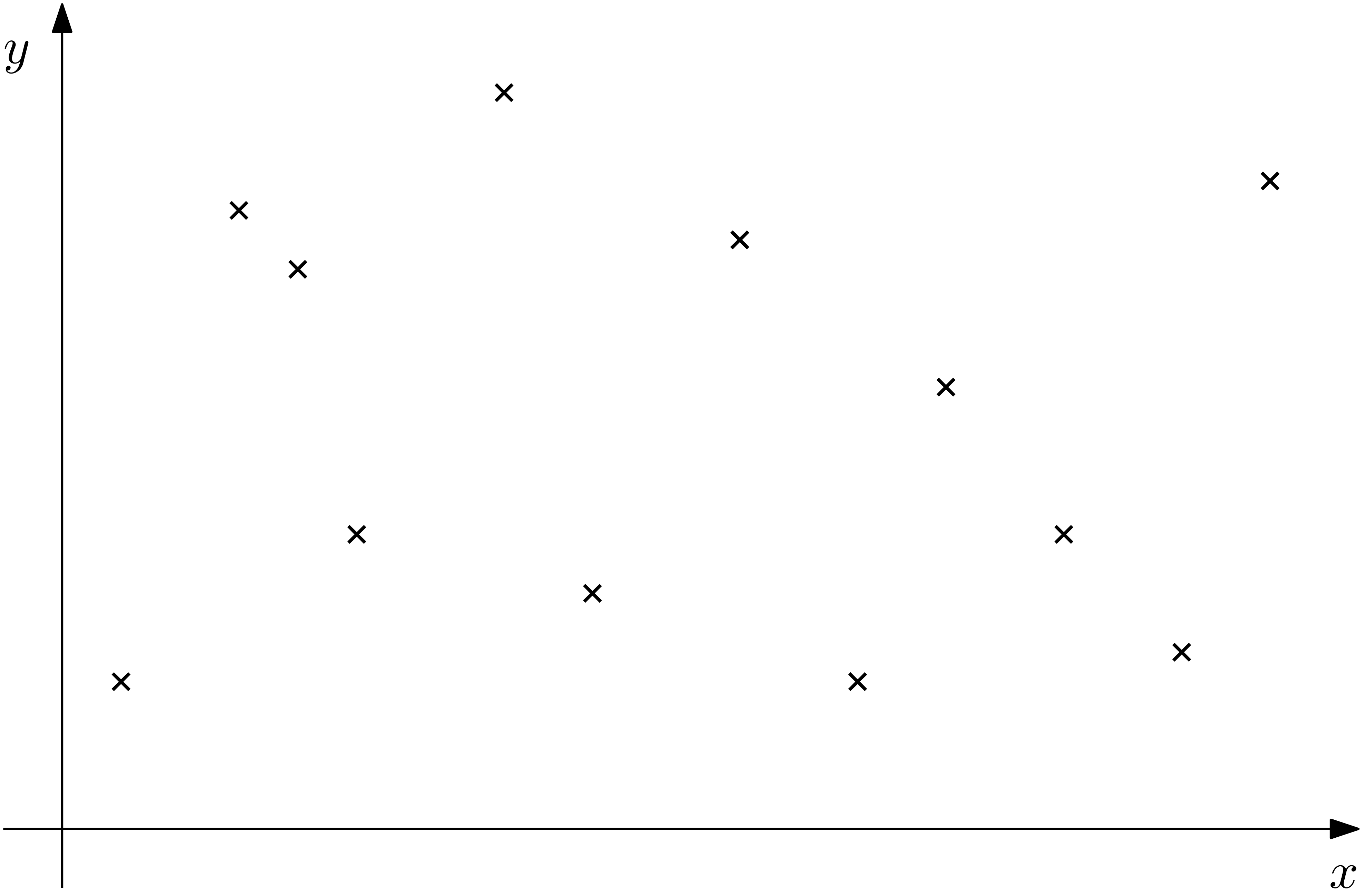
Space complexity: $O(n)$



Range Trees: $D = 2$



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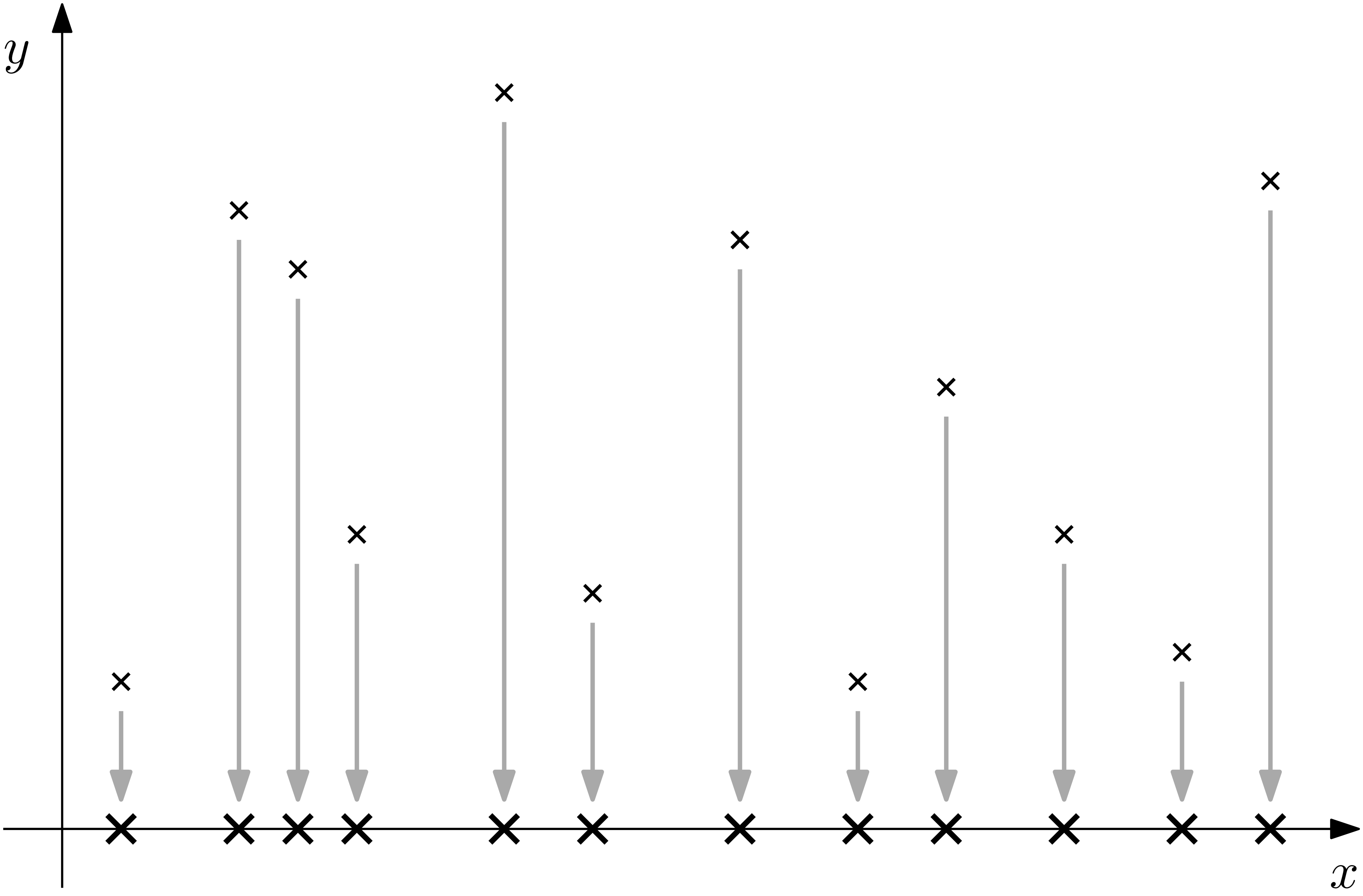

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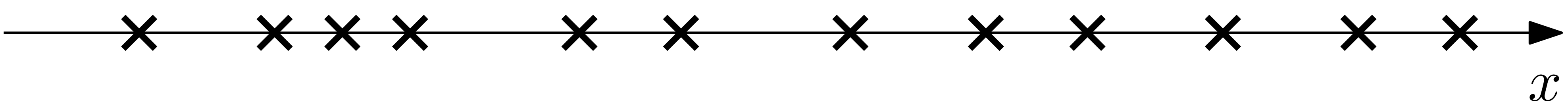
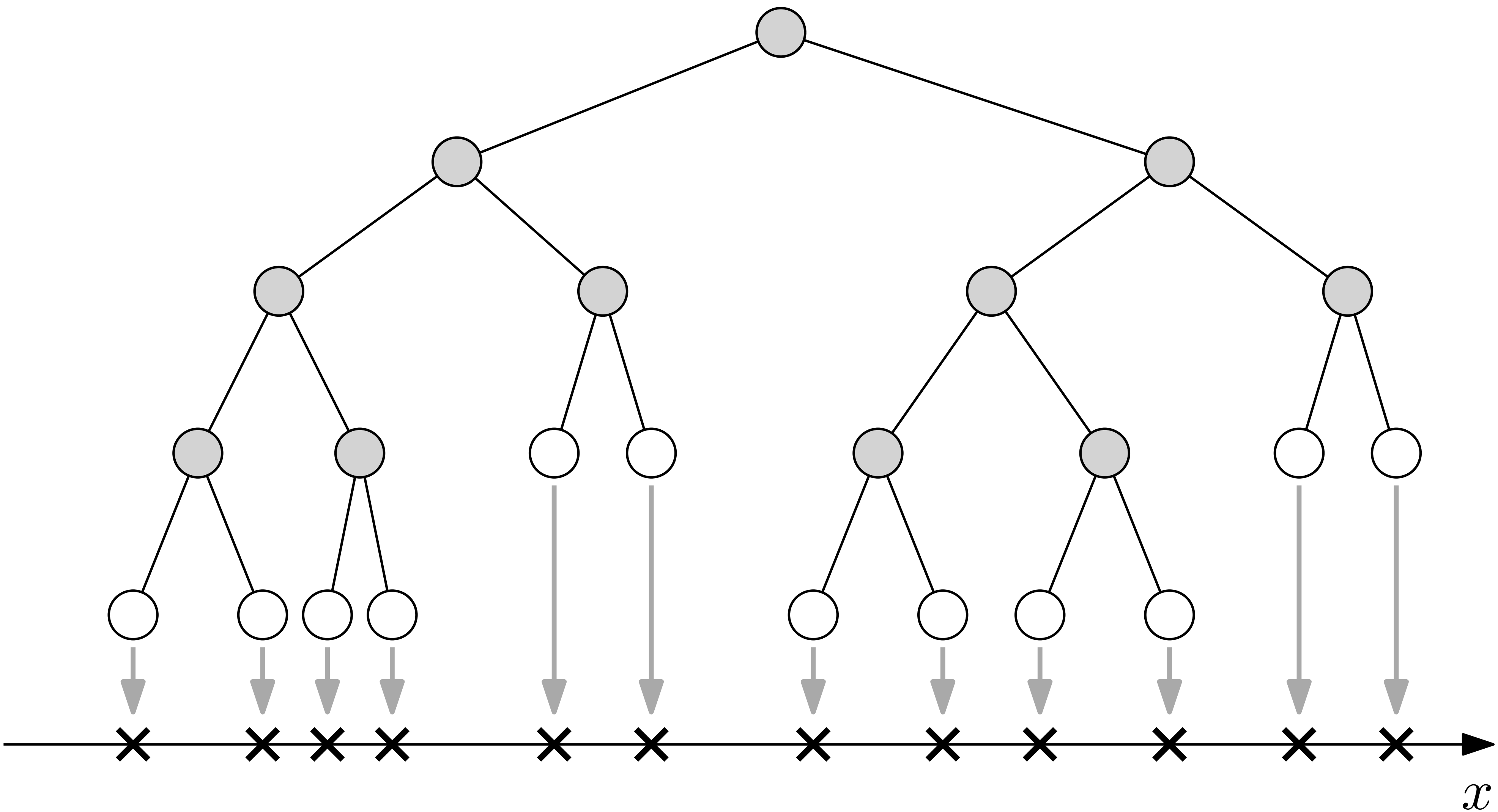


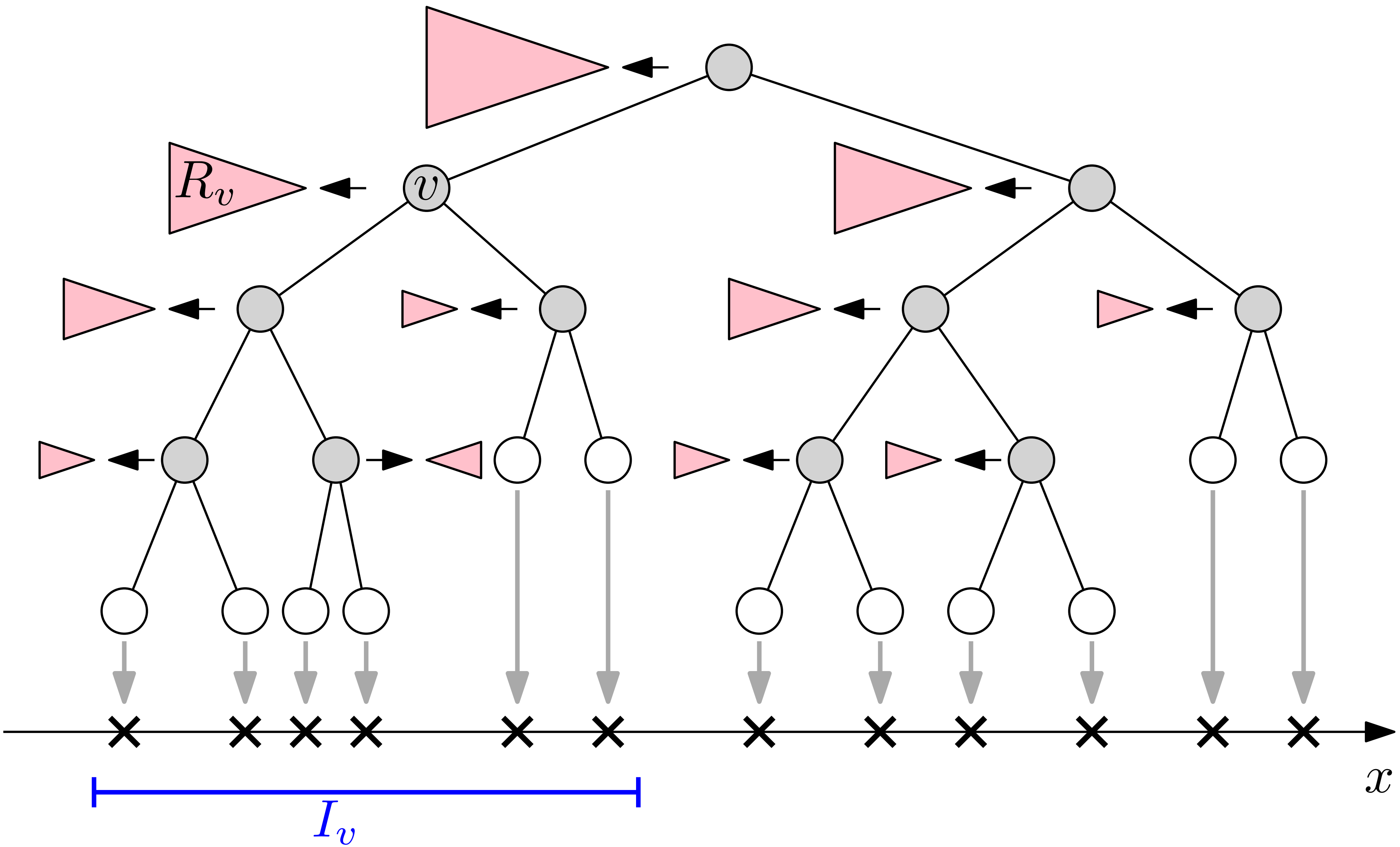
Range Trees: $D = 2$

Build a range tree on the set of x-coordinates of the points in S



Range Trees: $D = 2$

For each node v representing an interval $I_v = [x_1, x_2]$, build a range tree R_v on the y coodinates of the points in S whose x-coordinate is in I_v



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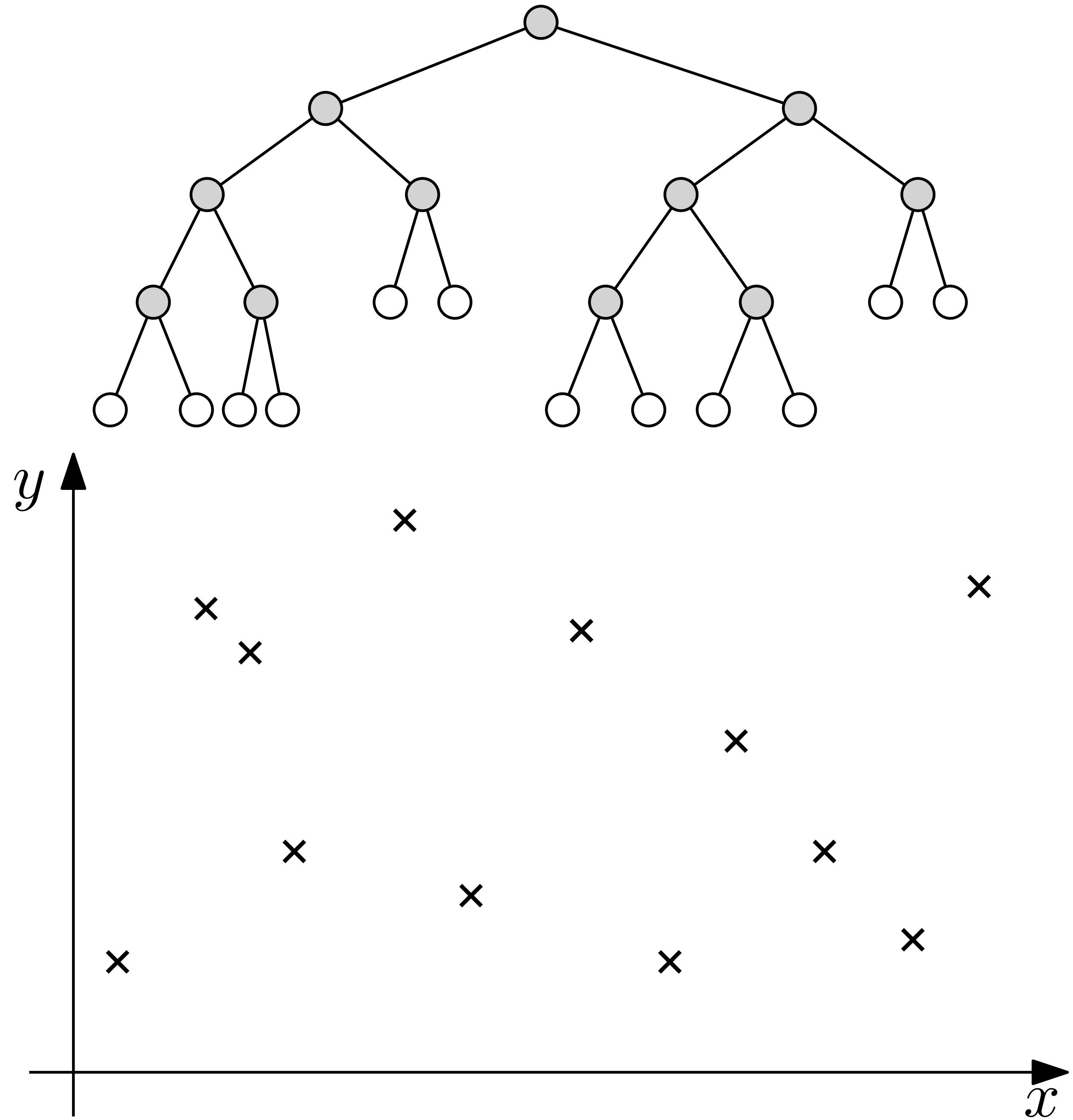

