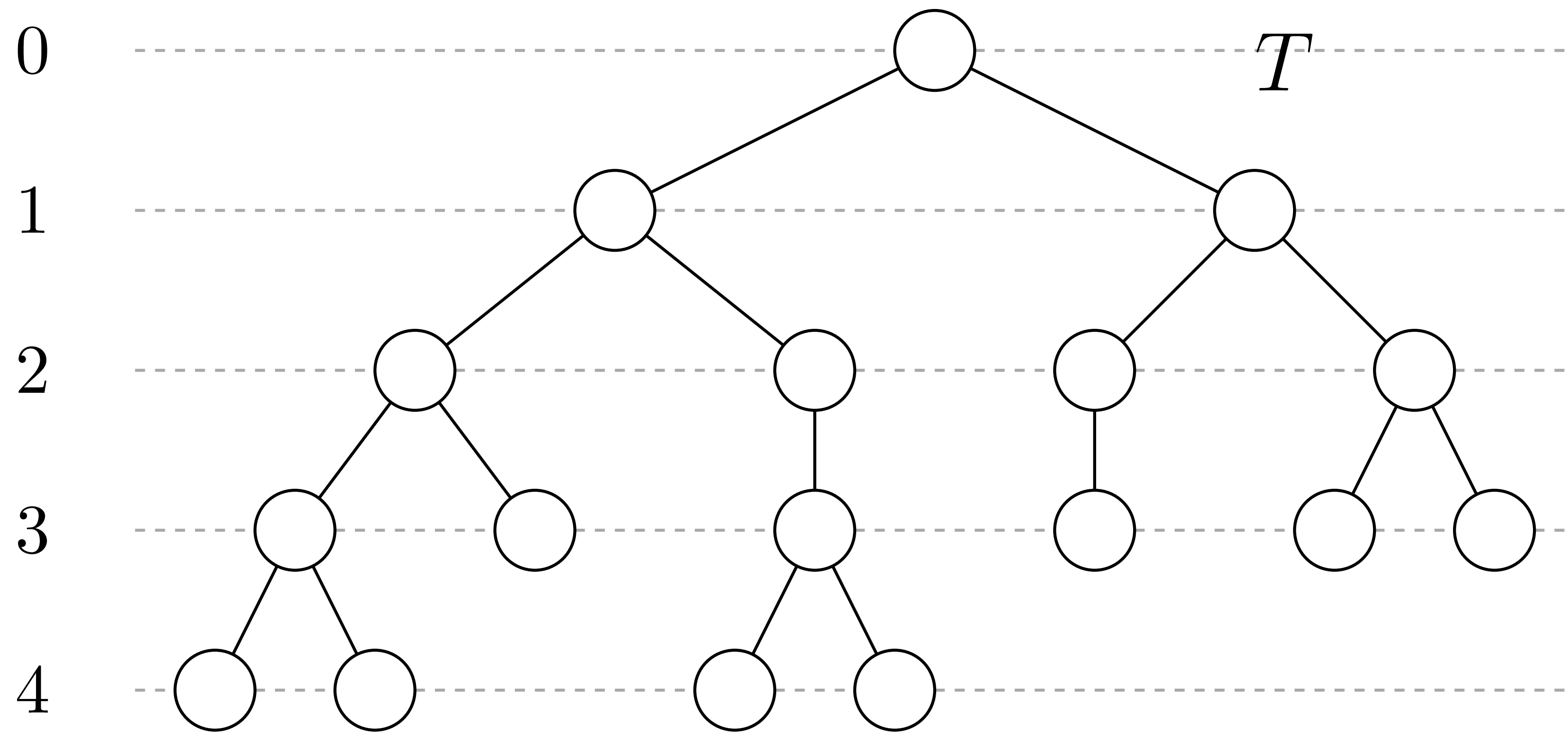
Level Ancestors

Level Ancestor Queries



Definition: Let v be a vertex at depth d_v in T. For $d \leq d_v$, a *level ancestor query* LA(v, d) on a vertex v asks to report the ancestor of v at depth d.

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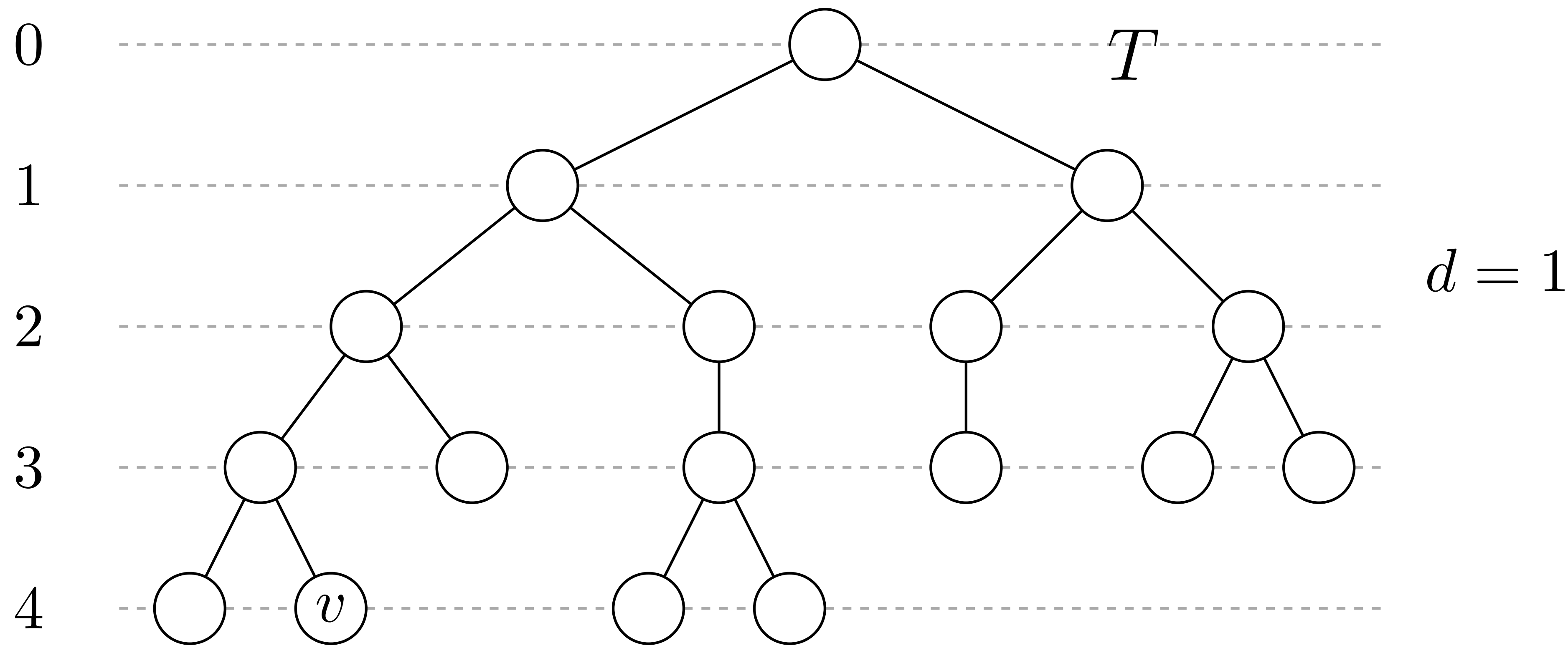