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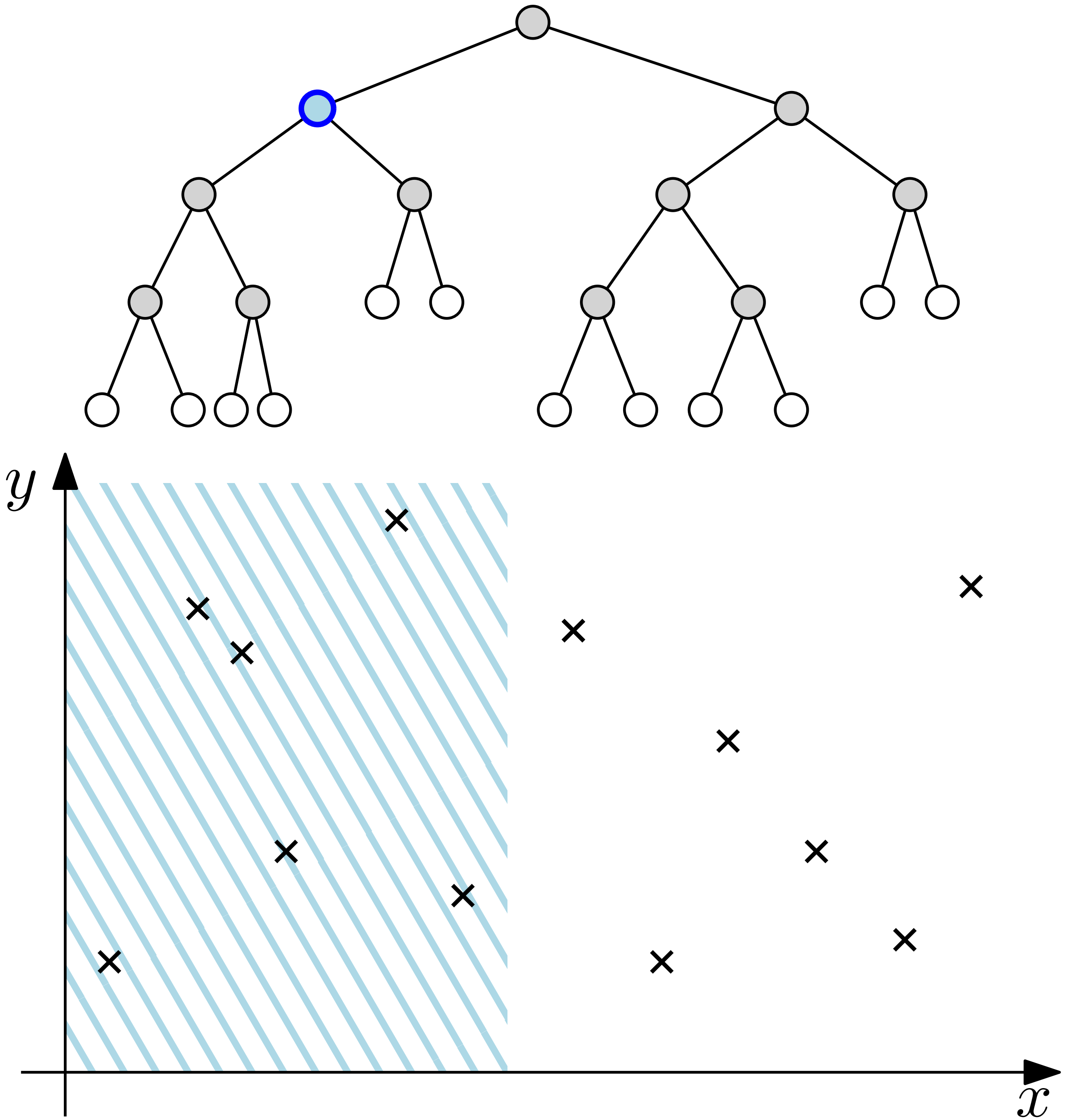

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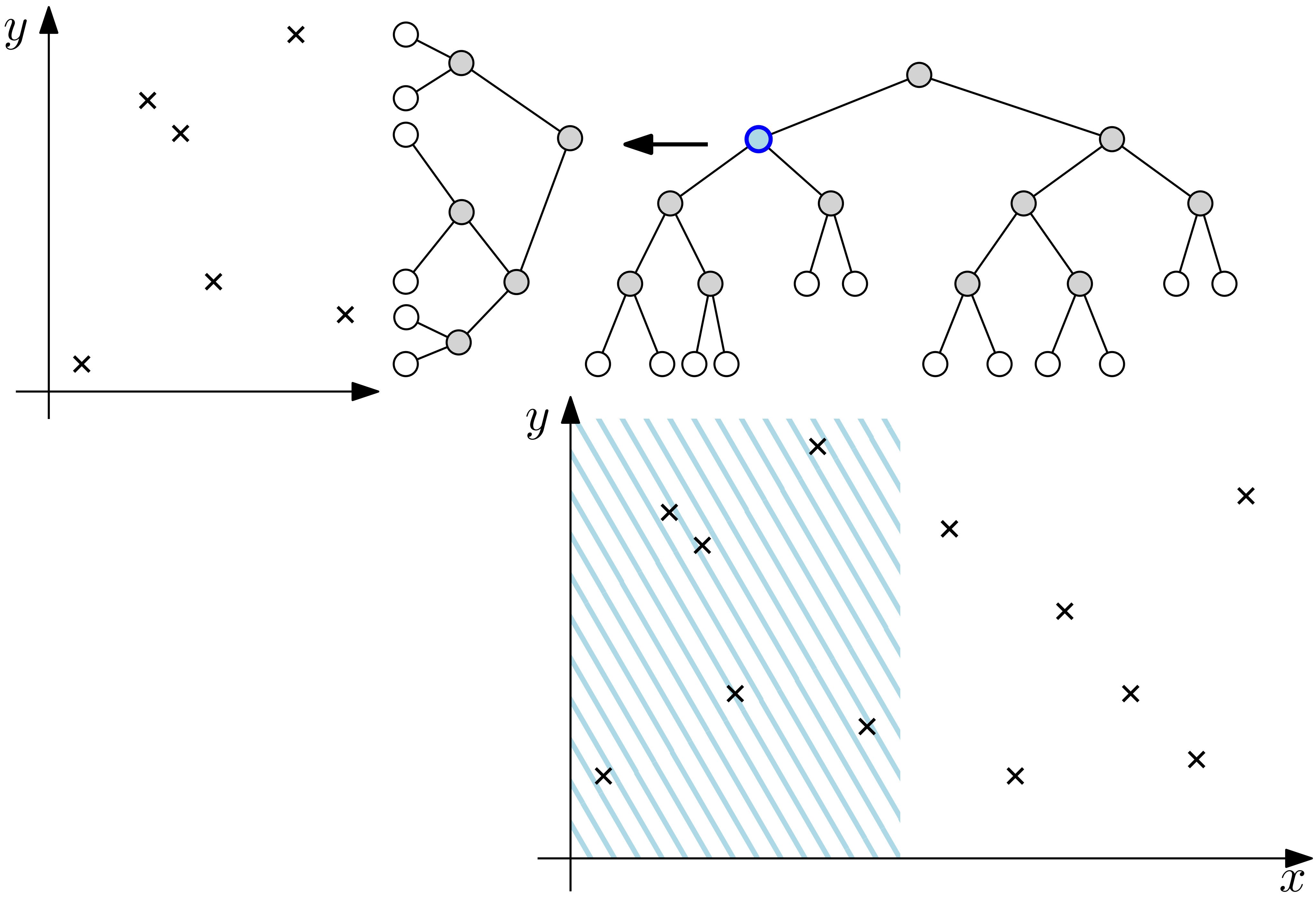

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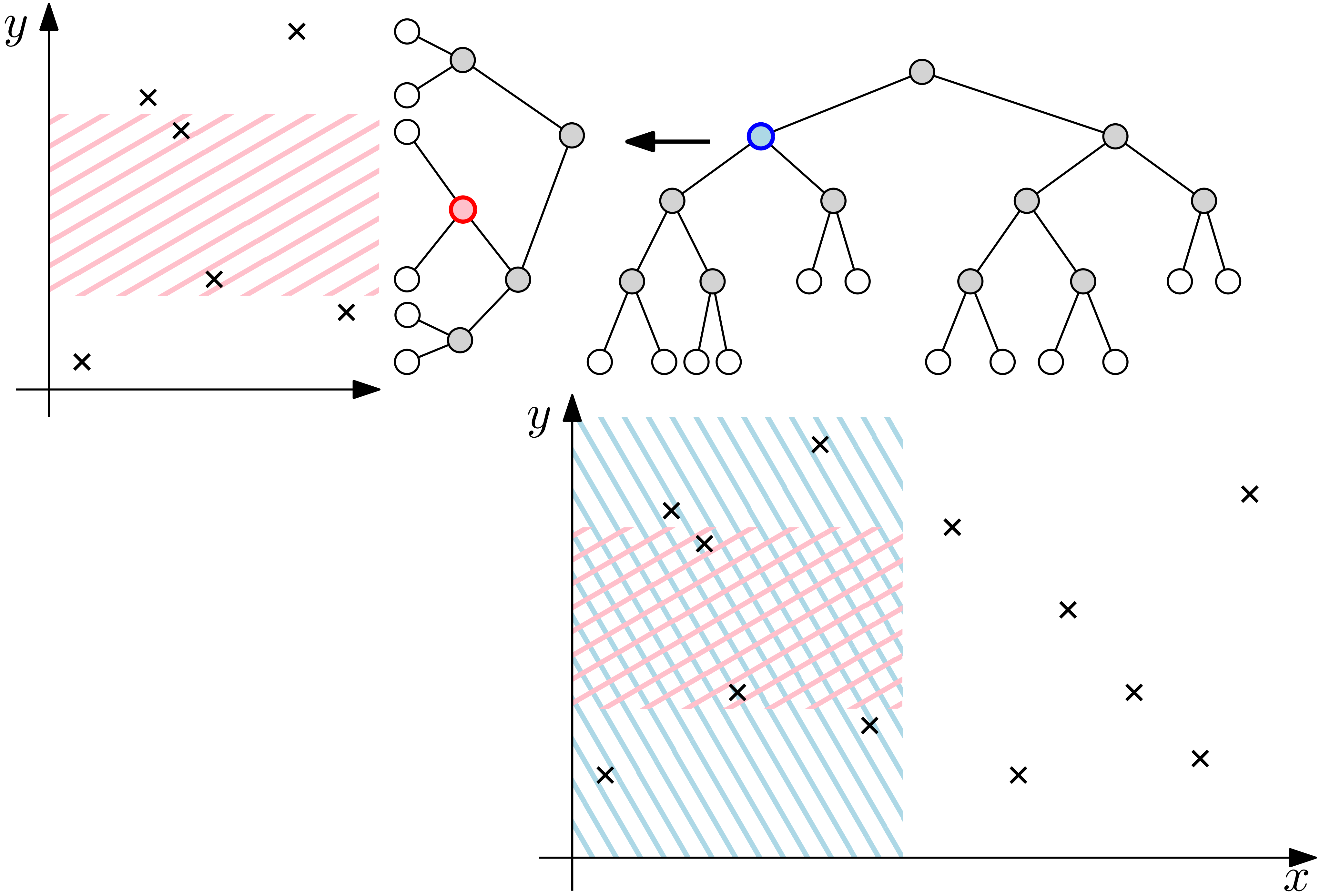

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Construction:

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- The root v of T has T_1 and T_2 as its left and right subtrees.
- Store, in v, a pointer to a new 1D Range Tree on S
- Return T

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Time: $O(n \log n) + T(n)$, where $T(n) = 2 \cdot T(\frac{n}{2}) + O(n \log n)$

$$O(n \log^2 n)$$

can we do better?

Range Trees: $D = 2$

Construction:

S^y is the set S sorted on the y-coordinate

- **Preliminarily** sort S on the x-coordinate.
- Split S into S_1 and S_2 of $\approx \frac{n}{2}$ elements each.
- Recursively build (T_1, S_1^y) and (T_2, S_2^y) from S_1 and S_2, respectively.
- The root v of T has T_1 and T_2 as its left and right subtrees.
- Merge S_1^y and S_2^y into S^y.
- Store, in v, a pointer to a new 1D Range Tree on S^y
- Return (T, S^y)

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Time: $O(n \log n) + T(n)$, where $T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$

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Time: $O(n \log n) + T(n)$, where $T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$

$$O(n \log n)$$

Range Trees: $D = 2$

To report the points $p_1 = (x_1, y_1) \leq q \leq p_2 = (x_2, y_2)$:

- Use T to find the $h = O(\log n)$ subtrees $R_1, \ldots, R_h$ that store the points $q = (x, y)$ with $x_1 \leq x \leq x_2$.

- For each tree $R_j \in \{R_1, \ldots, R_h\}$ representing the x-interval I_j:
  - Query R_j to report the number of/set of points $q = (x, y)$ with $x \in I_j$ and $y_1 \leq y \leq y_2$.

Range Trees: $D = 2$

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Time: $O(\log n) \cdot O(\log n) + O(k) = O(\log^2 n + k)$

Number of R_is | Time to query R_i | "size" of the output

Range Trees: $D = 2$

Preprocessing time: $O(n \log n)$

Query time: $O(\log^2 n + k)$

- $k = \#$ reported points.
- $k = \Theta(1)$ if we only care about the *number* of points.

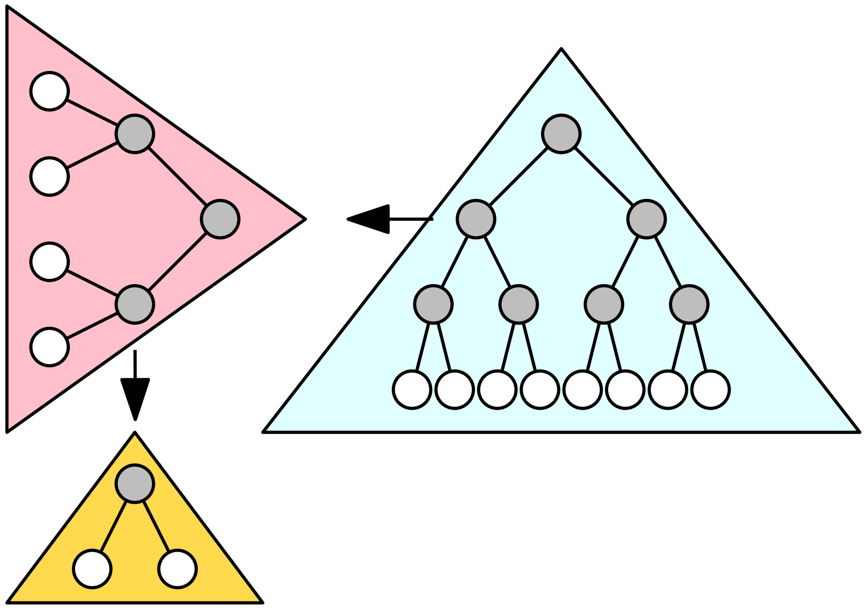
Space complexity:

- Bounded by the overall size of 1D Range Trees
- Each point belongs to $O(\log n)$ 1D Range Tees
- Total space: $O(n \log n)$

Higher dimensions: construction

To store points $p = (x, y, z, w, ...)$ in $D > 2$ dimensions:
Recursive construction:

- Build a Range Tree T on the first coordinate x of the points:
- For each subtree T_v of T associated with the interval $I_v = [x_1, x_2]$:
  - Construct a range tree R_v on the last $D-1$ coordinates $(y, z \ldots)$ of the set of points $p = (x, y, \ldots)$ with $x \in I_v$.
  - Store, in v, a pointer to R_v.



Time: $O(n \log^{D-1} n)$.

Space: $O(n \log^{D-1} n)$.

Higher dimensions: query

Let $p_1 = (x_1, y_1, z_1, \dots)$, $p_2 = (x_2, y_2, z_2, \dots)$.

To report the points $p_1 \leq q \leq p_2$:

- Use T to find the $h = O(\log n)$ subtrees $R_1, \dots, R_h$ that store the points $q = (x, y, z, \dots)$ with $x_1 \leq x \leq x_2$.
- For each tree $R_j \in \{R_1, \dots, R_h\}$ representing the x-interval I_j:
  - Recursively query R_i to report the number/set of points q s.t. $x \in I_j$ and $(y_1, z_1, \dots) \leq q \leq (y_2, z_2, \dots)$.

Query time: $O(\log^D n + k)$.

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	

Fractional Cascading

Fractional Cascading: The problem

Input:

k sorted arrays $A_1, \ldots, A_k$ of n elements each:

A_1	4	9	15	22	23	38	41	50	53	58
A_2	3	7	10	11	15	17	20	36	62	64
A_3	21	23	29	35	37	40	52	57	61	66
A_4	2	5	6	15	24	27	39	50	54	76

$k = 4$

Query:

Given x report, for $i = 1, \ldots, k$, x if $x \in A_i$ or its *predecessor* if $x \notin A_i$.

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$k = 4$

$x = 31$

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$k = 4$

$x = 58$

Query:

Given x report, for $i = 1, \ldots, k$, x if $x \in A_i$ or its *predecessor* if $x \notin A_i$.

Fractional Cascading: A Trivial solution

- For $i = 1, \ldots, k$:
  - Binary search for x in A_i

Time: $O(k \log n)$

Fractional Cascading: A Trivial solution

- For $i = 1, \ldots, k$:
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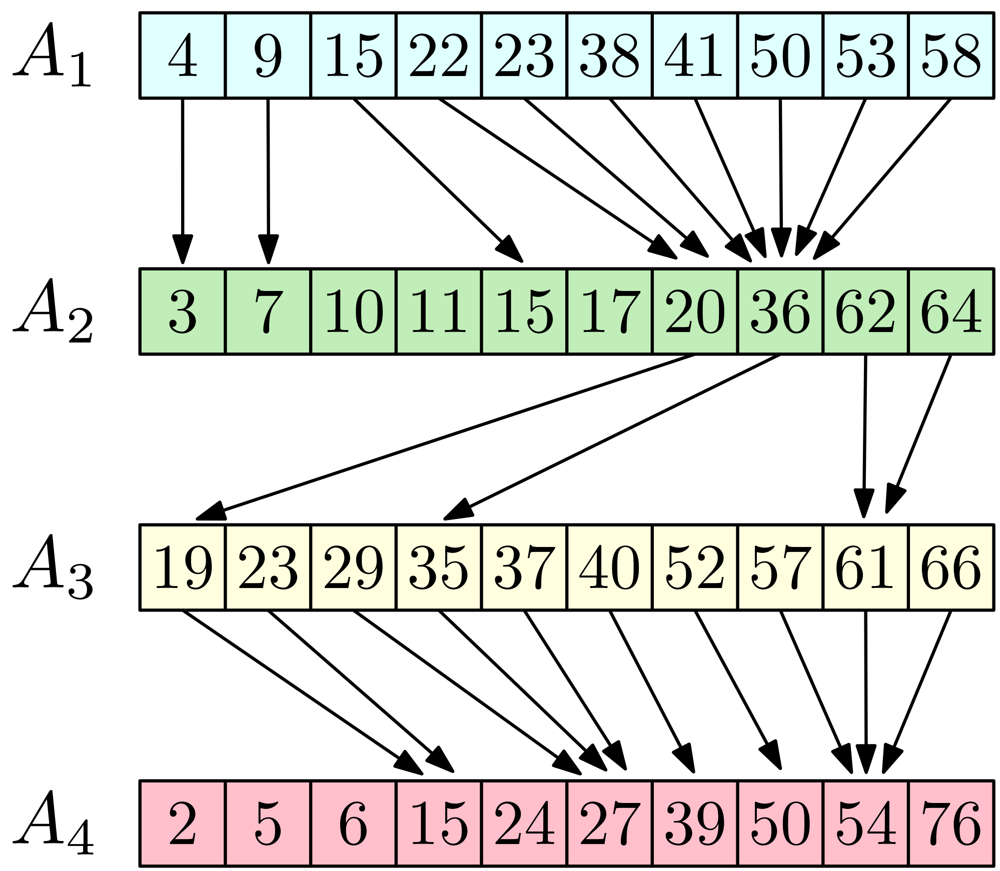
Time: $O(k \log n)$

We can do better!

Fractional Cascading

First idea: cross linking

Keep pointers from $A_i[j]$ to the predecessor of $A_i[j]$ in A_{i+1}.



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Fractional Cascading

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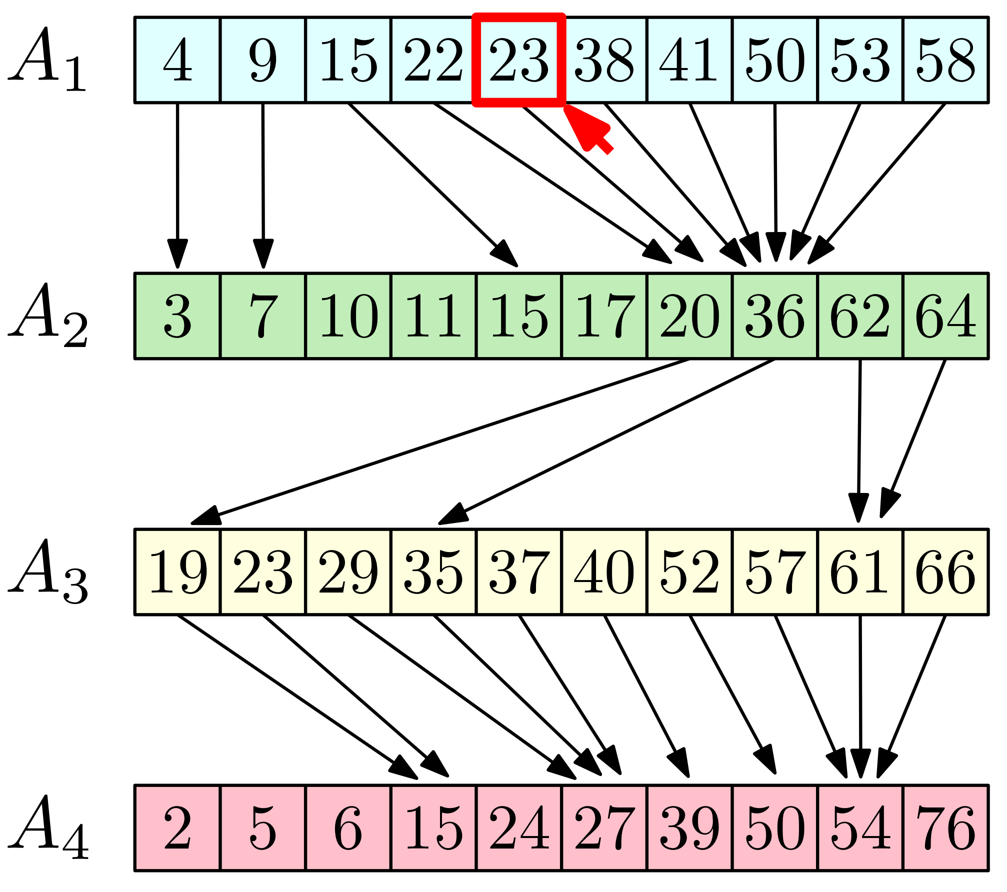

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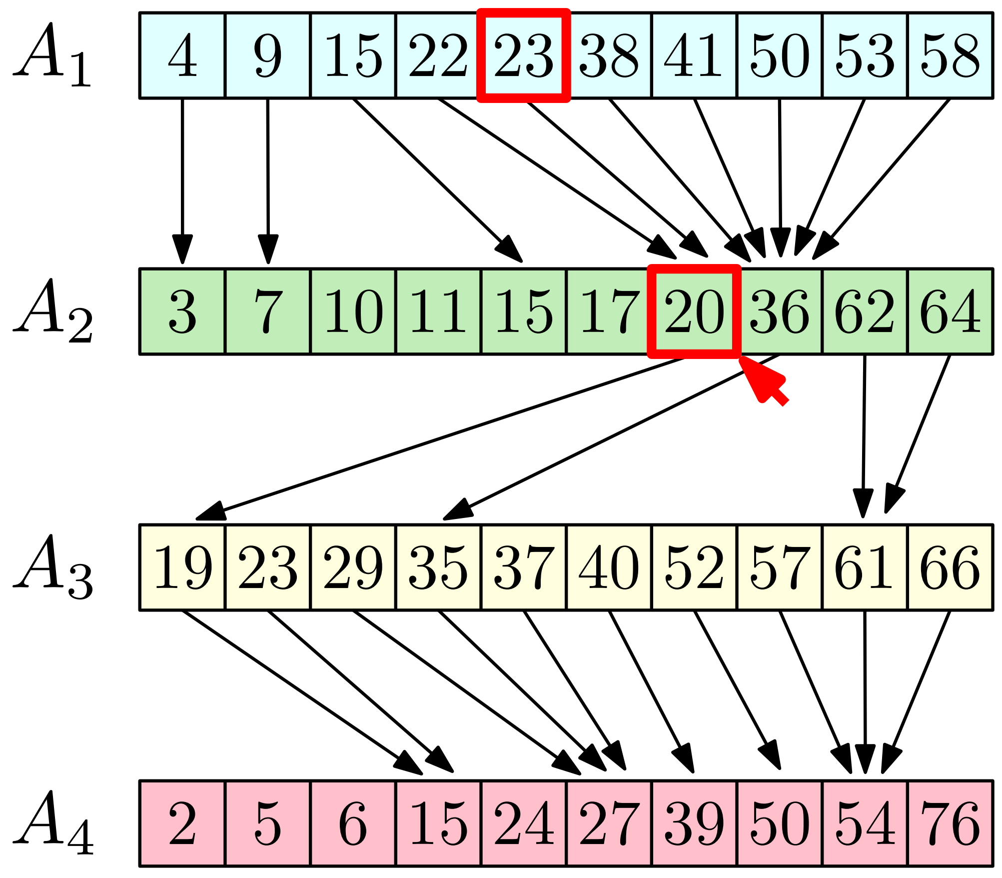

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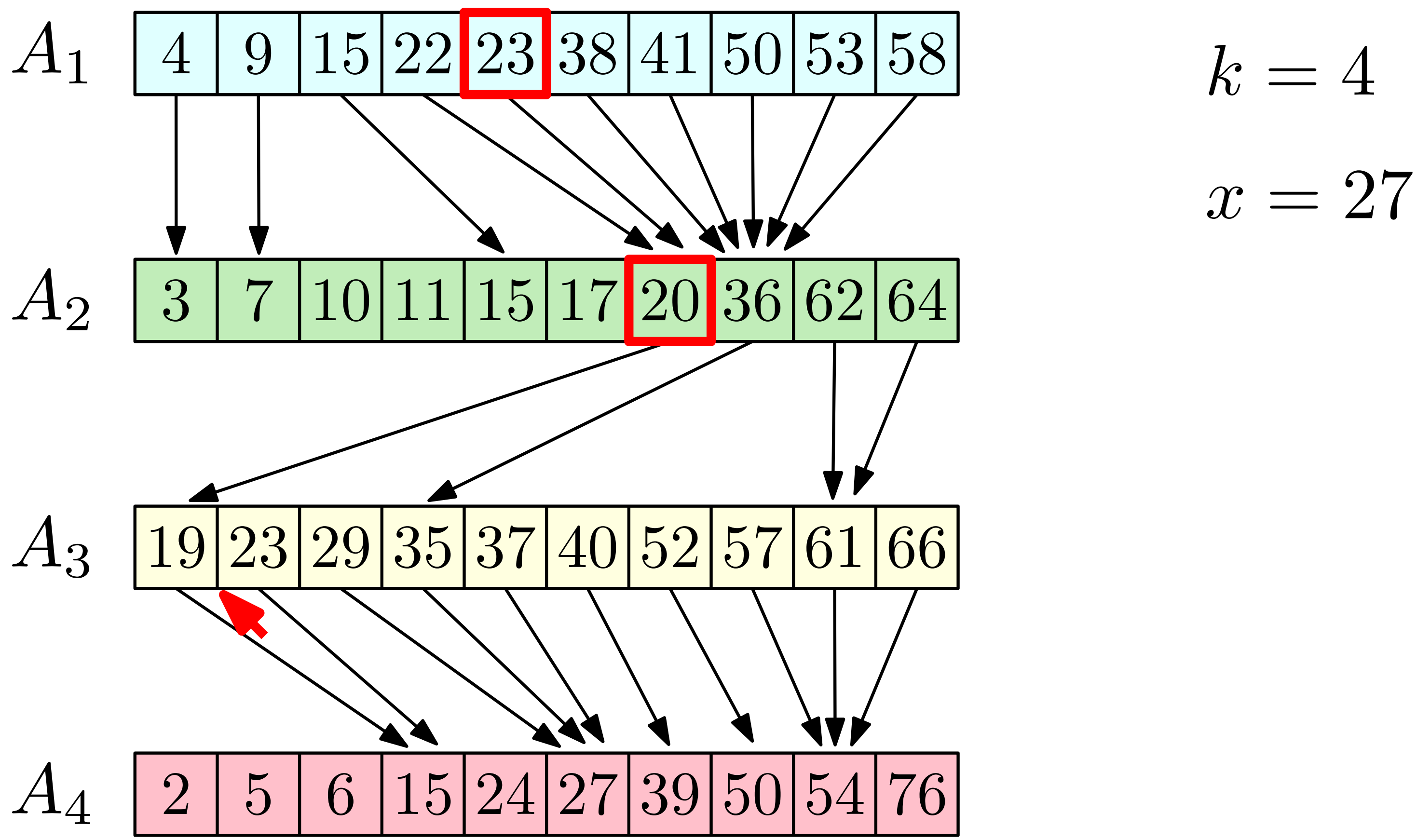

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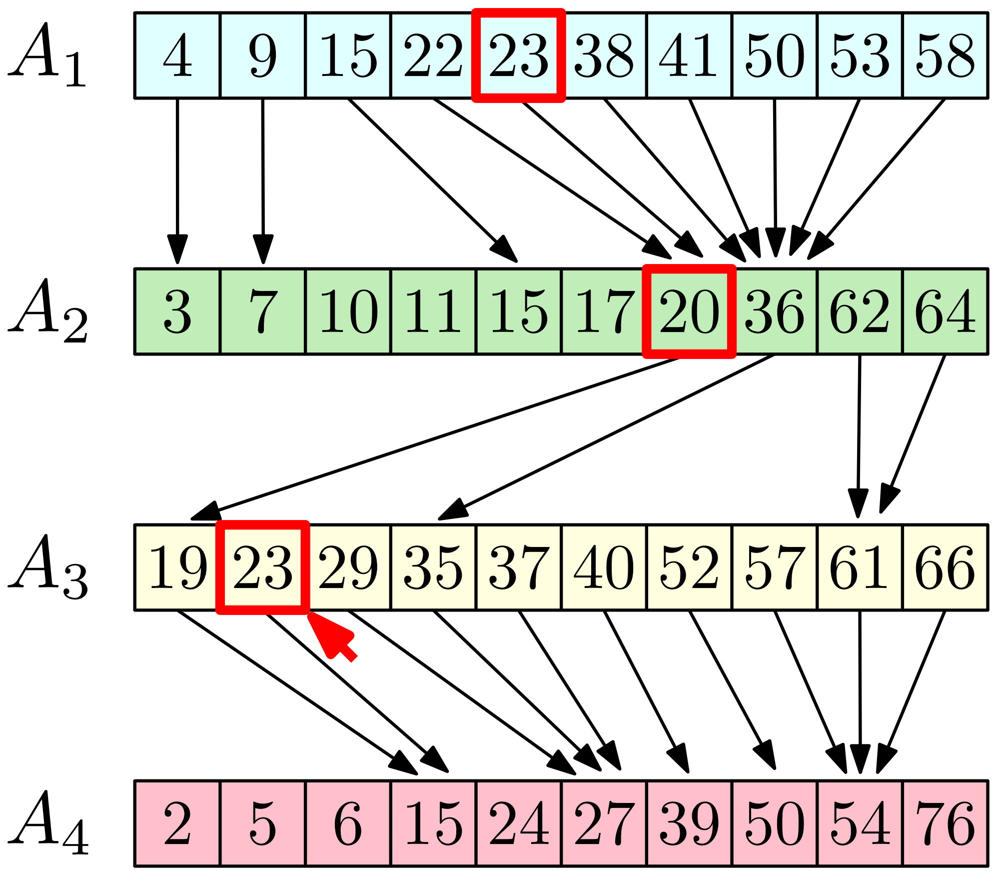

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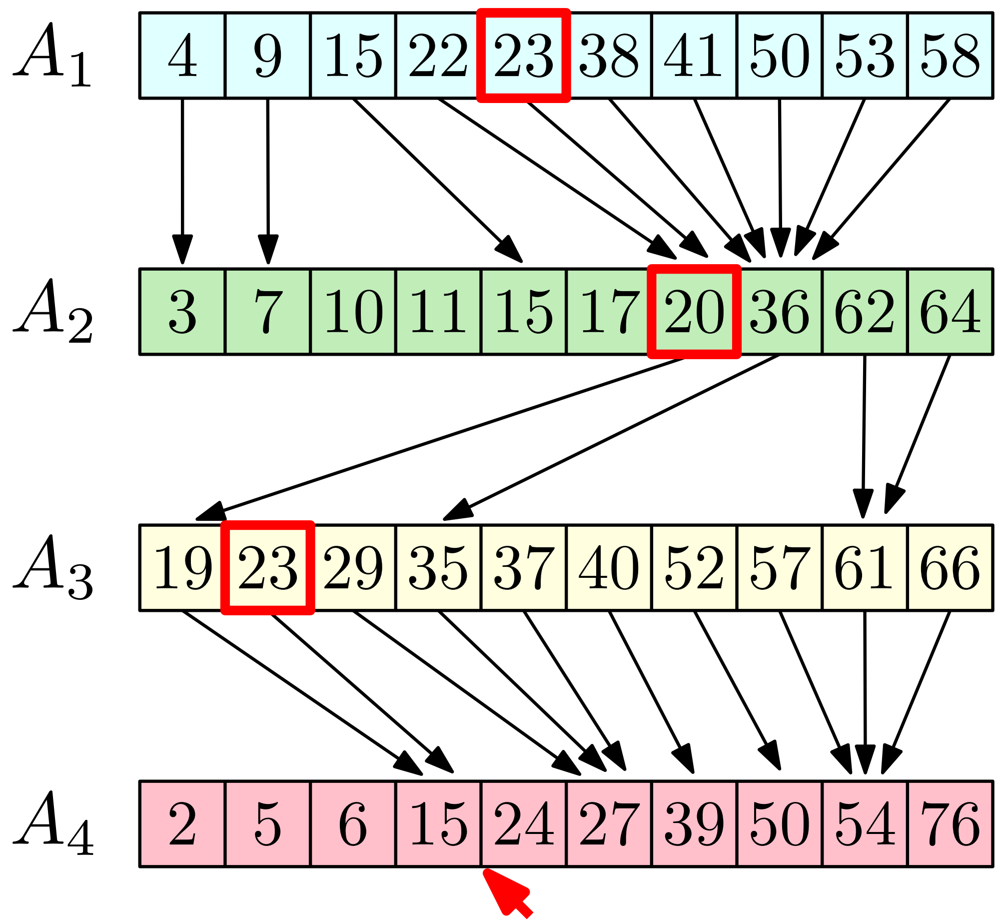

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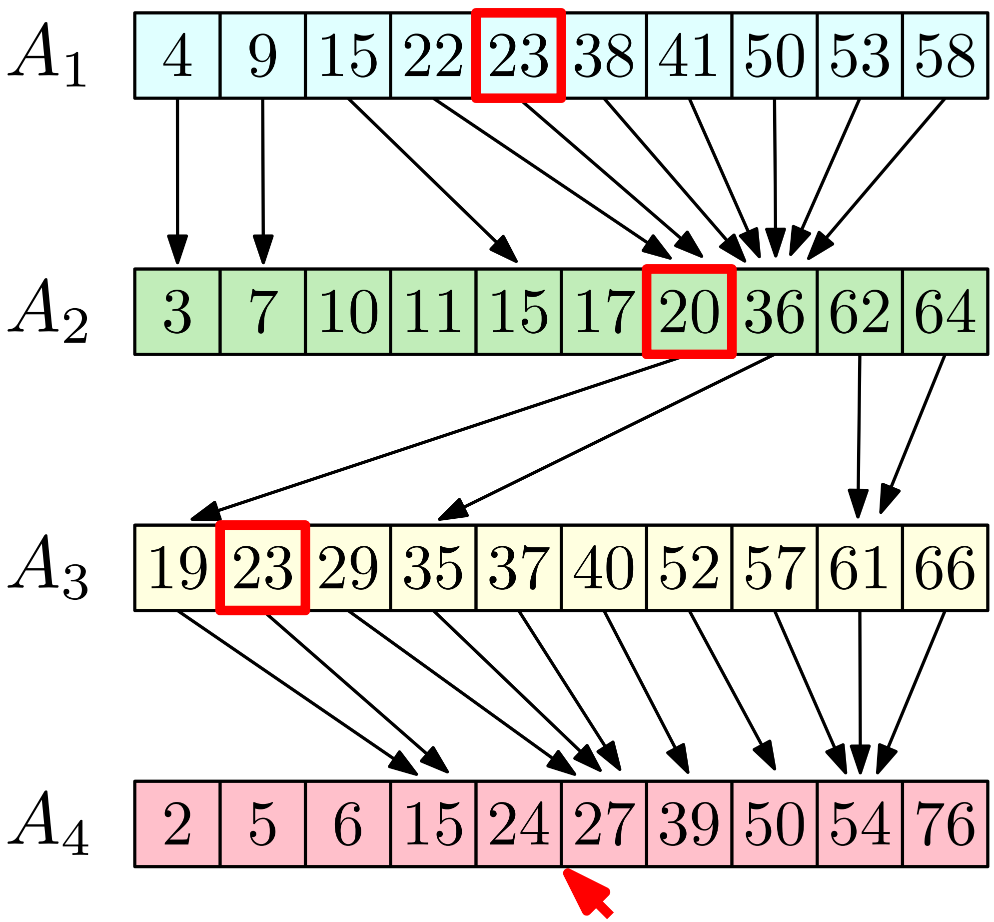

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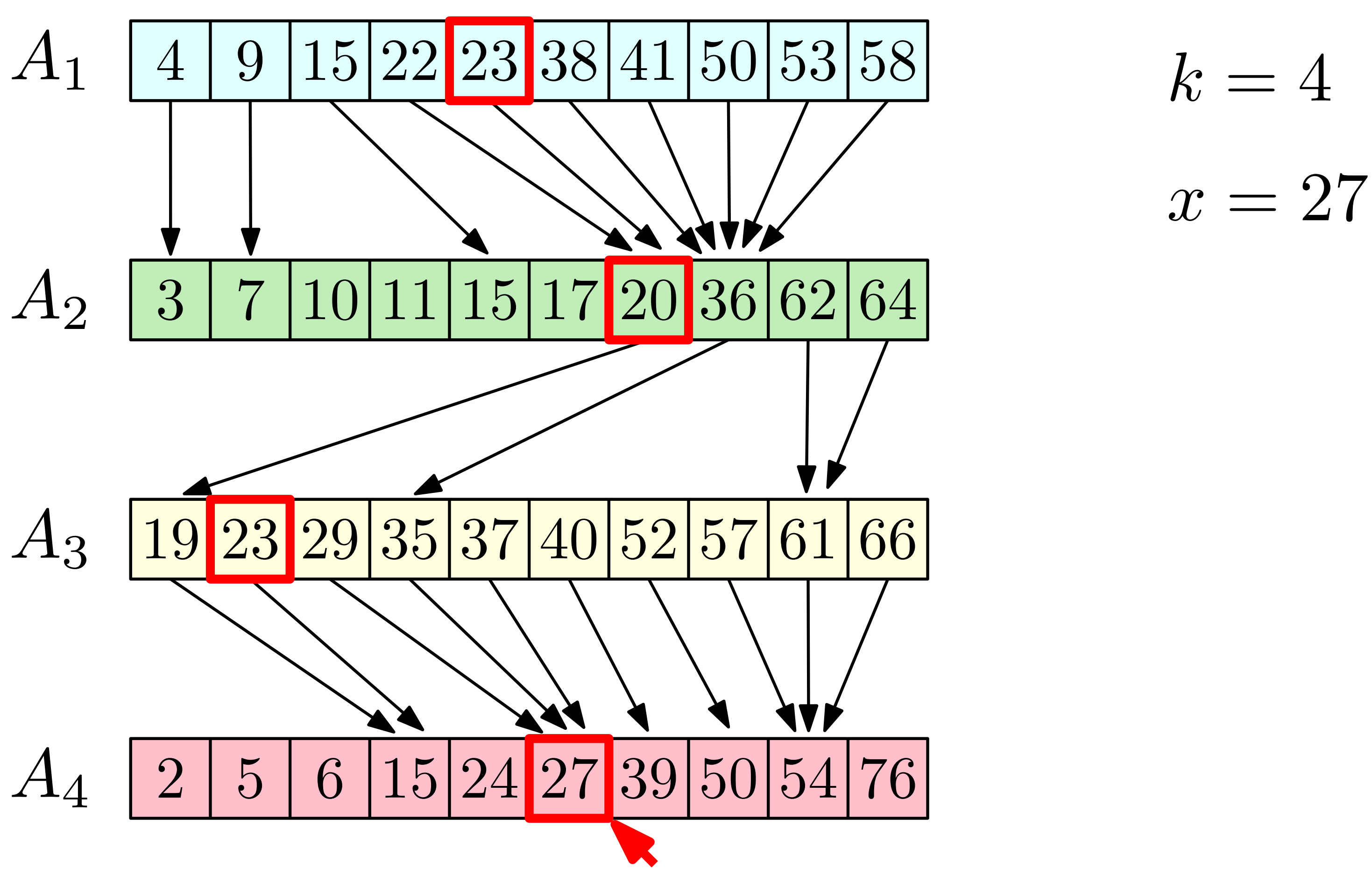
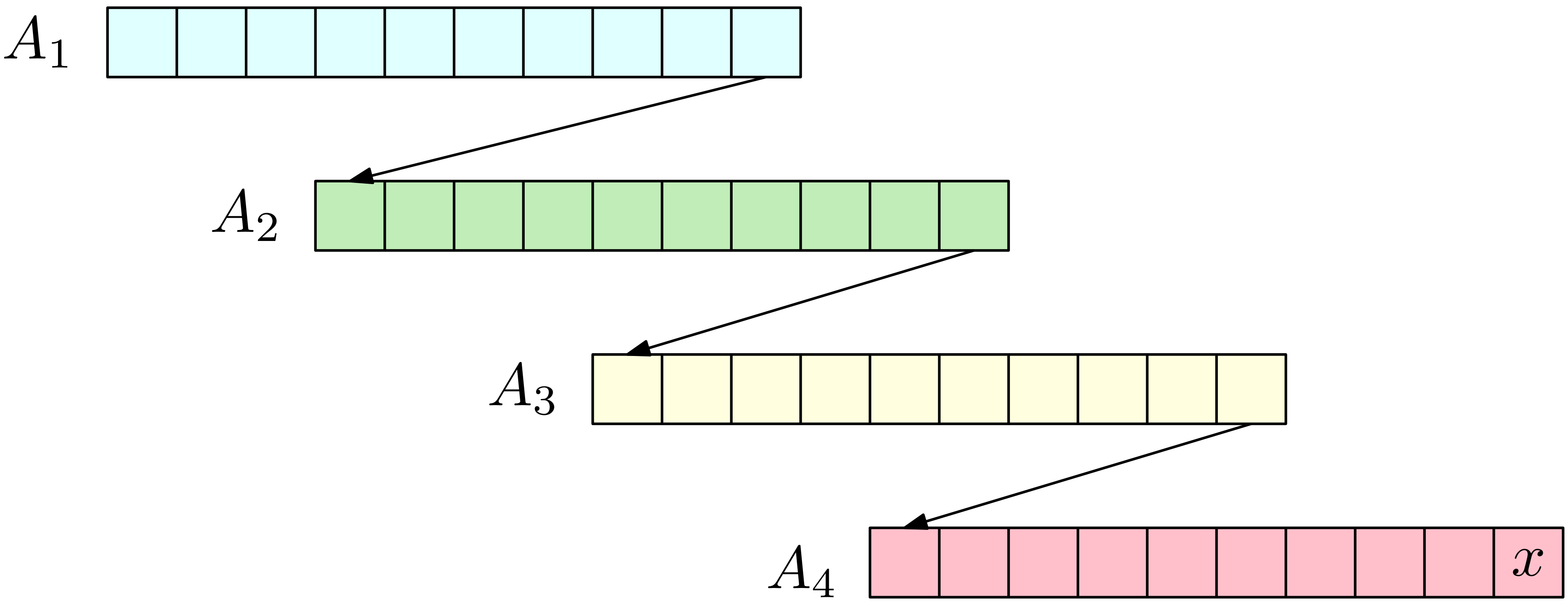
First idea: cross linking

Keep pointers from $A_i[j]$ to the predecessor of $A_i[j]$ in A_{i+1}.