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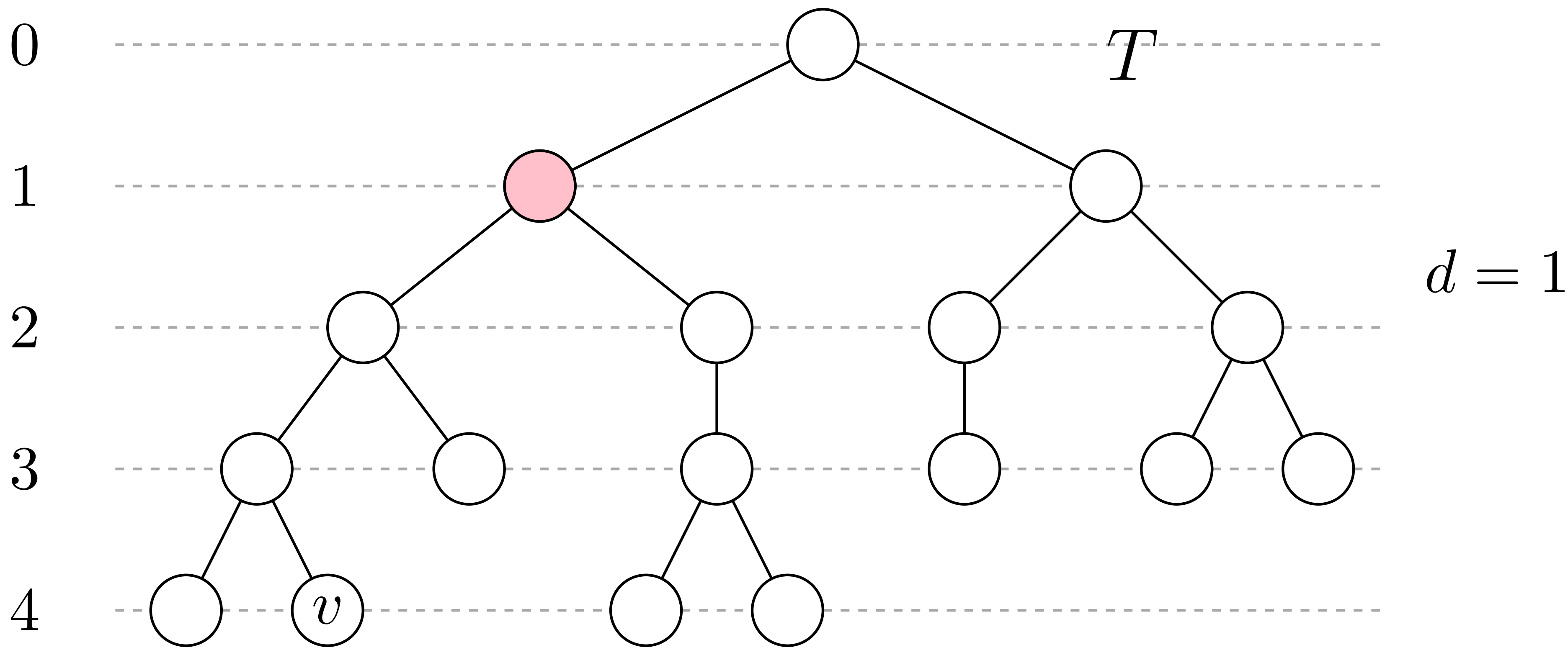

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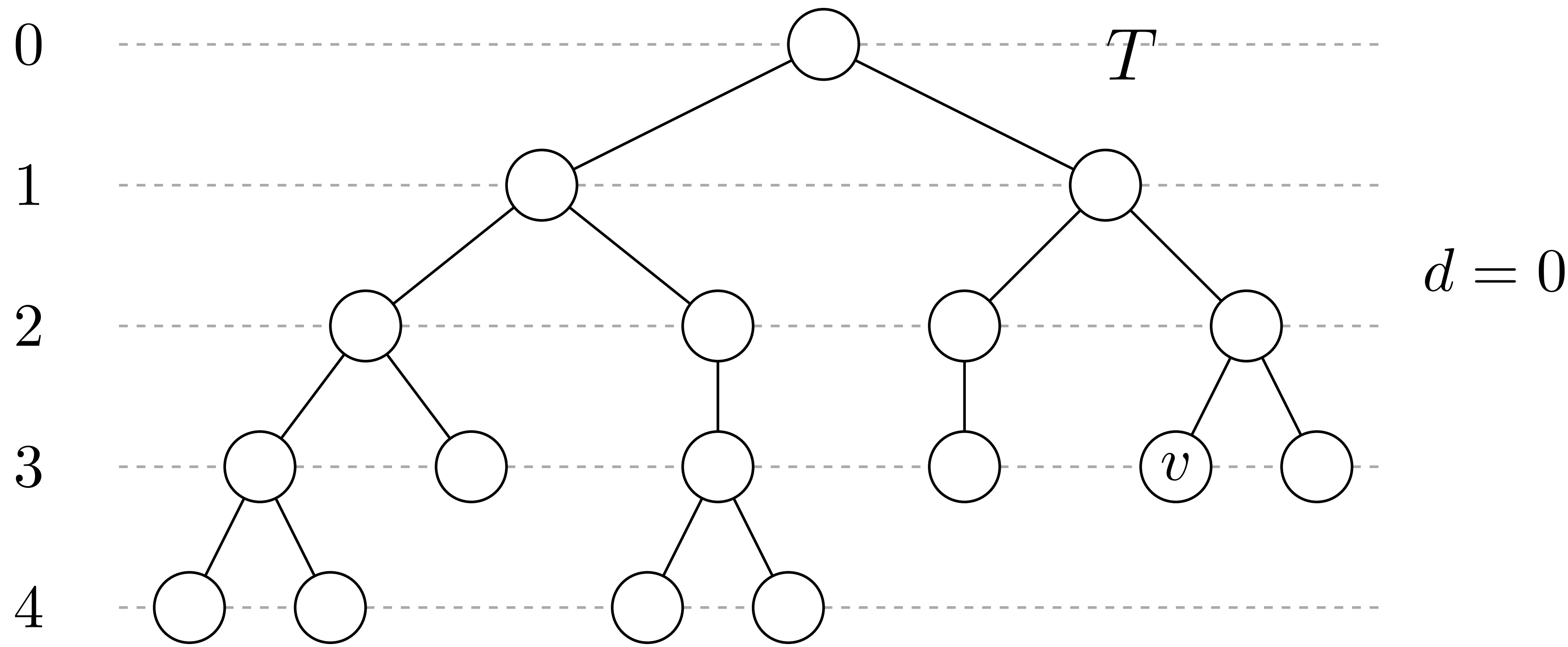