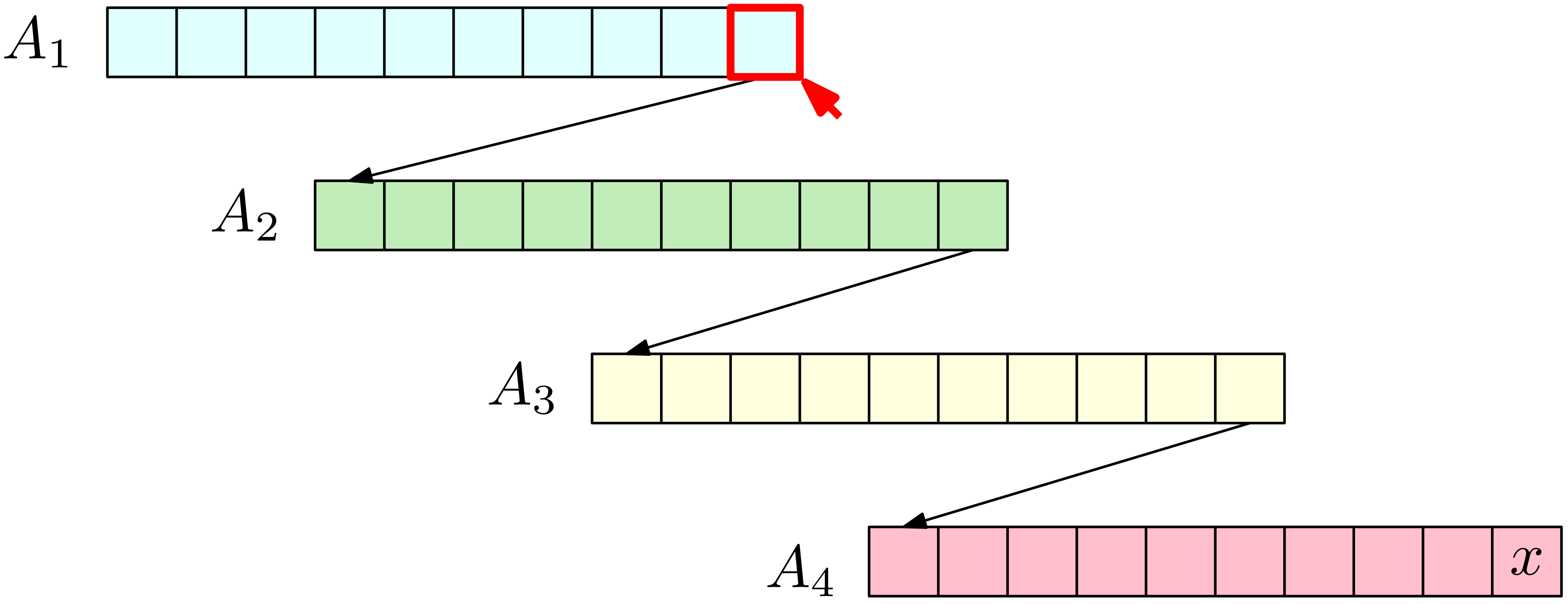
Fractional Cascading

How much time does it take?



Fractional Cascading

How much time does it take?



Fractional Cascading

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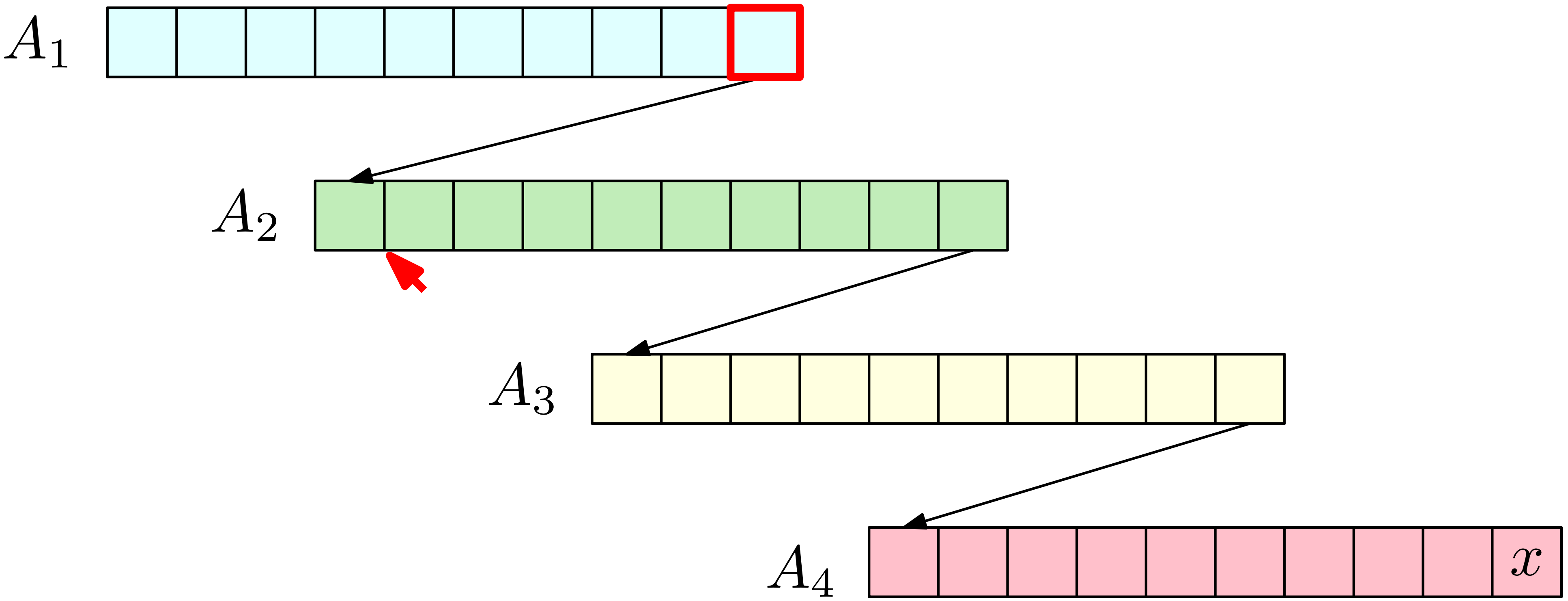
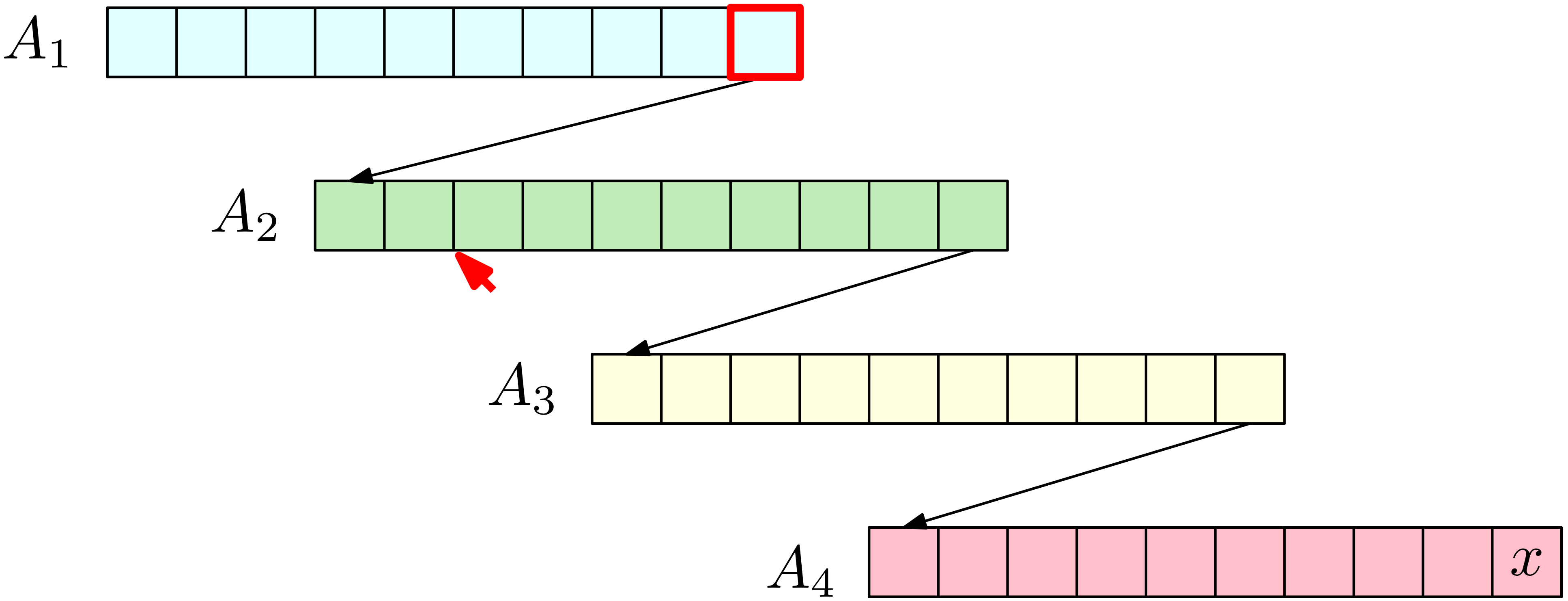


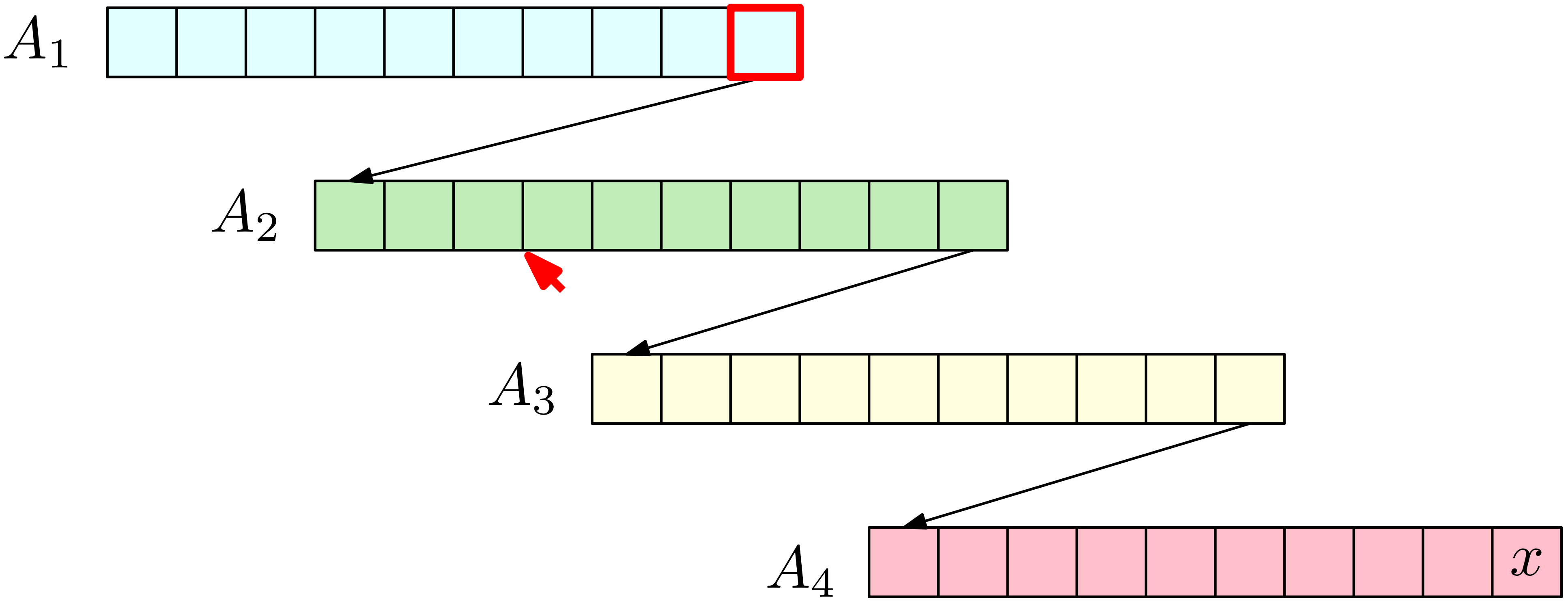
Fractional Cascading

How much time does it take?



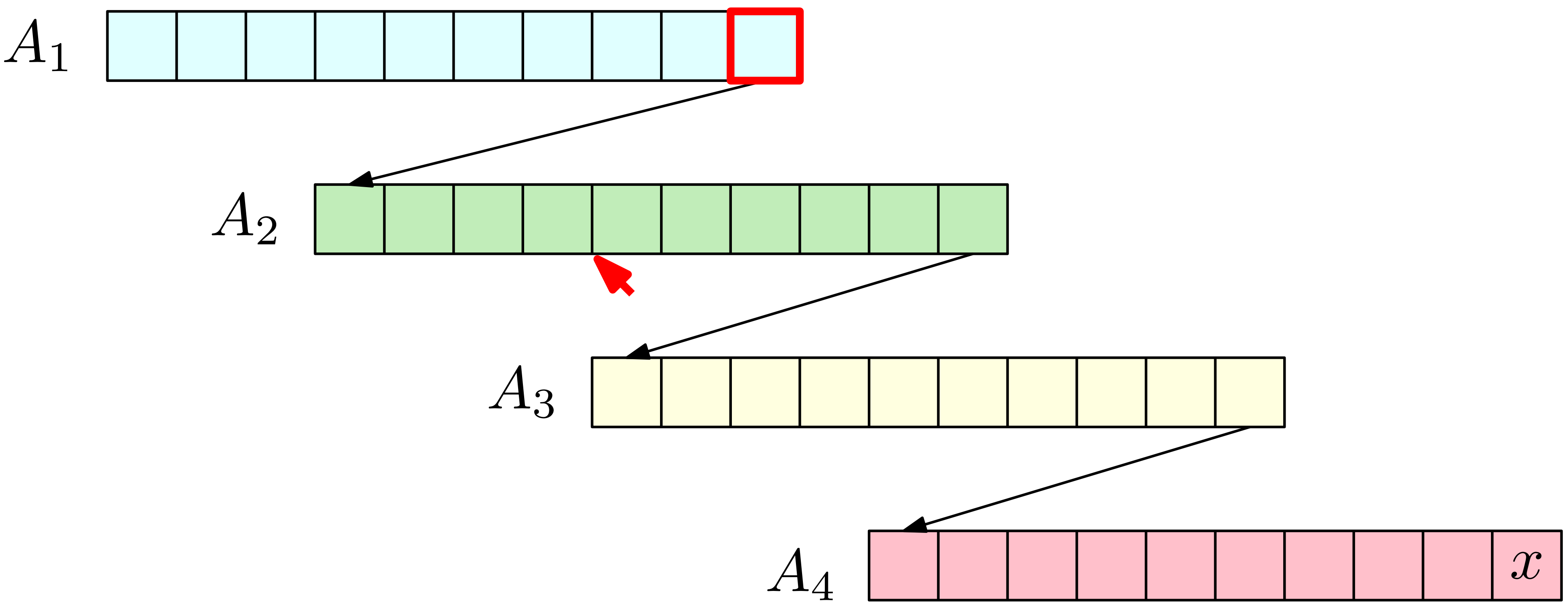
Fractional Cascading

How much time does it take?



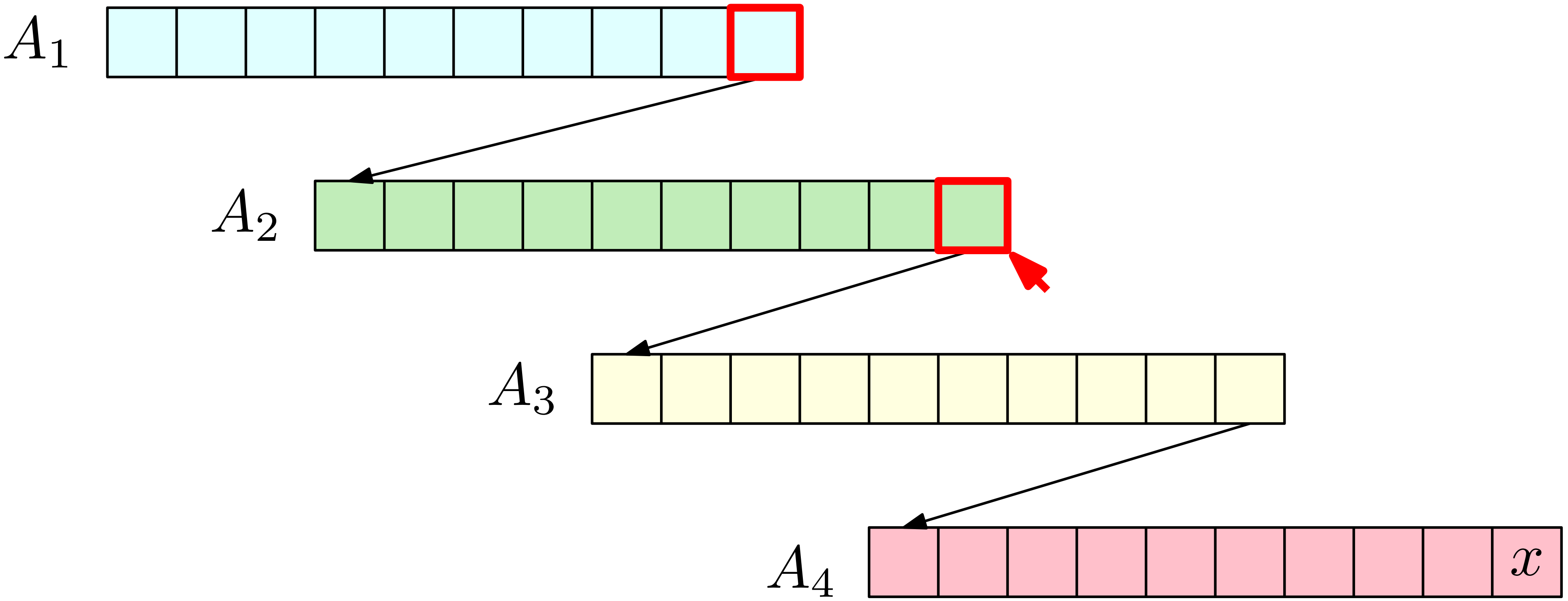
Fractional Cascading

How much time does it take?



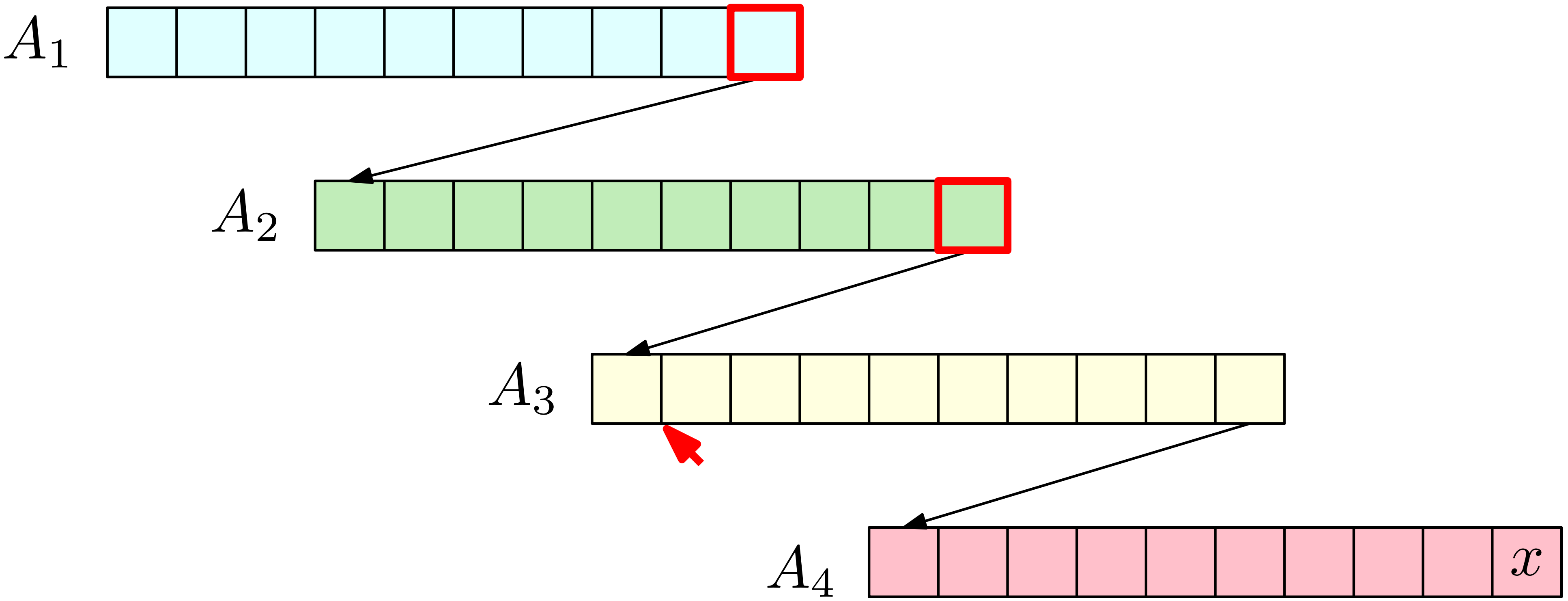
Fractional Cascading

How much time does it take?



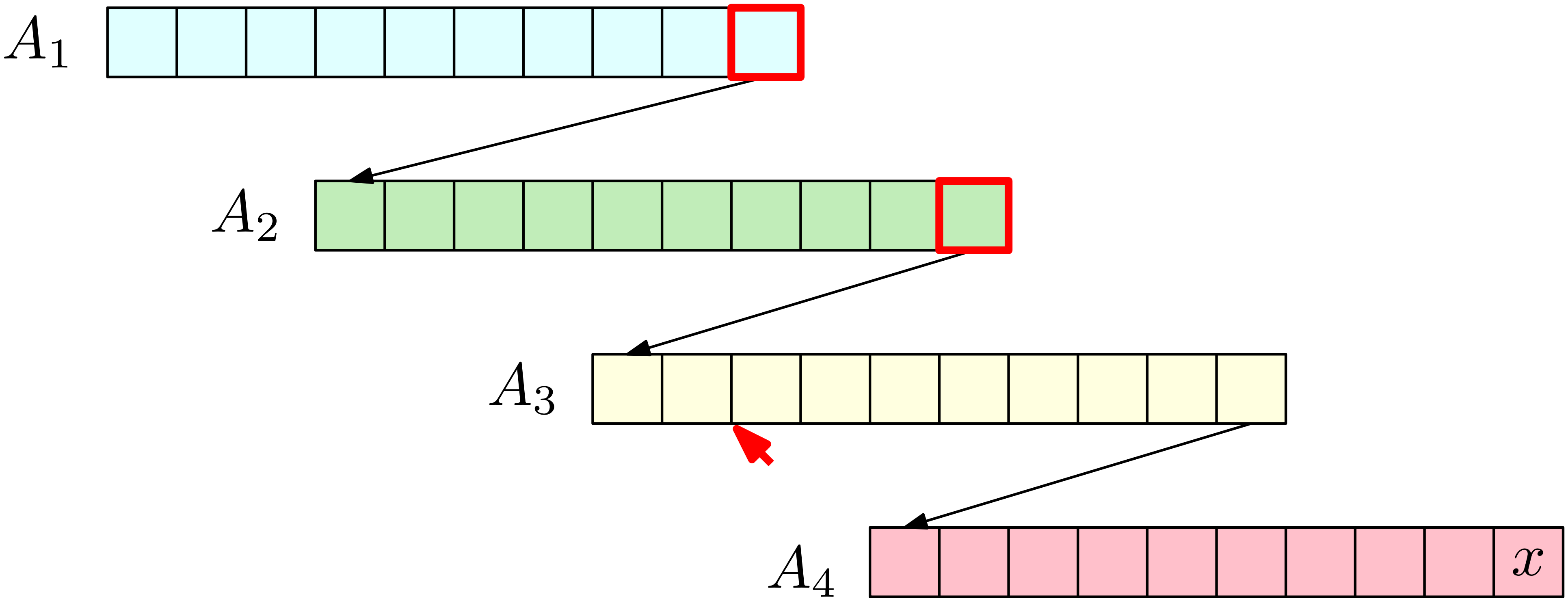
Fractional Cascading

How much time does it take?



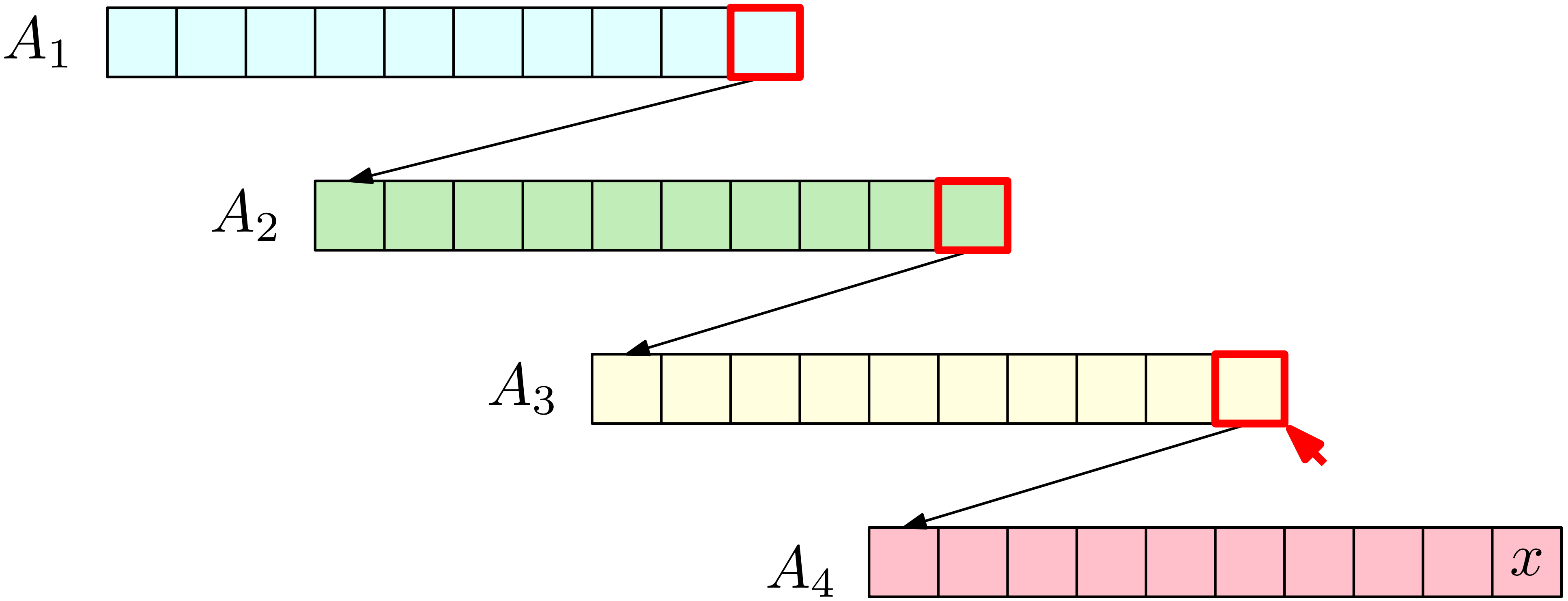
Fractional Cascading

How much time does it take?



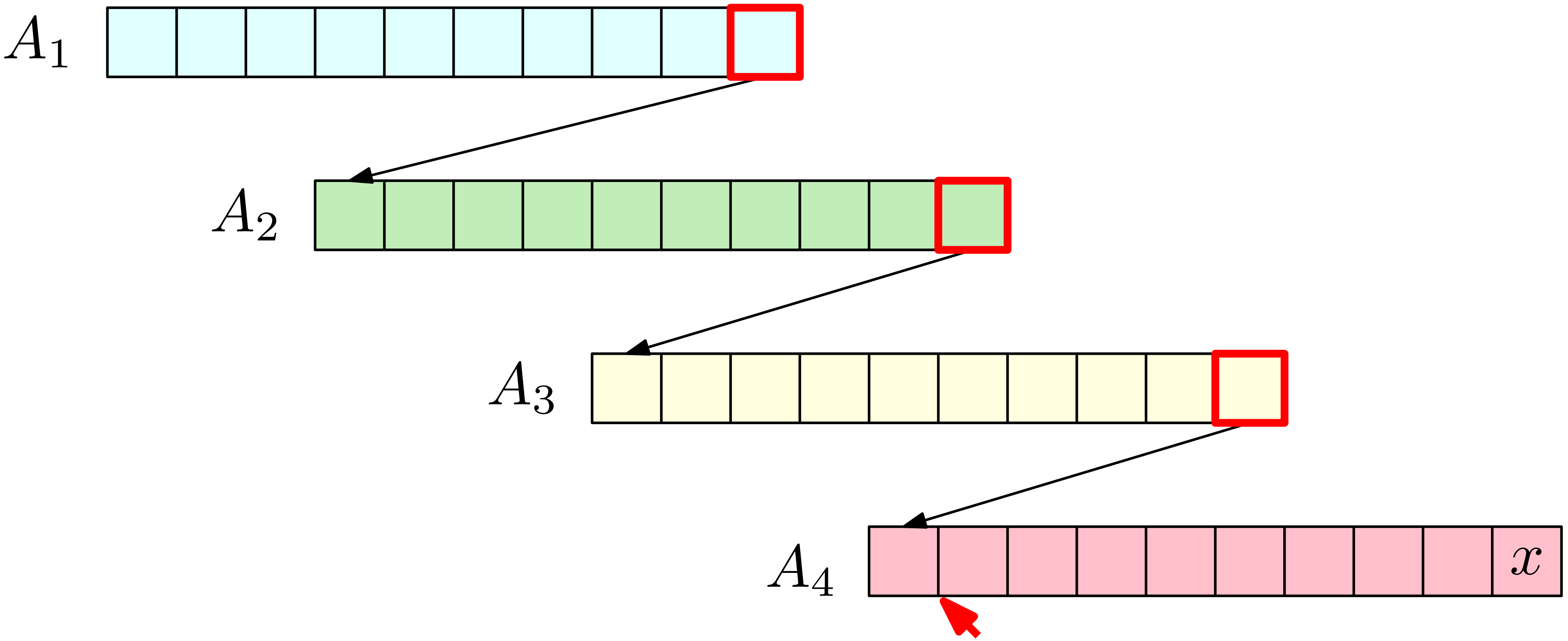
Fractional Cascading

How much time does it take?



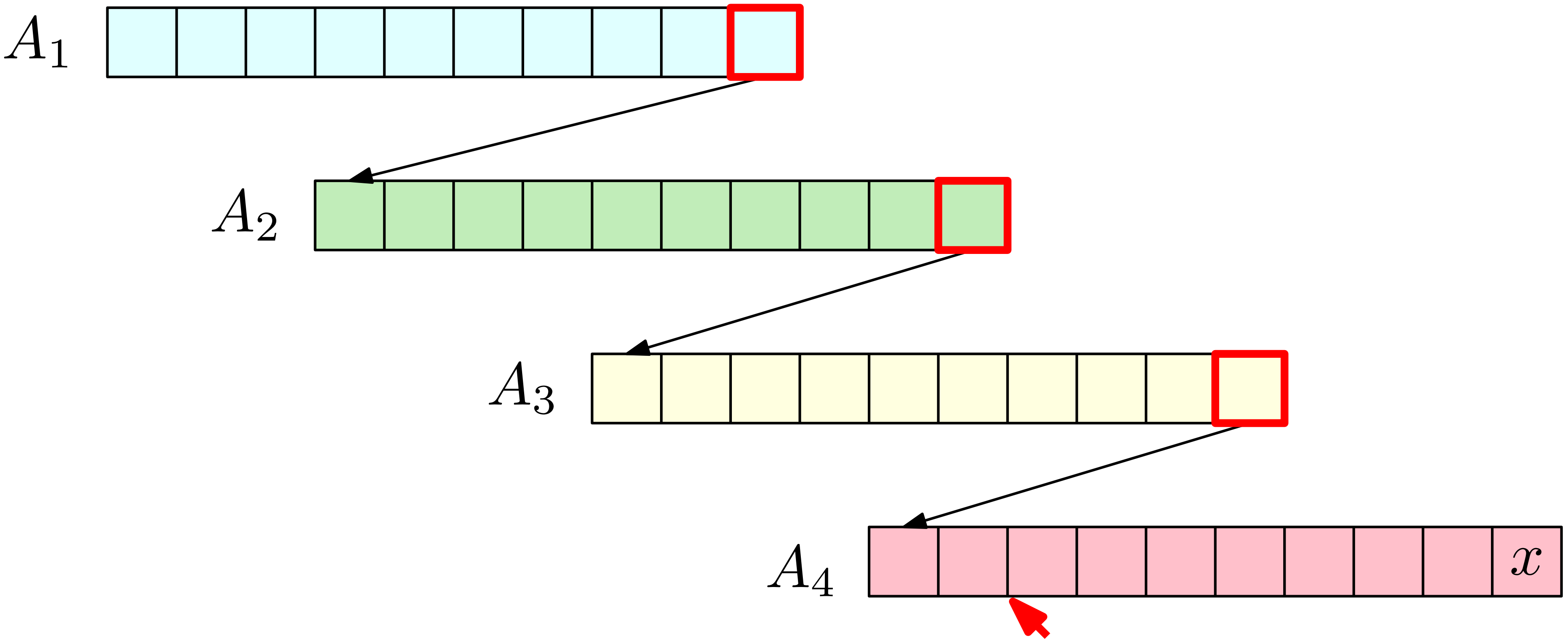
Fractional Cascading

How much time does it take?



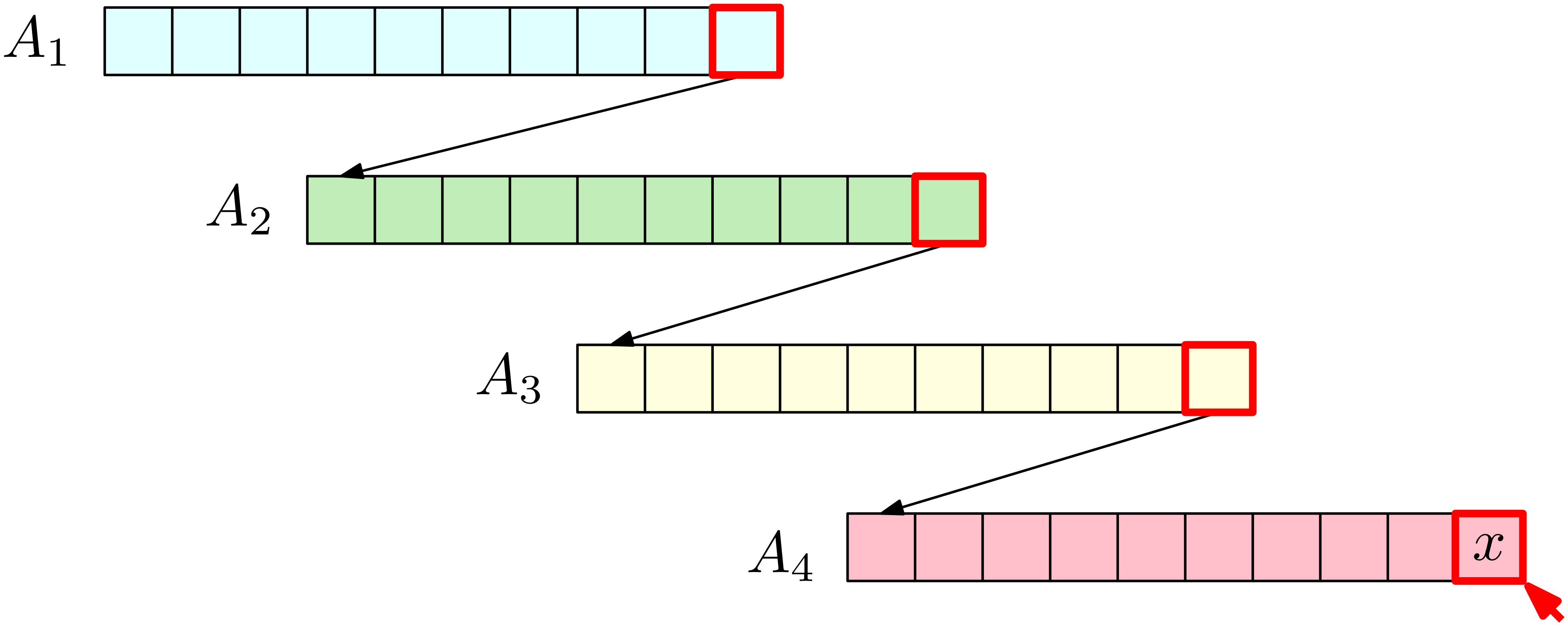
Fractional Cascading

How much time does it take?



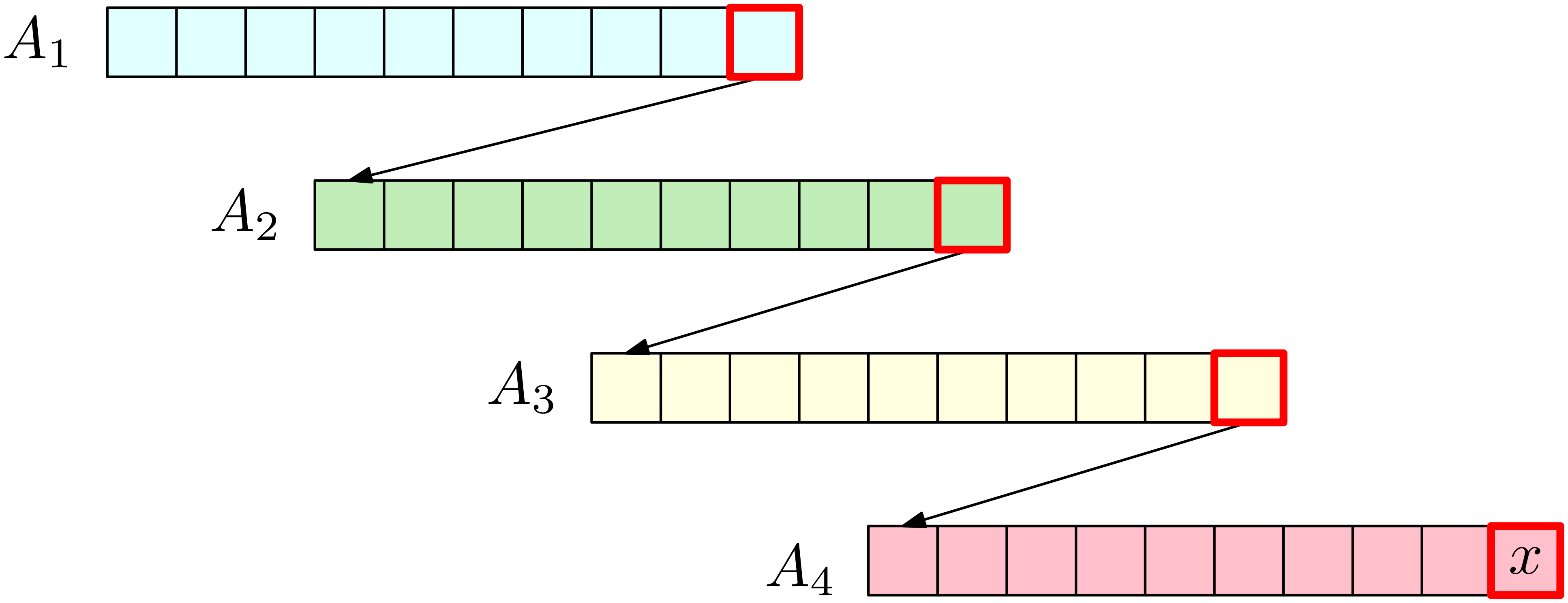
Fractional Cascading

How much time does it take?



Fractional Cascading

How much time does it take?