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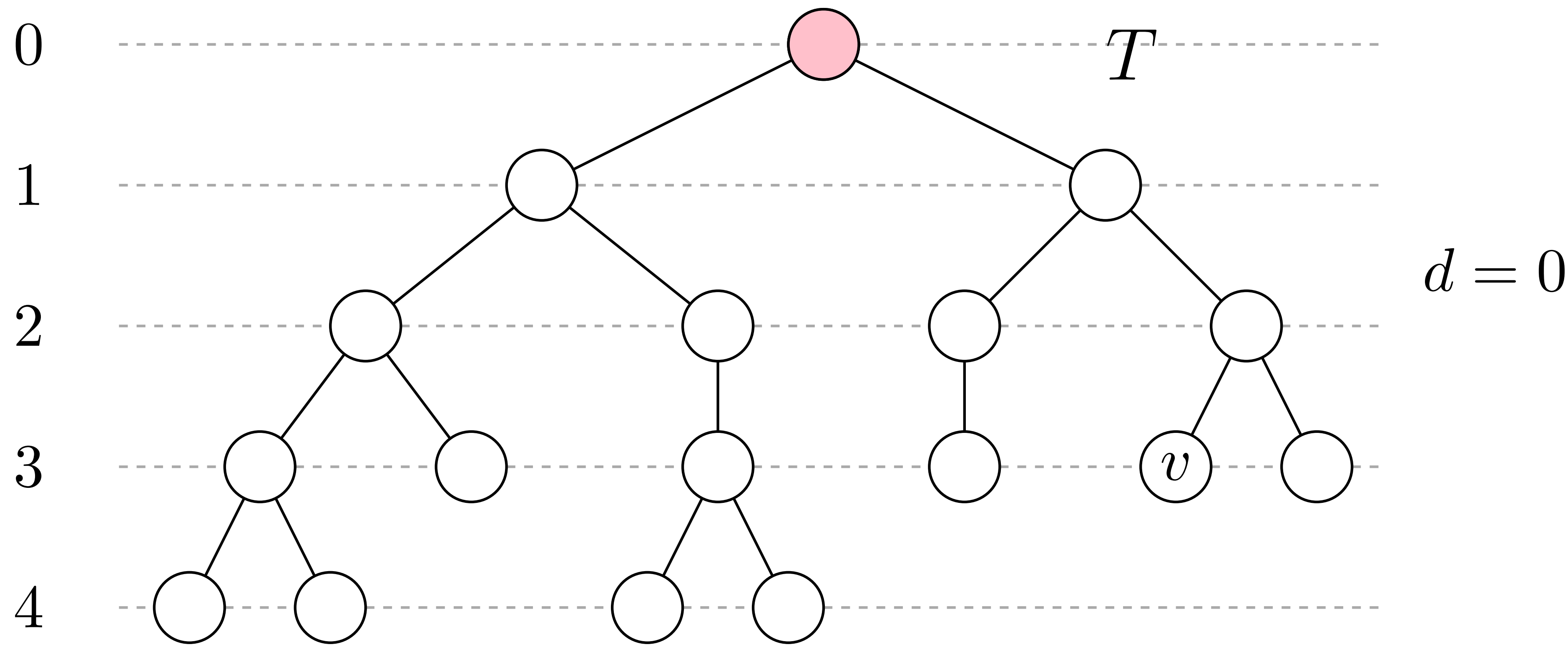

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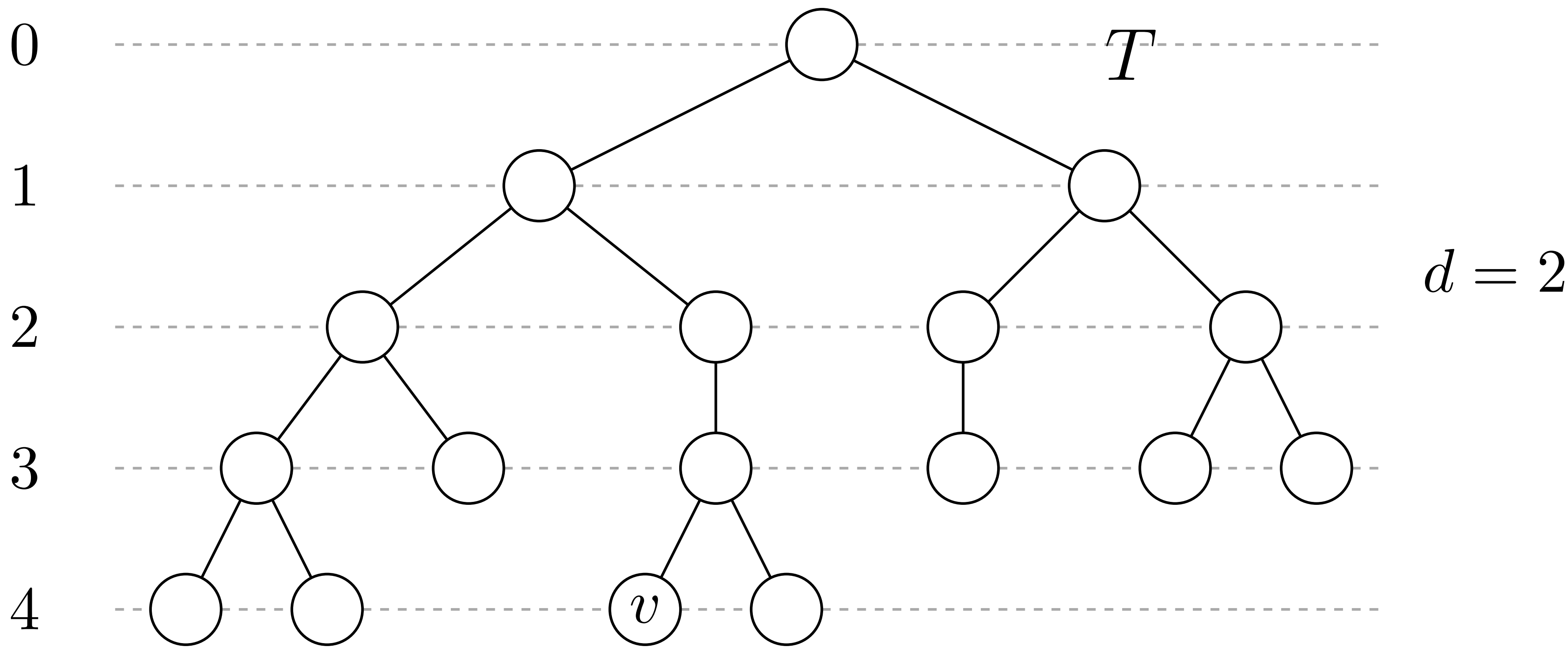

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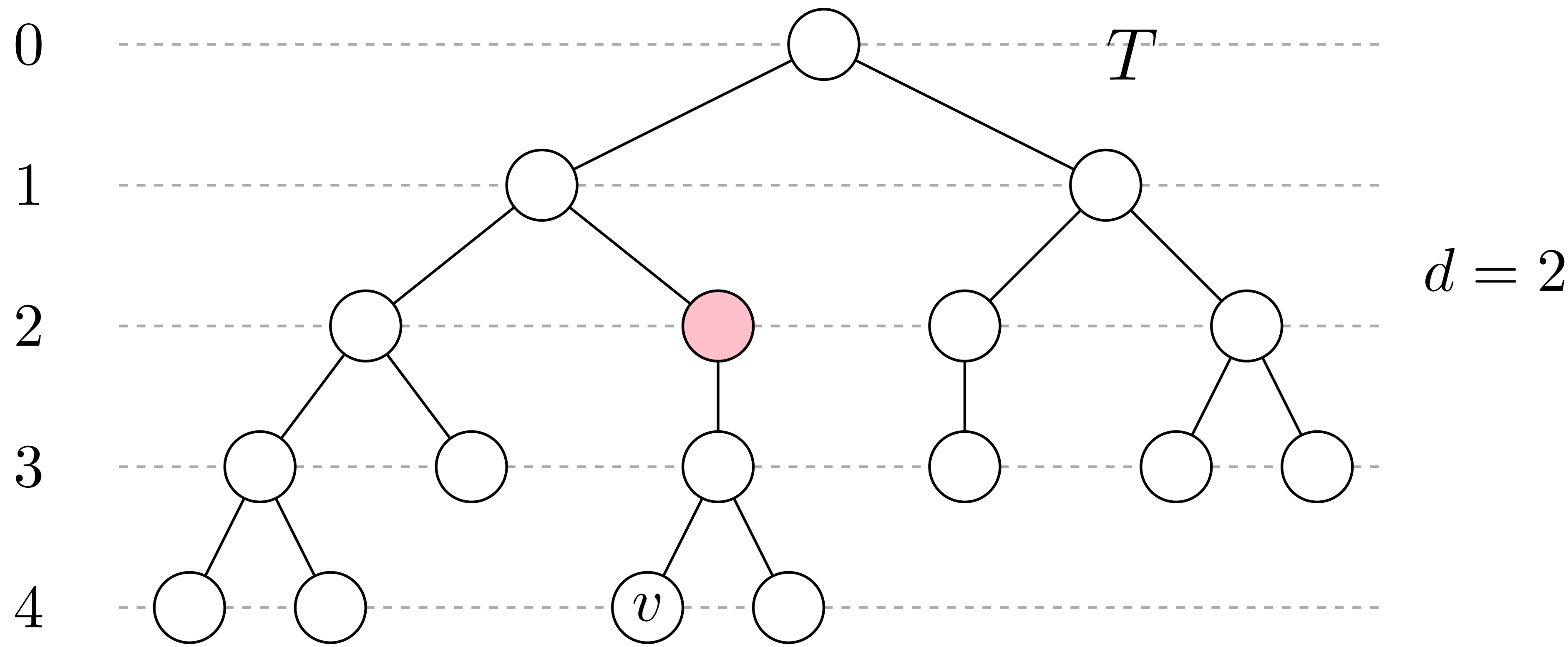

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The Problem

Given T, design a data structure that is able to preprocess T to answer level ancestors queries.