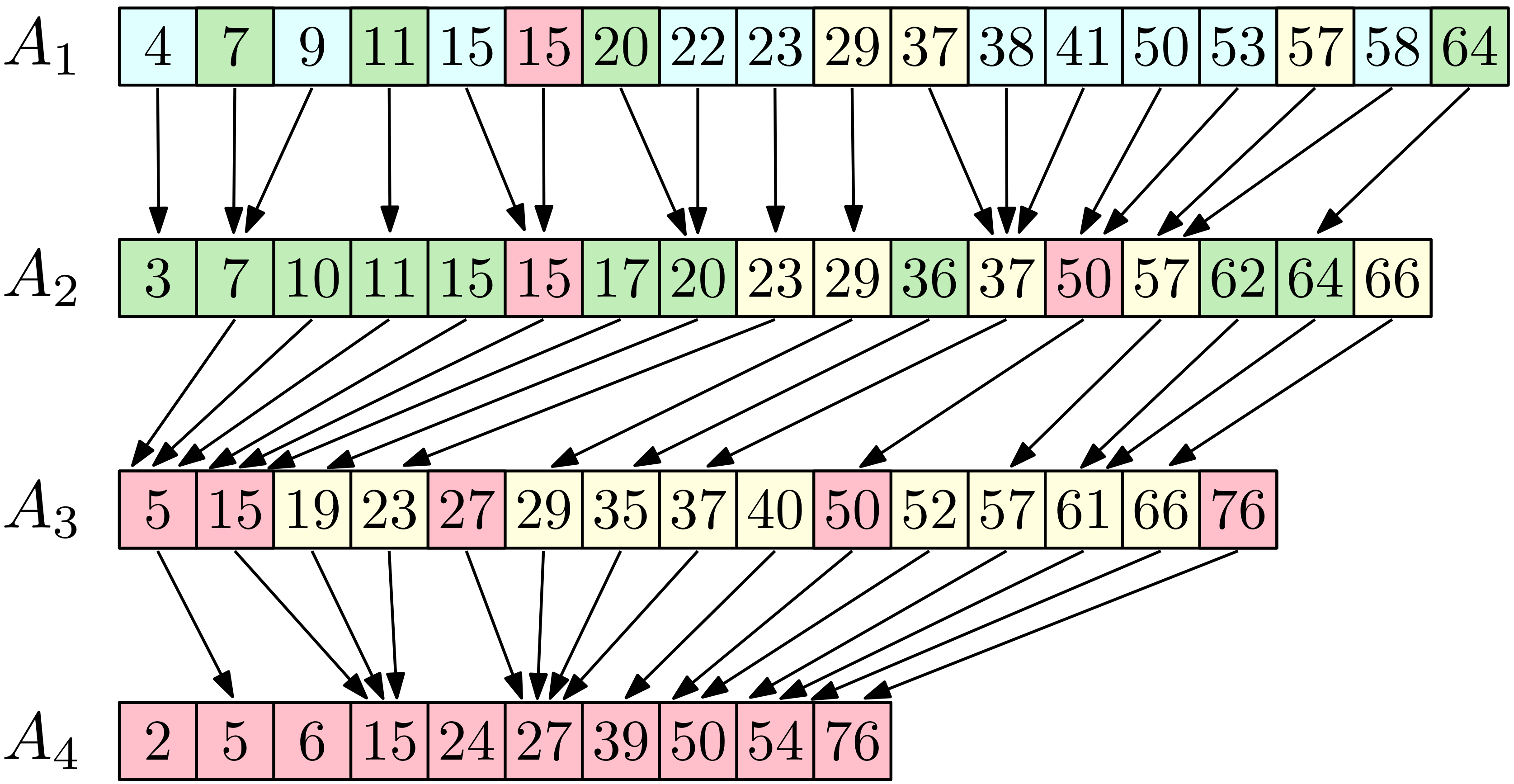


Worst-case time: $O(kn)$

Fractional Cascading

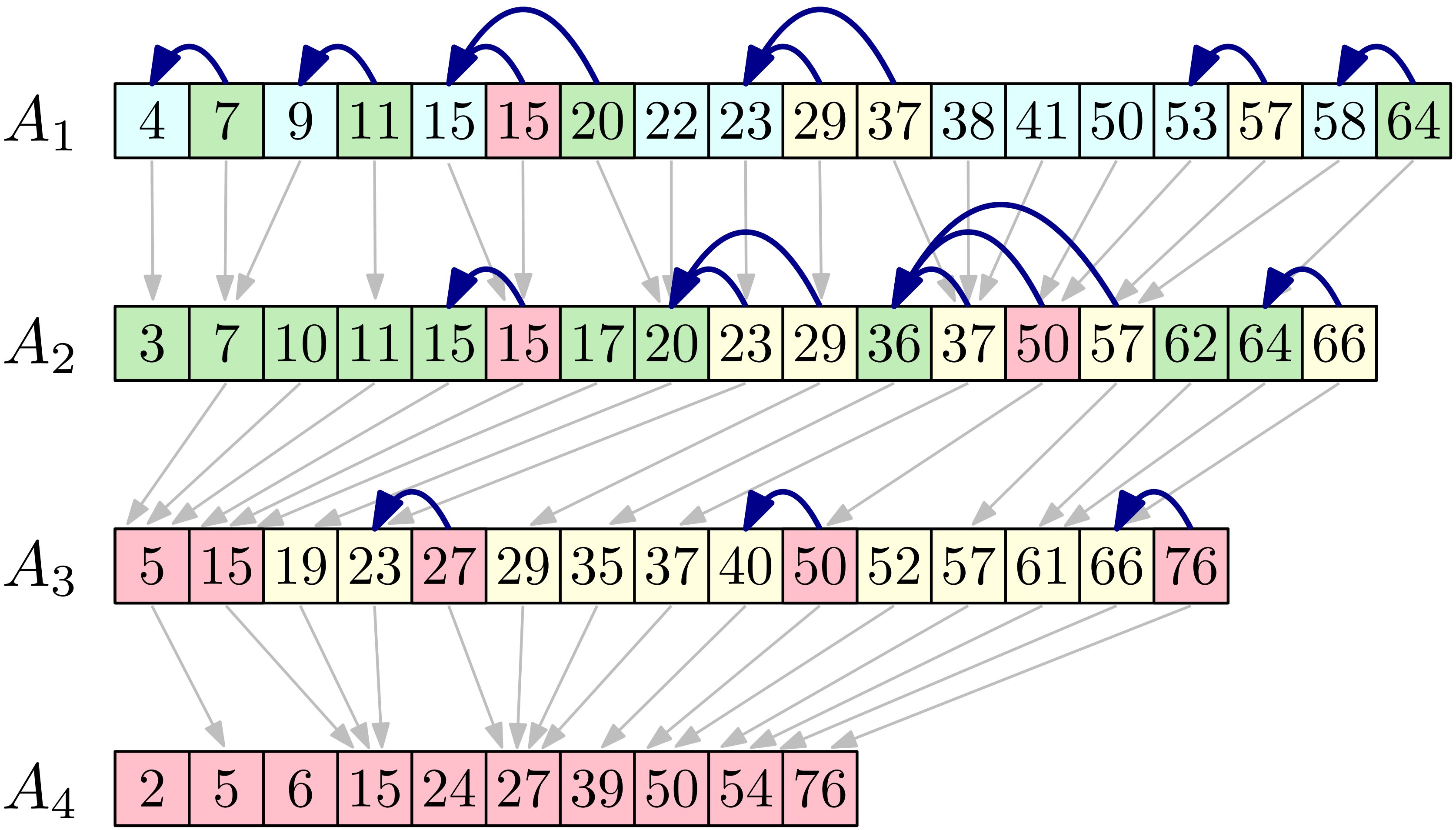
Second idea: fractional cascading

For $i = k, k-1, \ldots, 2$: Add every other element of A_i to A_{i-1}.



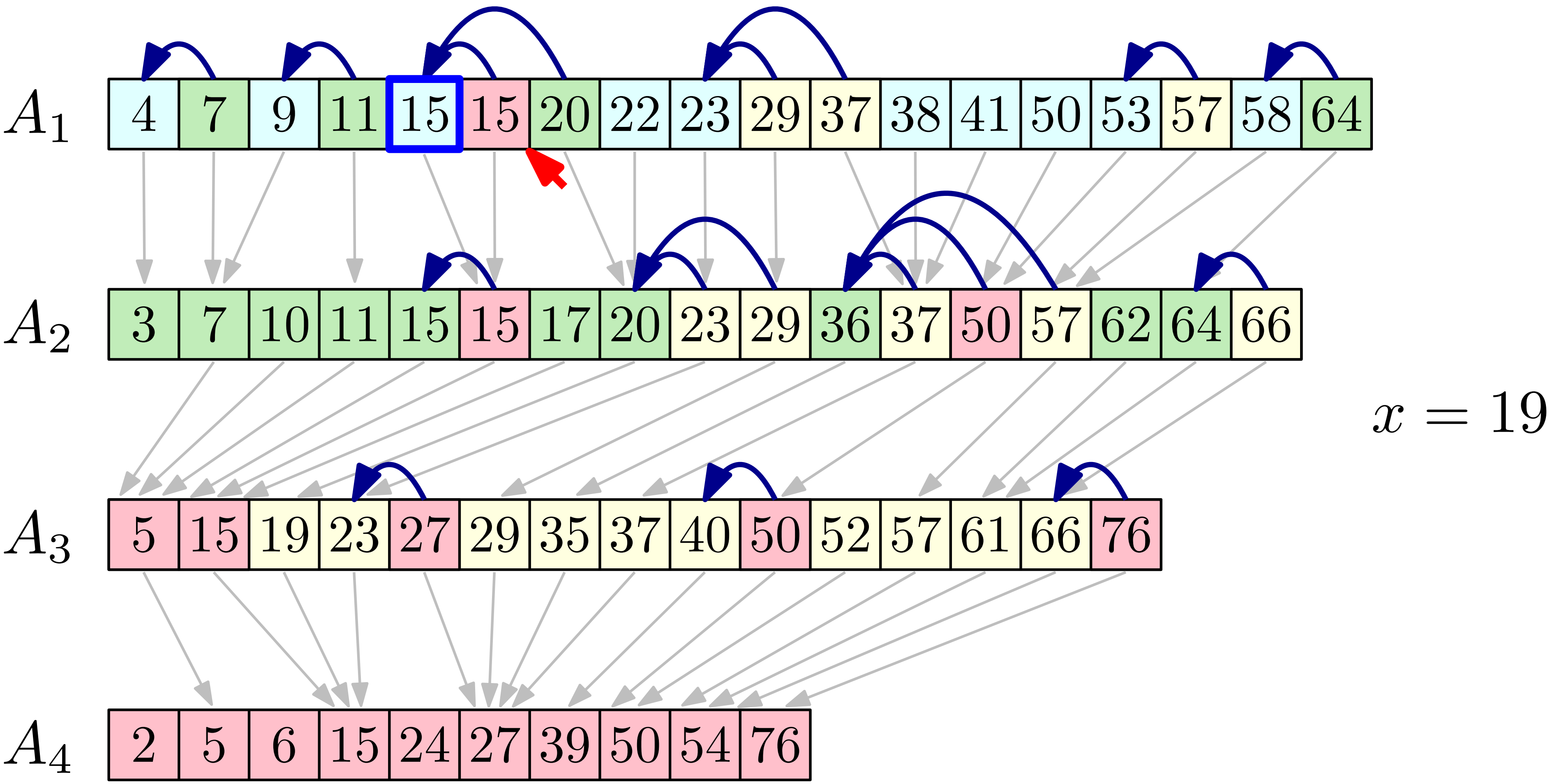
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



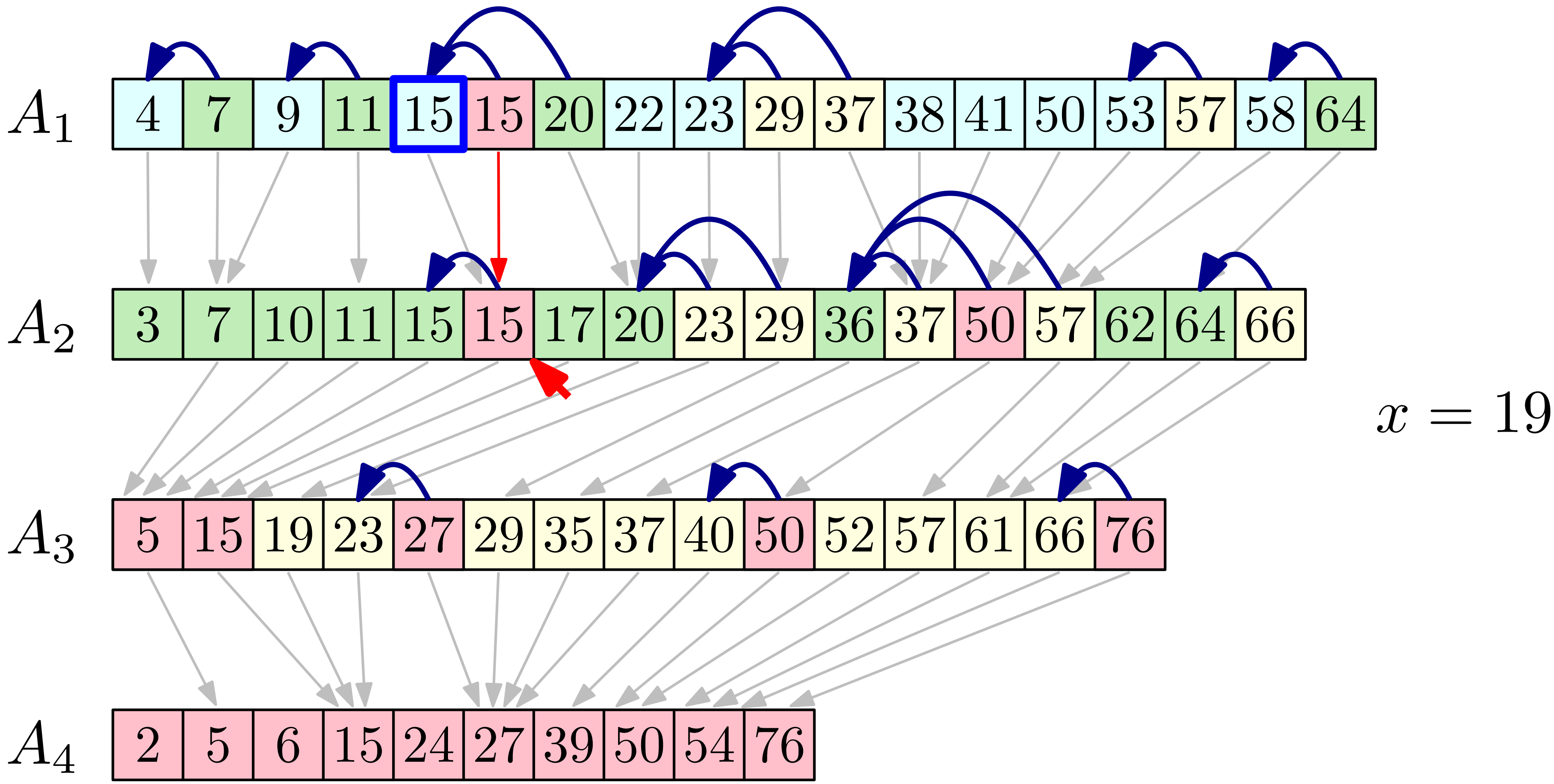
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



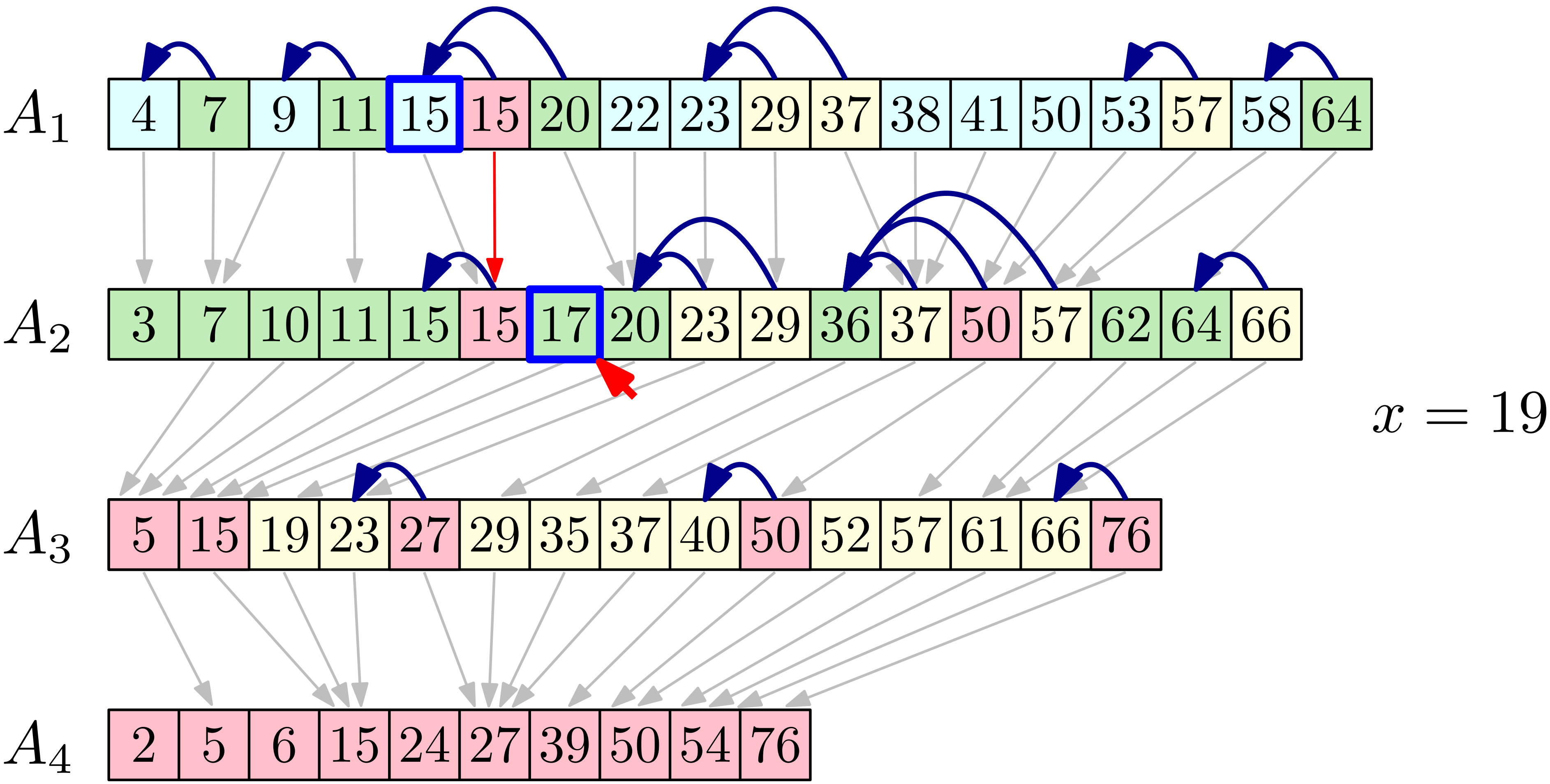
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



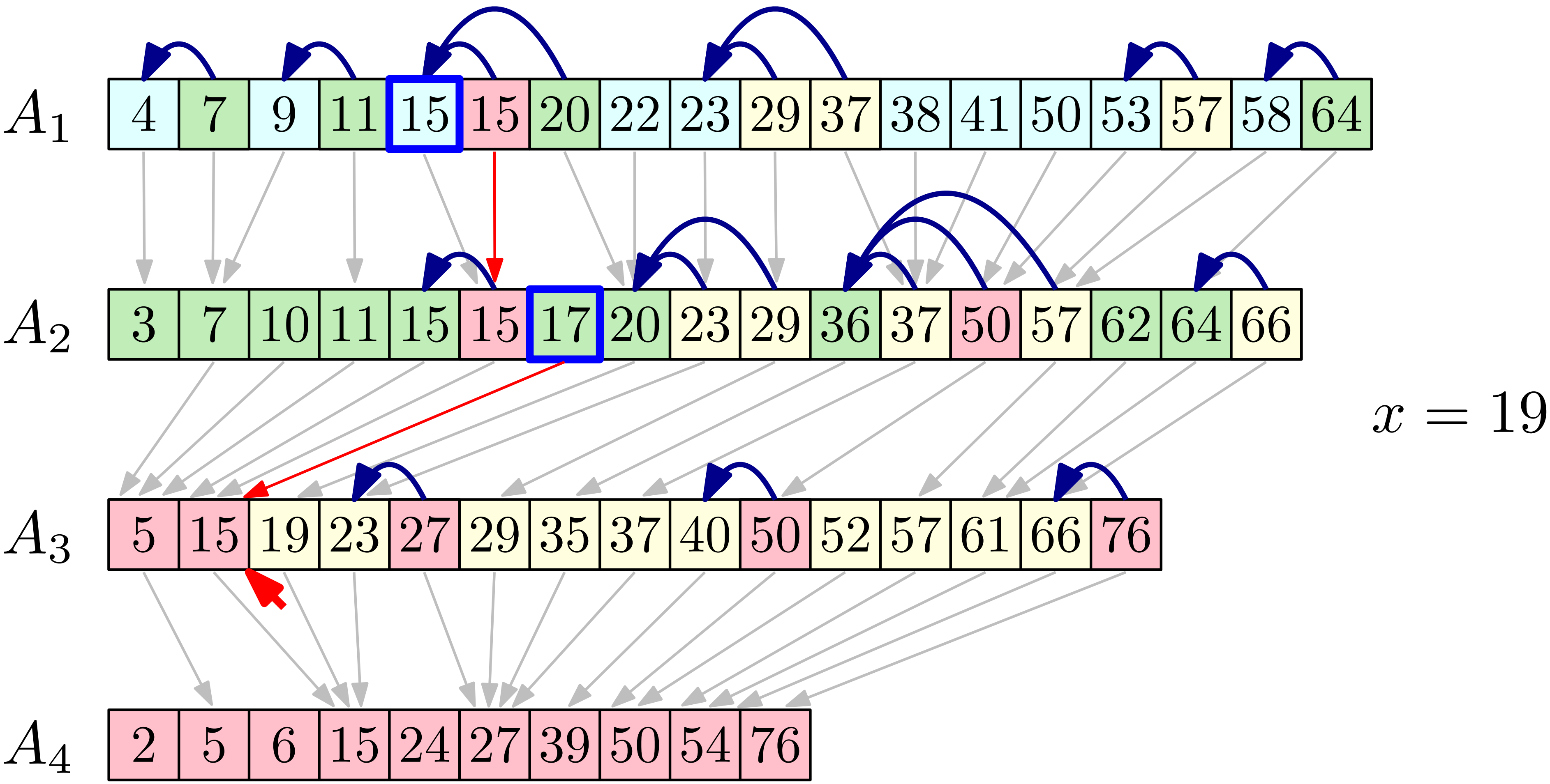
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



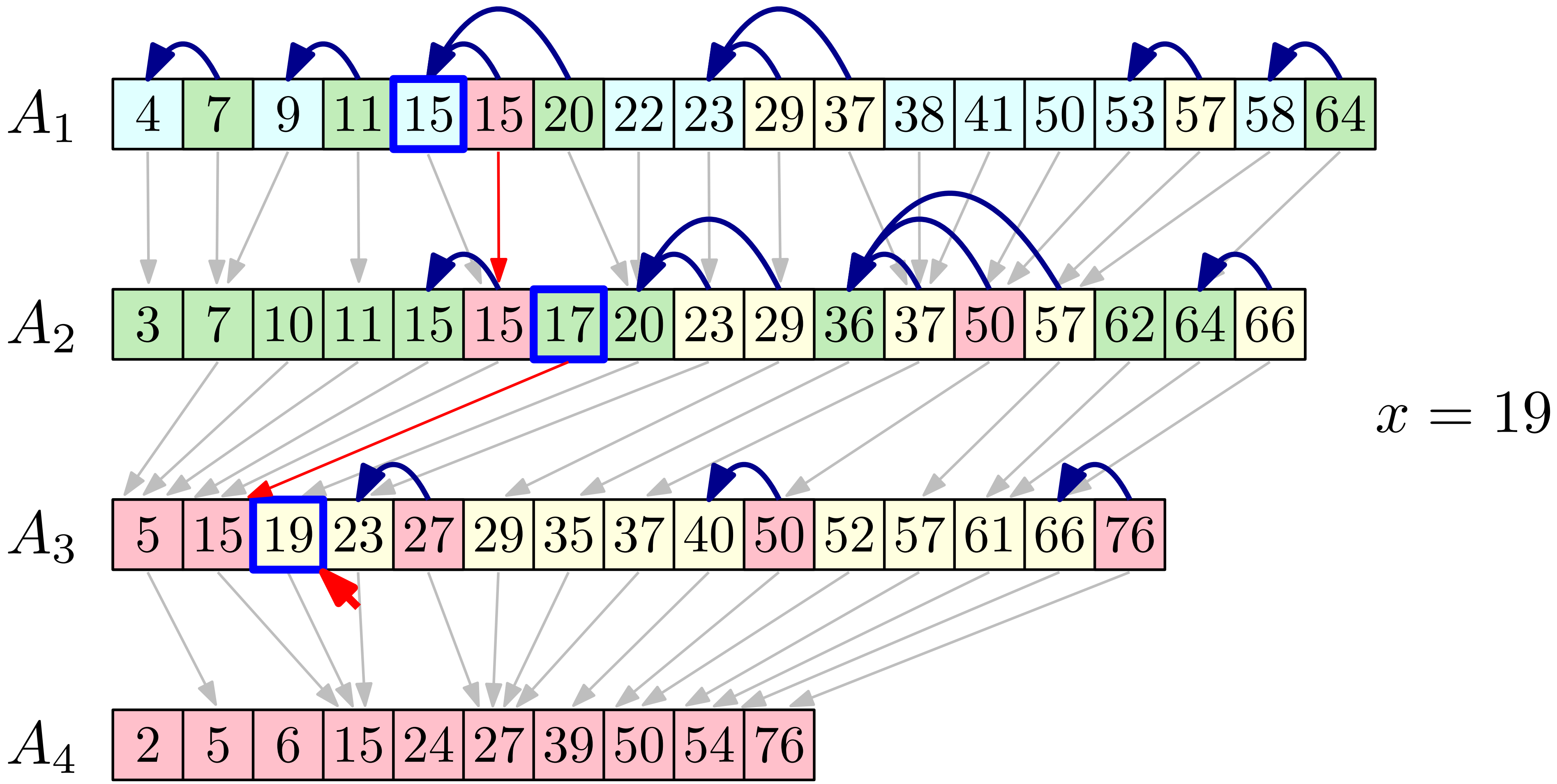
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



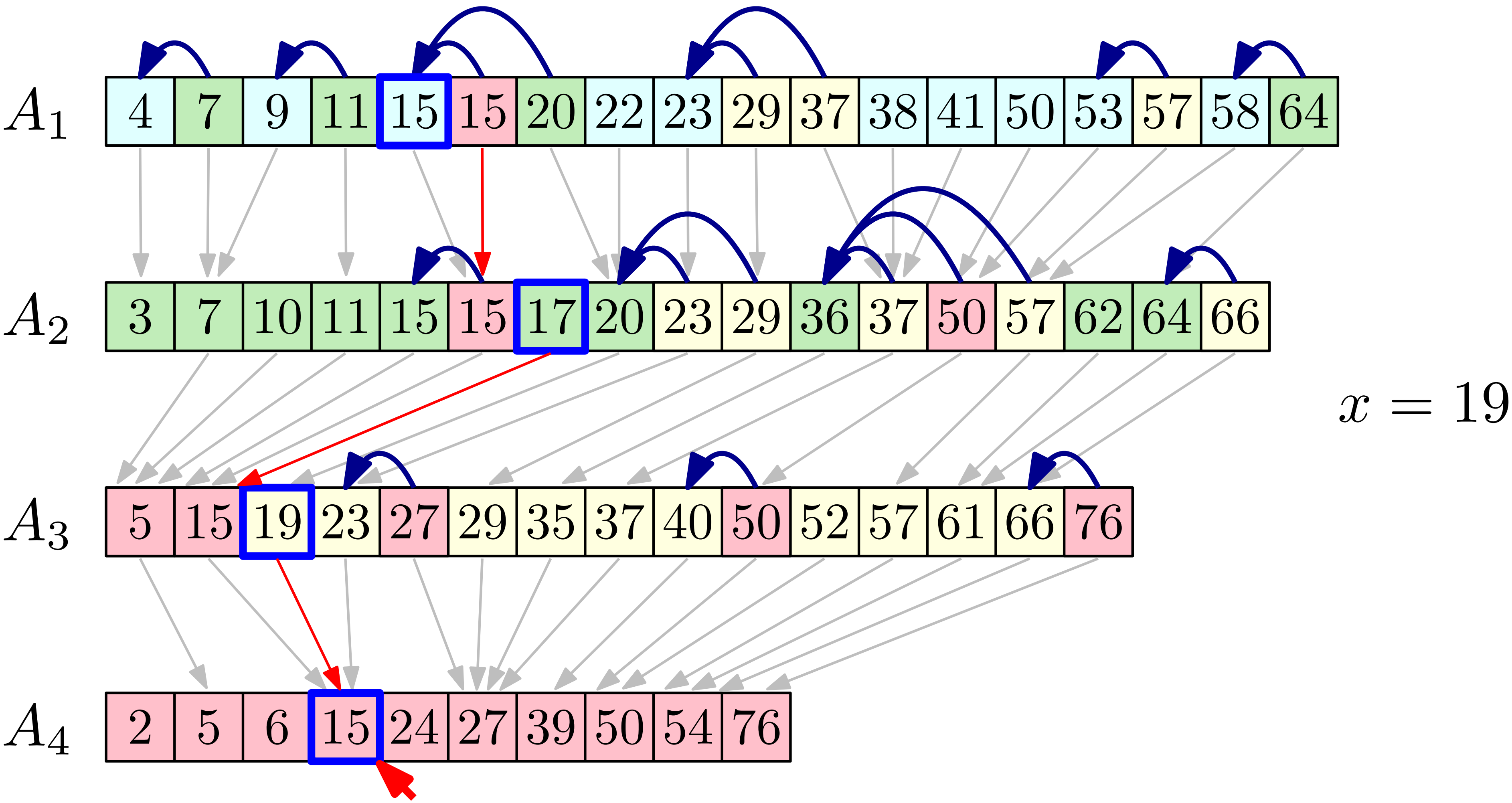
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



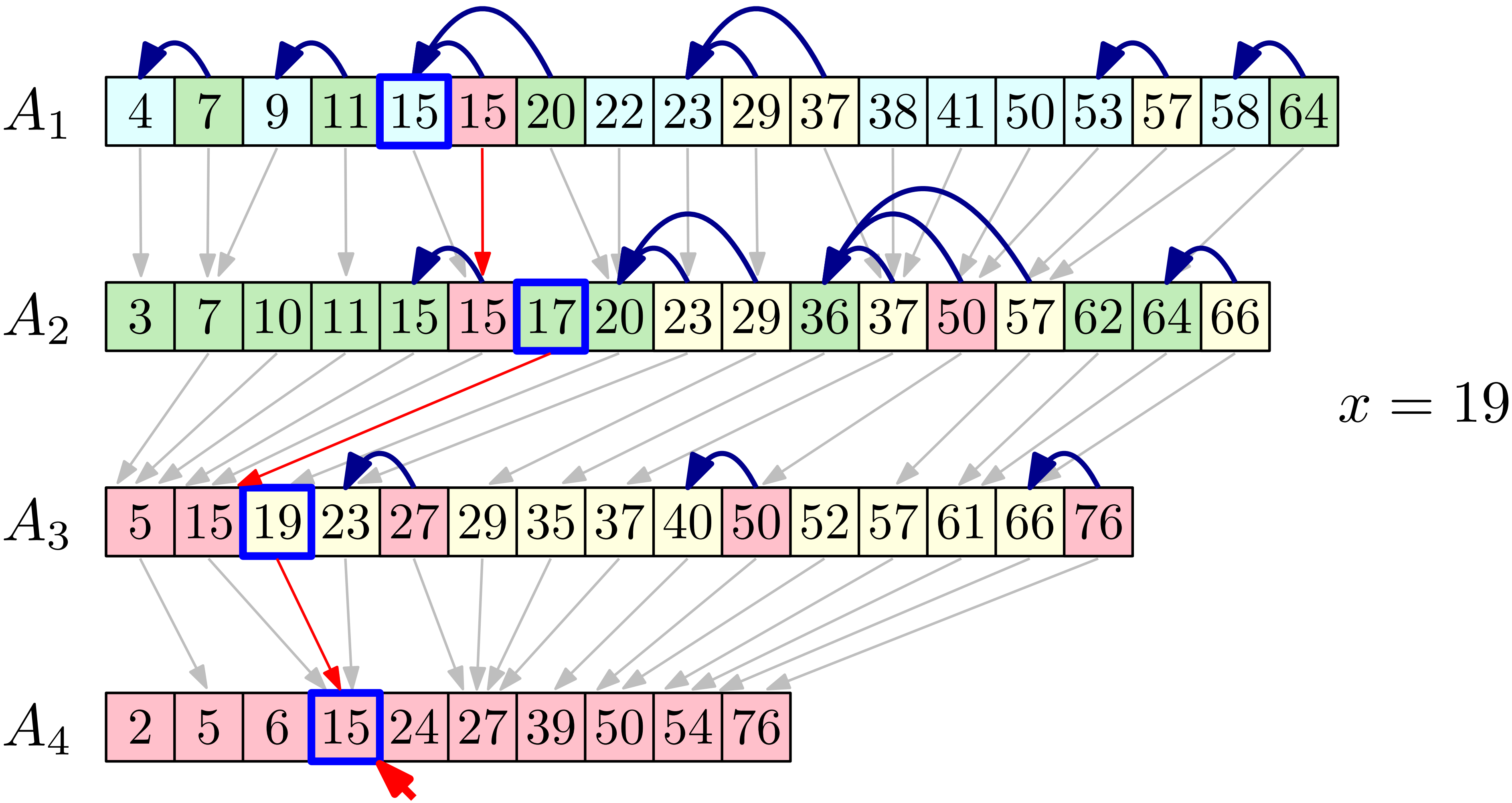
Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



Fractional Cascading

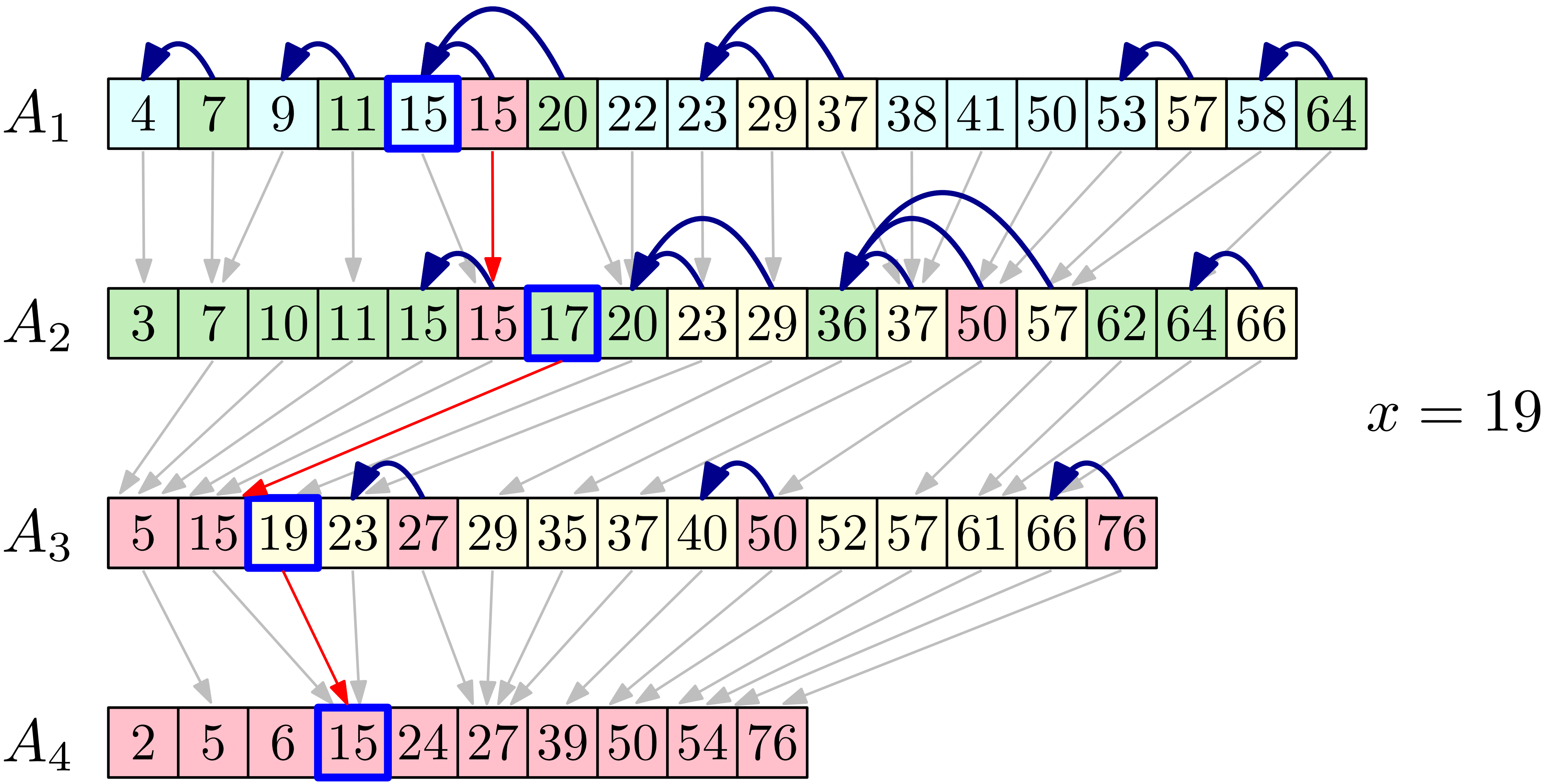
Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



Observation: the red pointer advances at most once per array

Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i

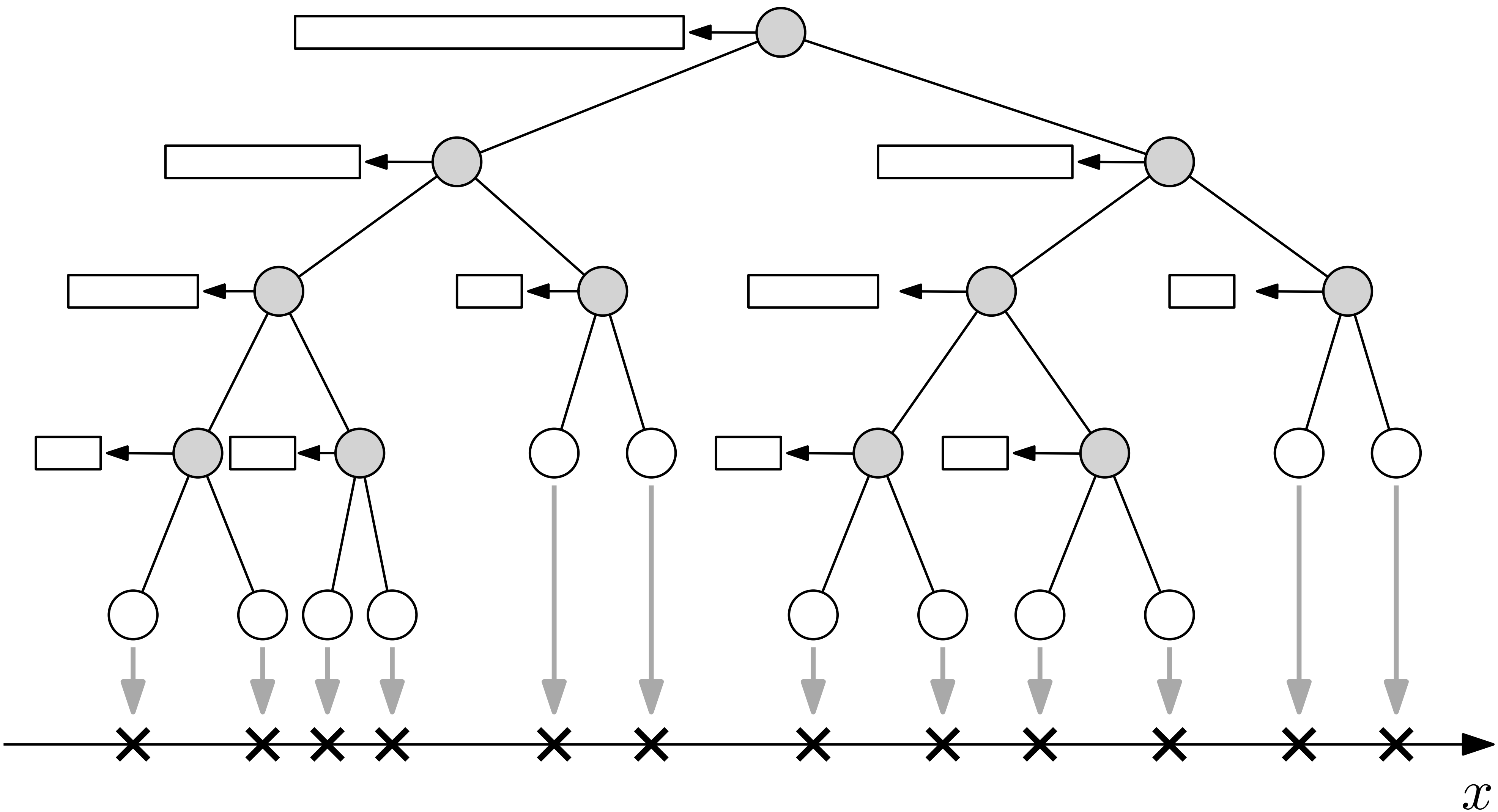


Size: $O(kn)$ **Preprocessing:** $O(kn)$ **Query:** $O(k + \log n)$

Layered Range Trees

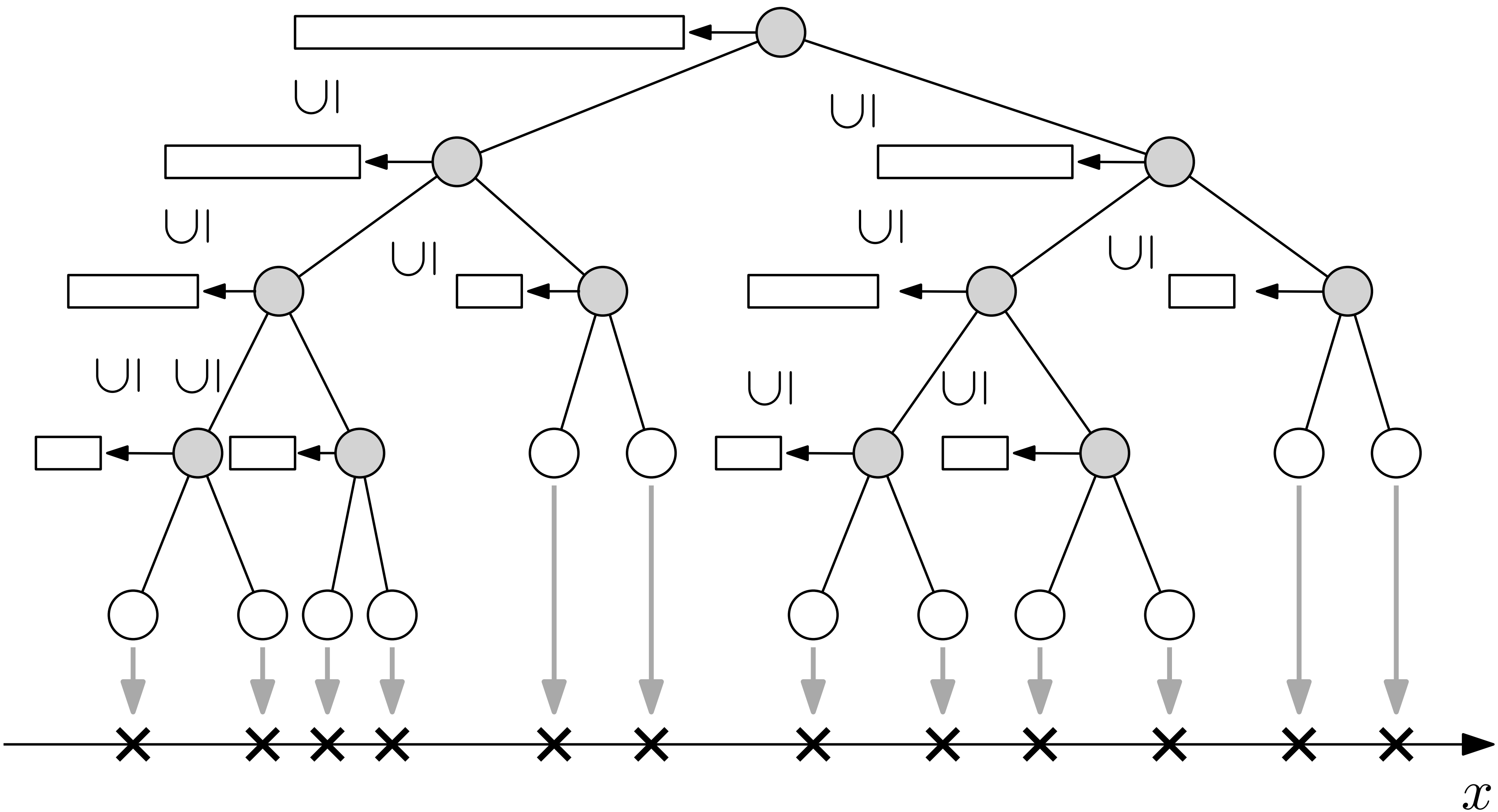
Layered Range Trees, $D = 2$

Build a 2D range tree in which the inner 1D range trees are implemented with arrays