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$n = \#$ of nodes

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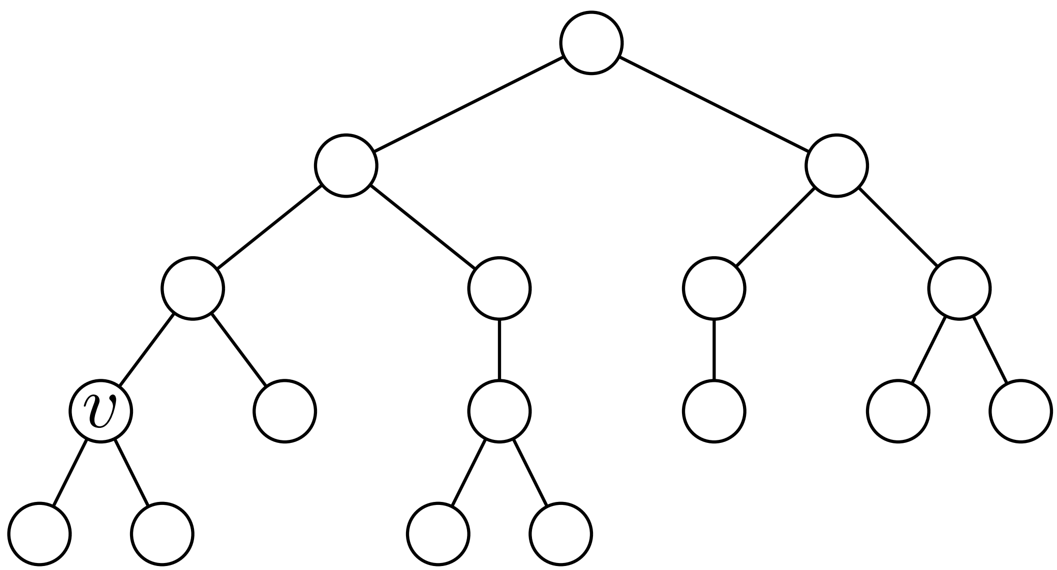

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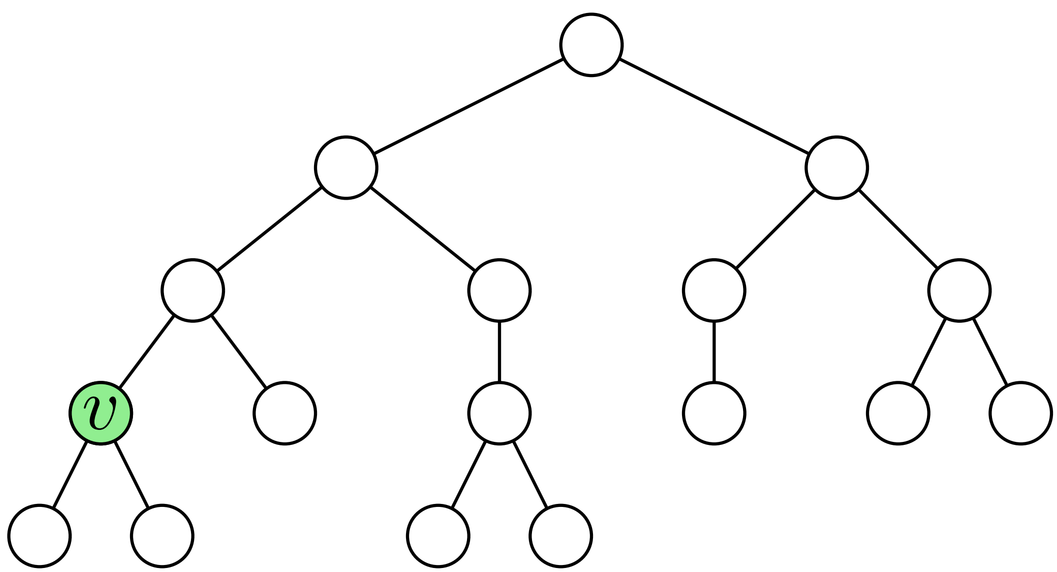

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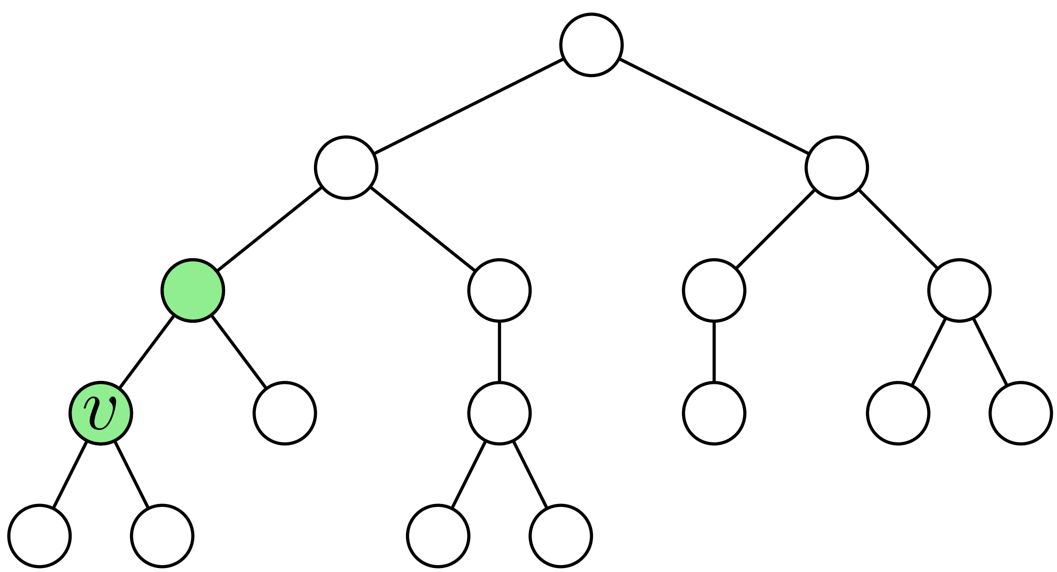

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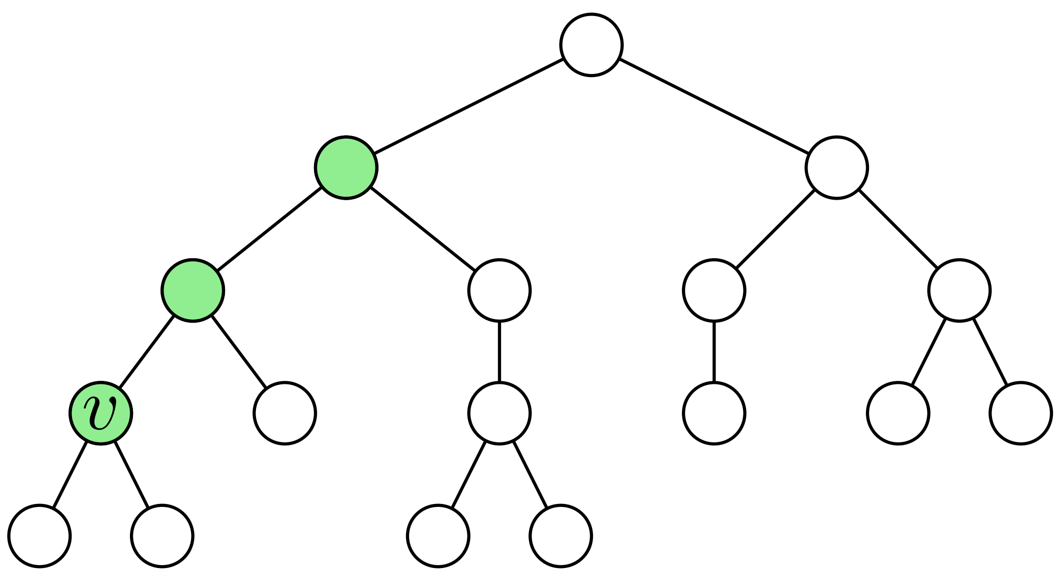