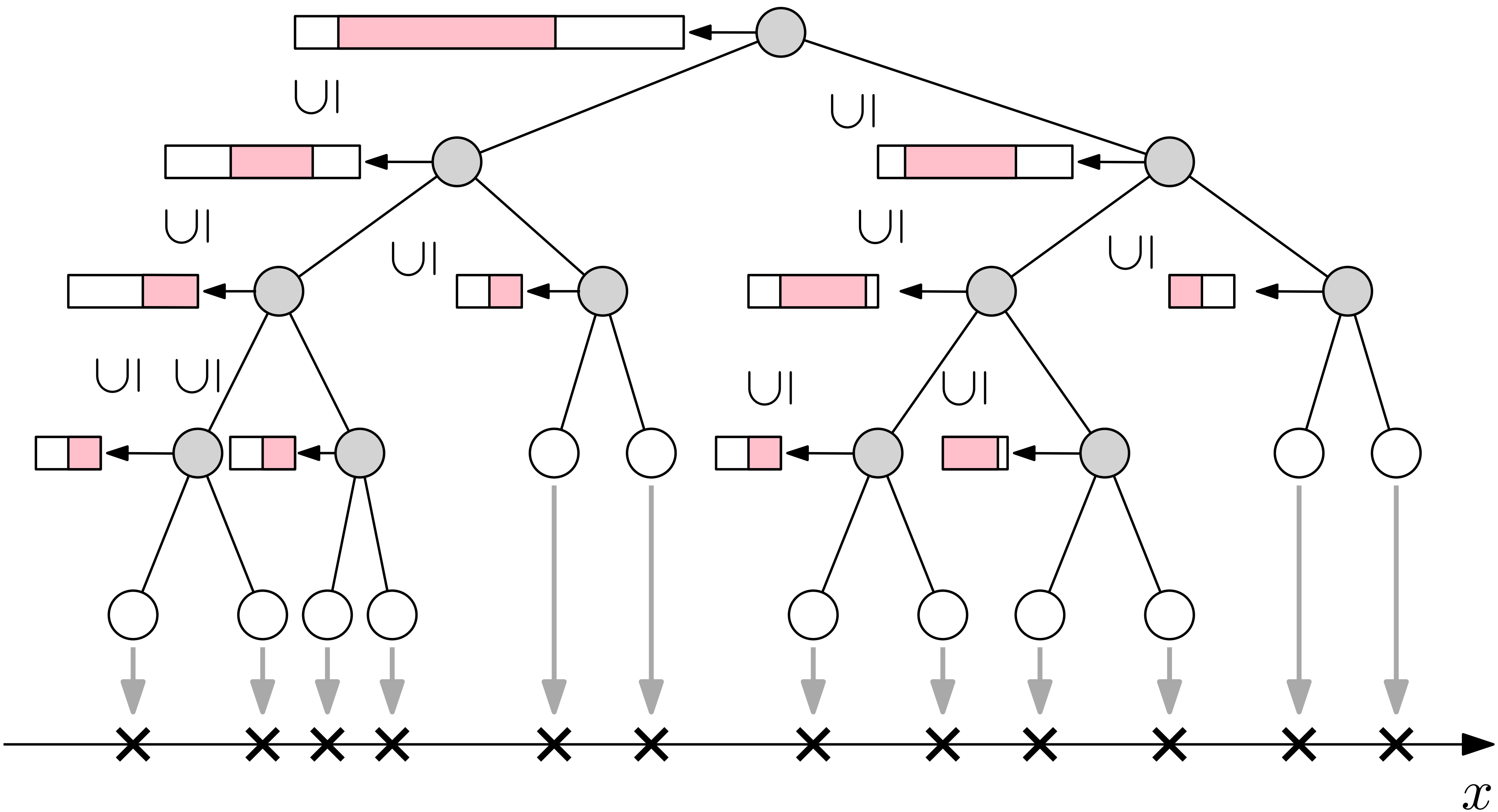
Layered Range Trees, $D = 2$

Reuse the cross-linking idea from fractional cascading



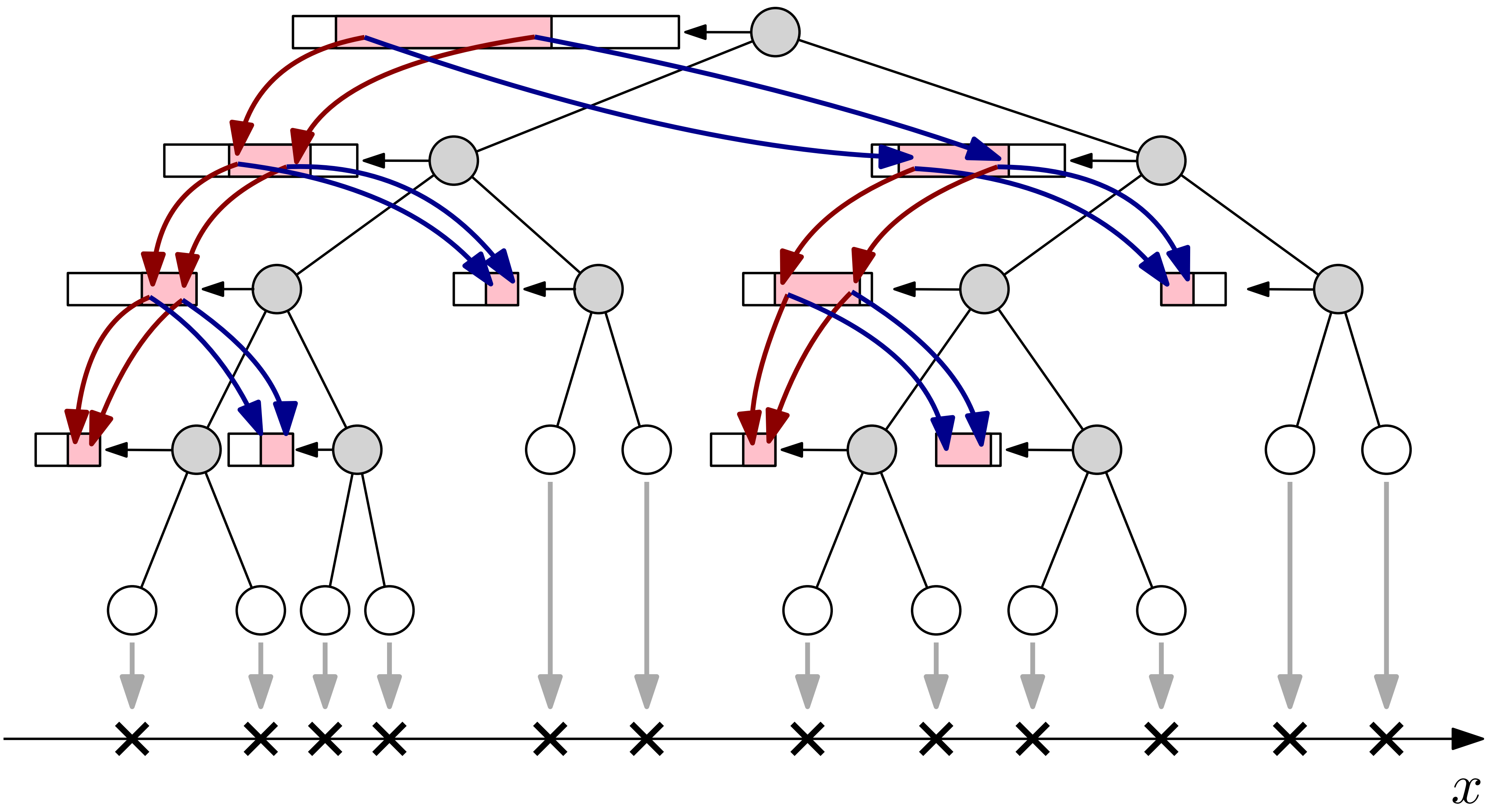
Layered Range Trees, $D = 2$

Reuse the cross-linking idea from fractional cascading



Layered Range Trees, $D = 2$

$\forall$ element y in the 1D range tree of v, store a pointer to the predecessor of y in the 1D range tree of the left/right child of v.



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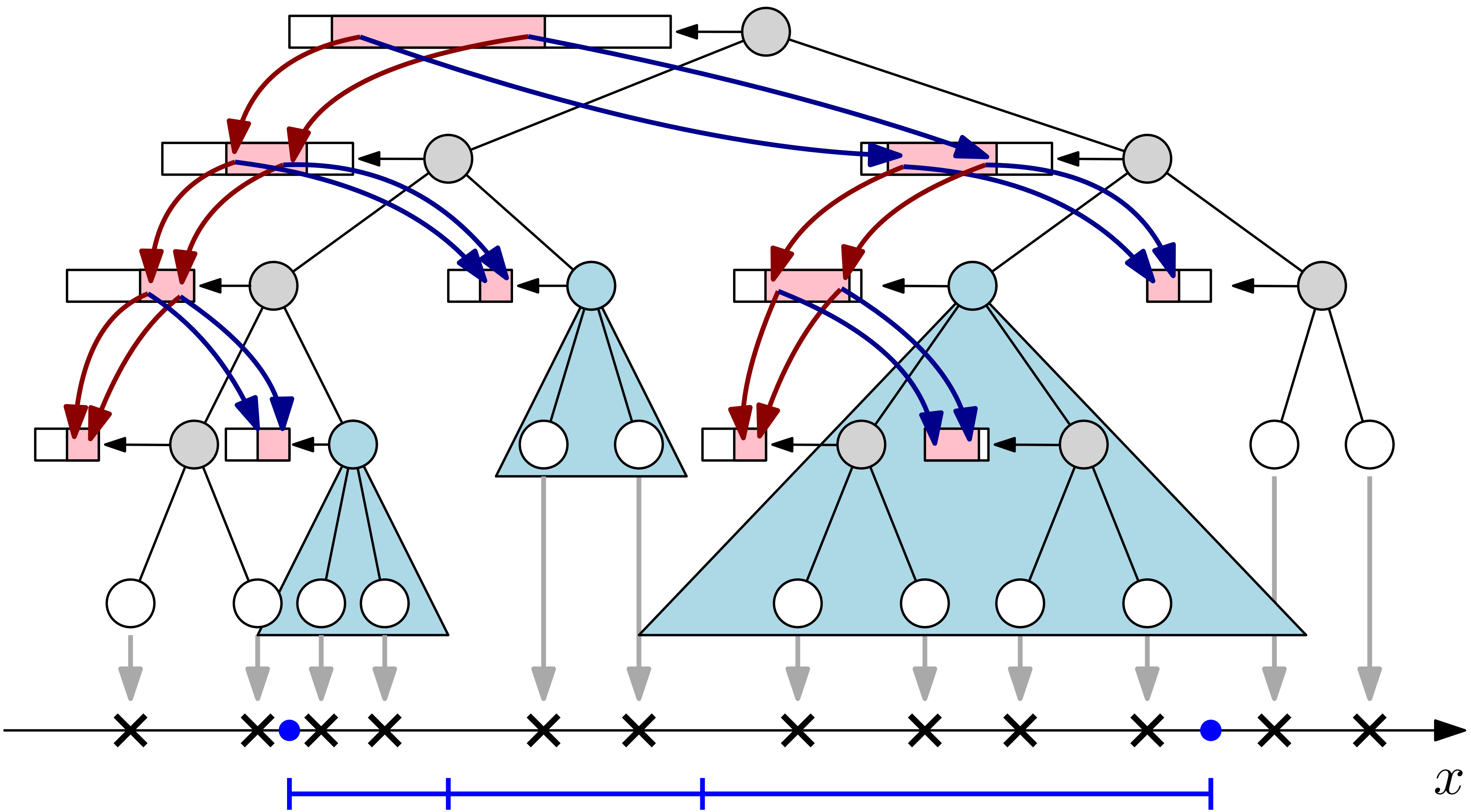

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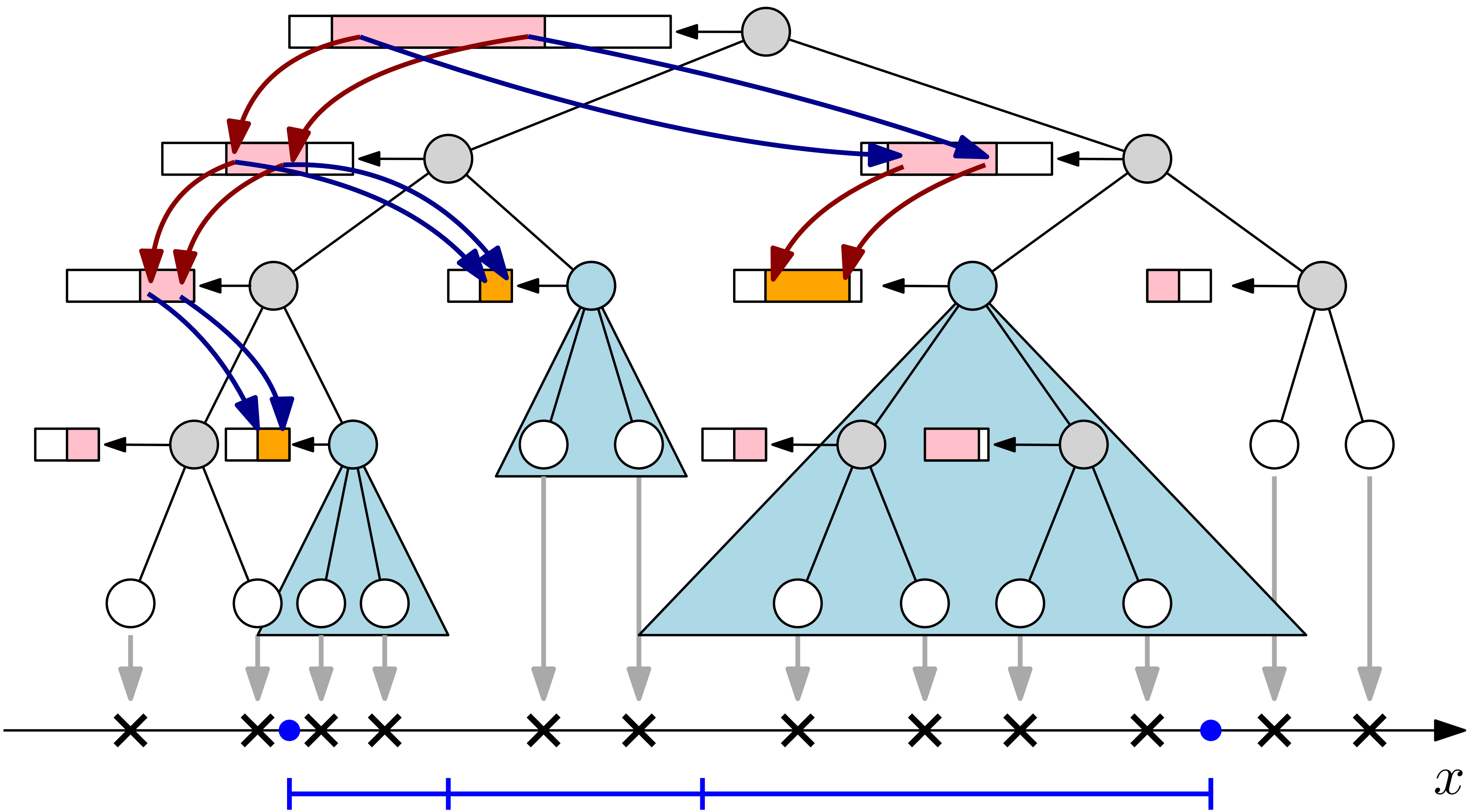

Layered Range Trees, $D = 2$

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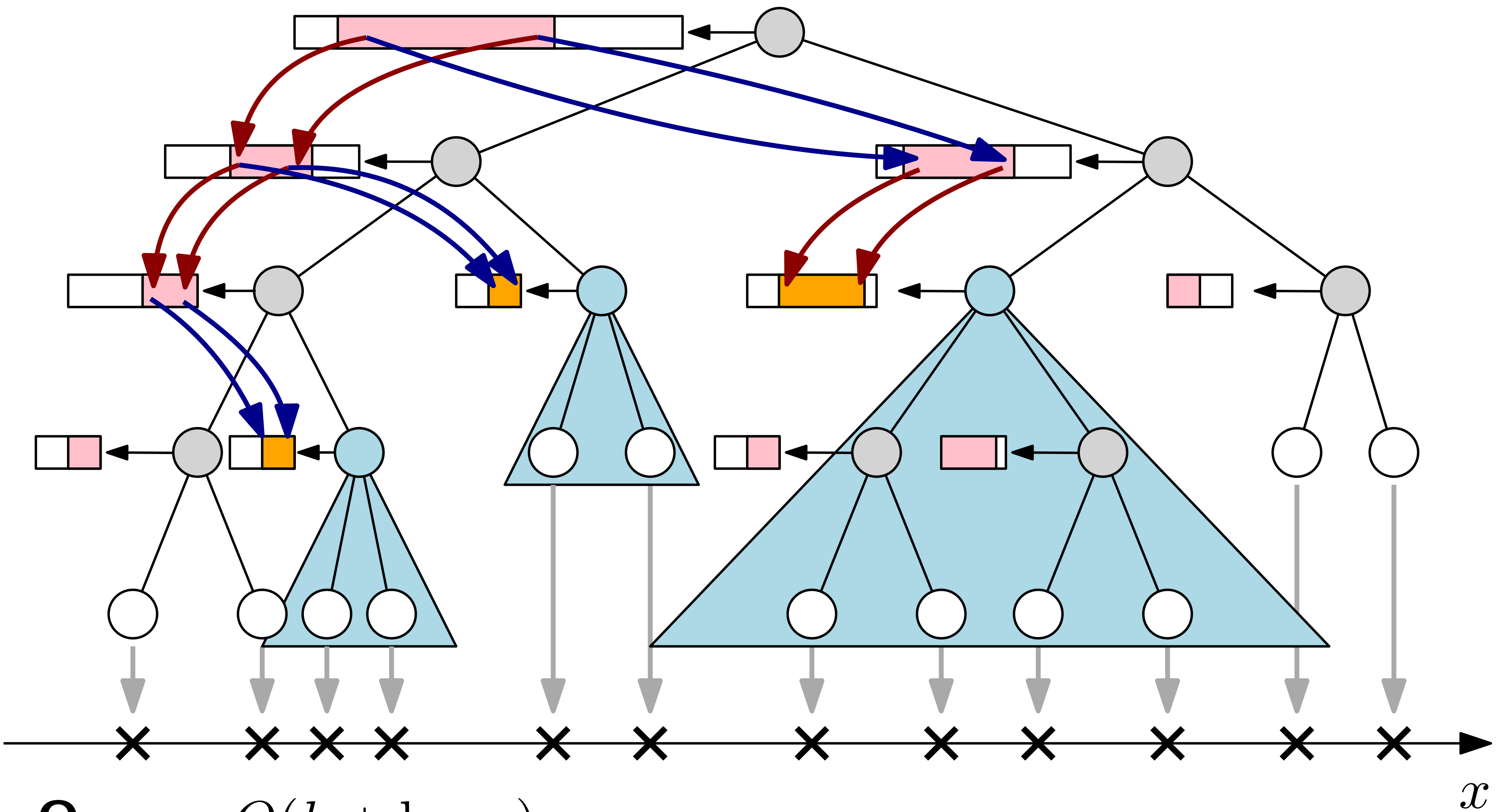

Query: $O(k + \log n)$

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log n + k)$	with cross-linking

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log n + k)$	with cross-linking
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^{D-1} n + k)$	with cross-linking

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log n + k)$	with cross-linking
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^{D-1} n + k)$	with cross-linking

Can be made dynamic (supports point insertion / deletion) in $O(\log^D n)$ amortized time per update.