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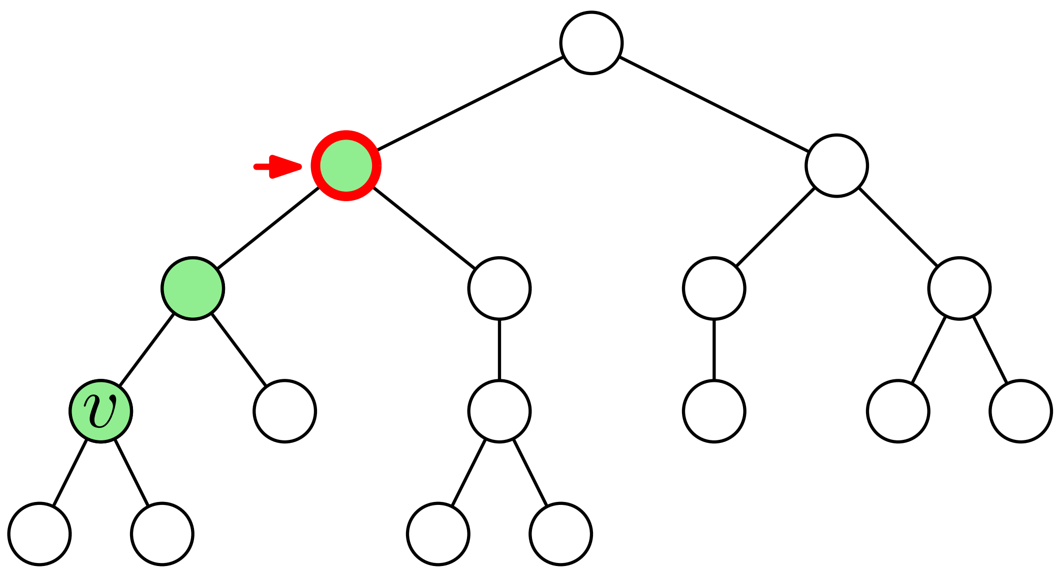

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(precompute the answer to all possible queries)

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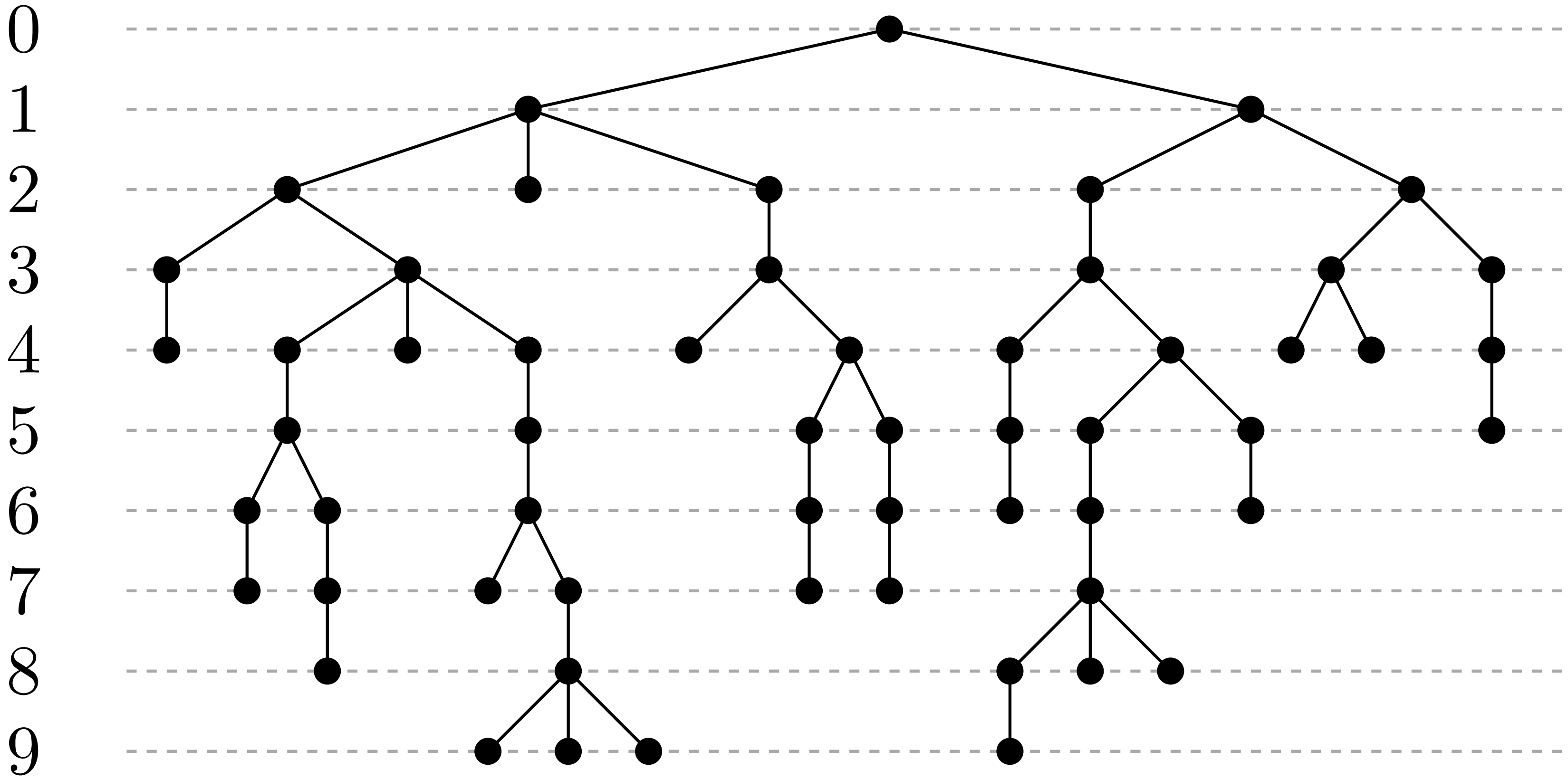
$$\mathsf{LA}(v,d) = \begin{cases} v & \text{if } d = d_v \\ \mathsf{LA}(\text{parent}(v),\, d) & \text{if } d < d_v \end{cases}$$

Jump Pointers: Idea



For each vertex v and $\ell = 0, 1, \ldots, \lfloor \log d_v \rfloor$, store:

$$J(v, \ell) = \mathsf{LA}(v, d_v - 2^{\ell})$$

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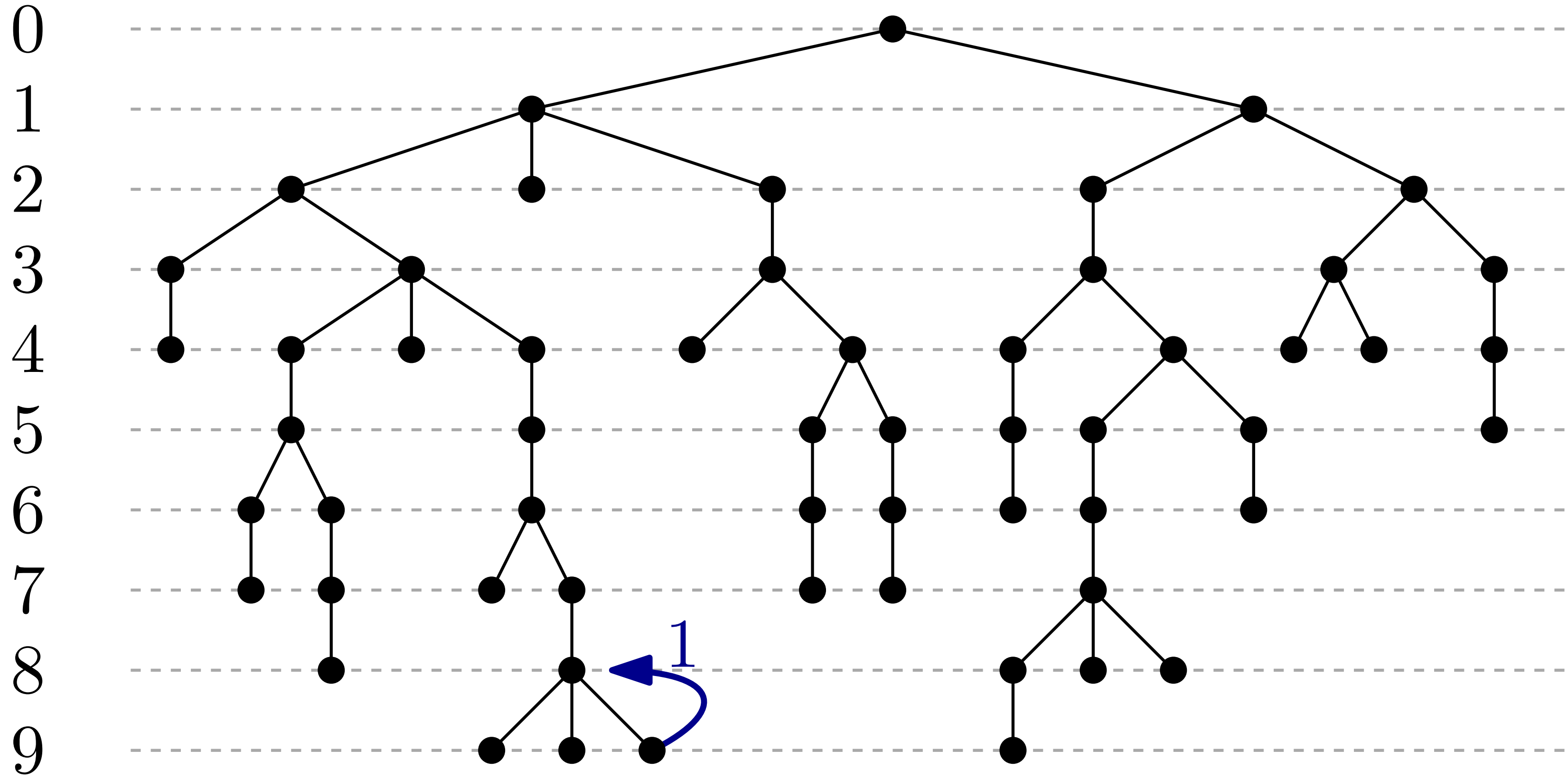

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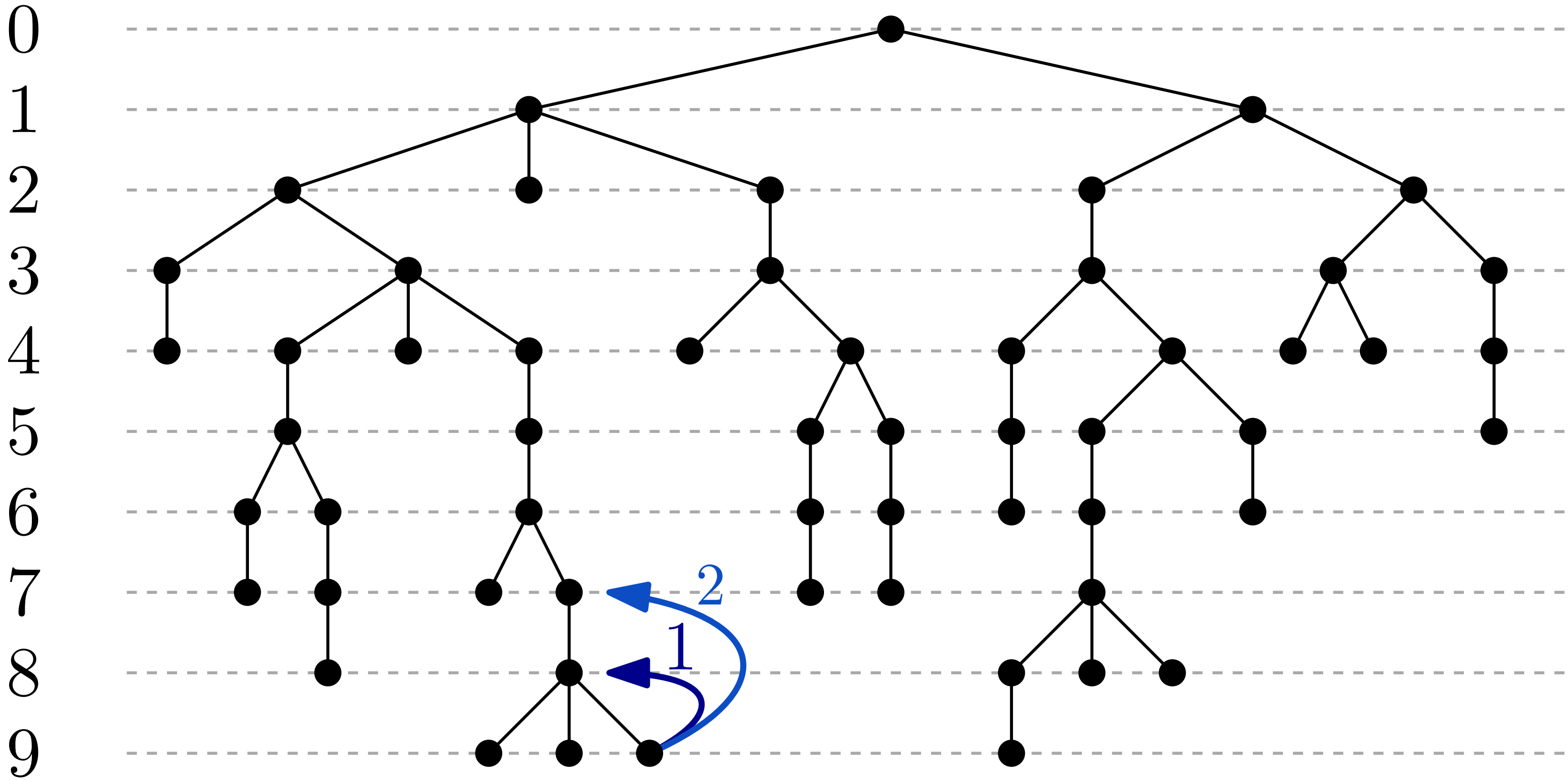

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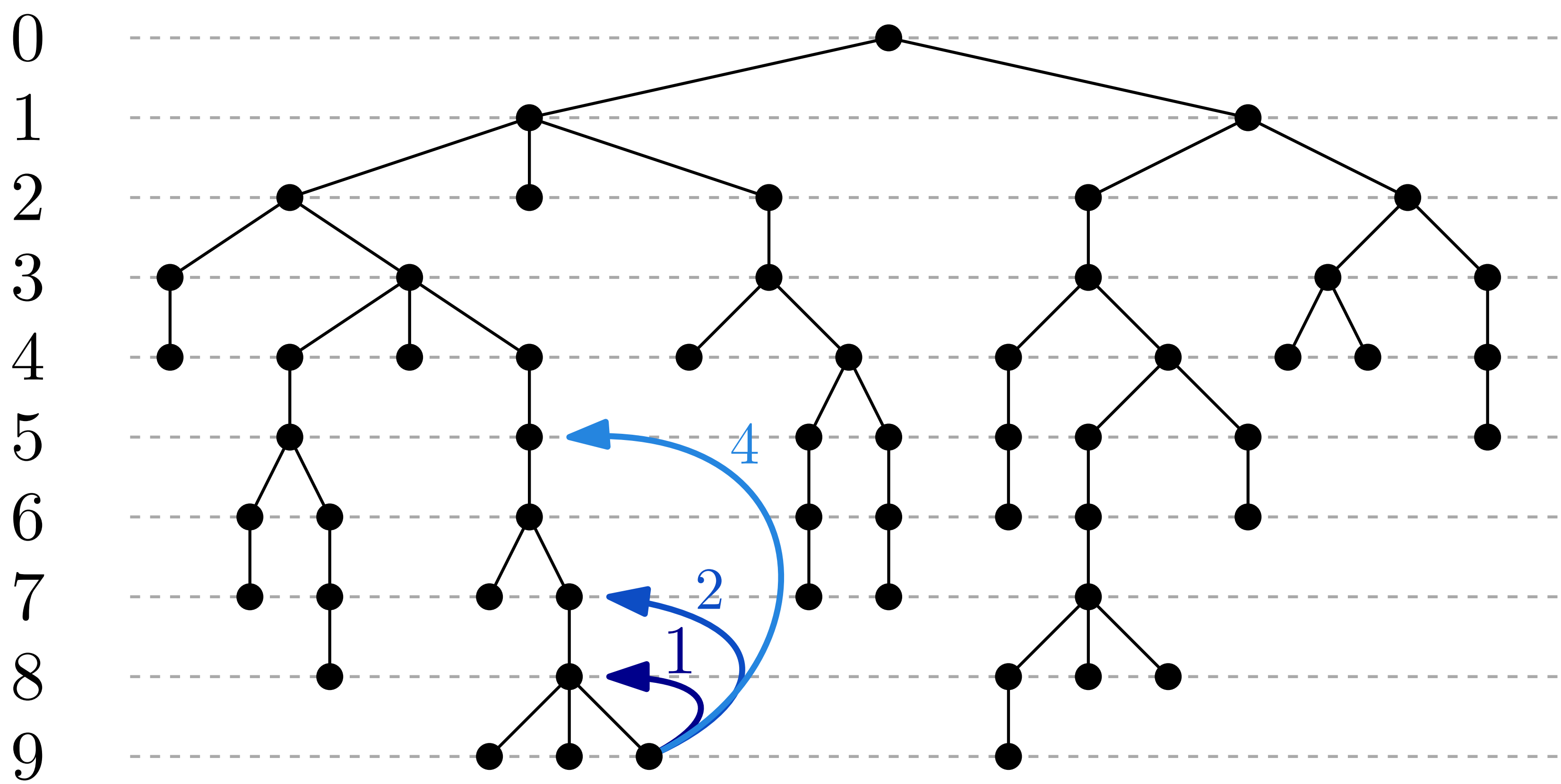

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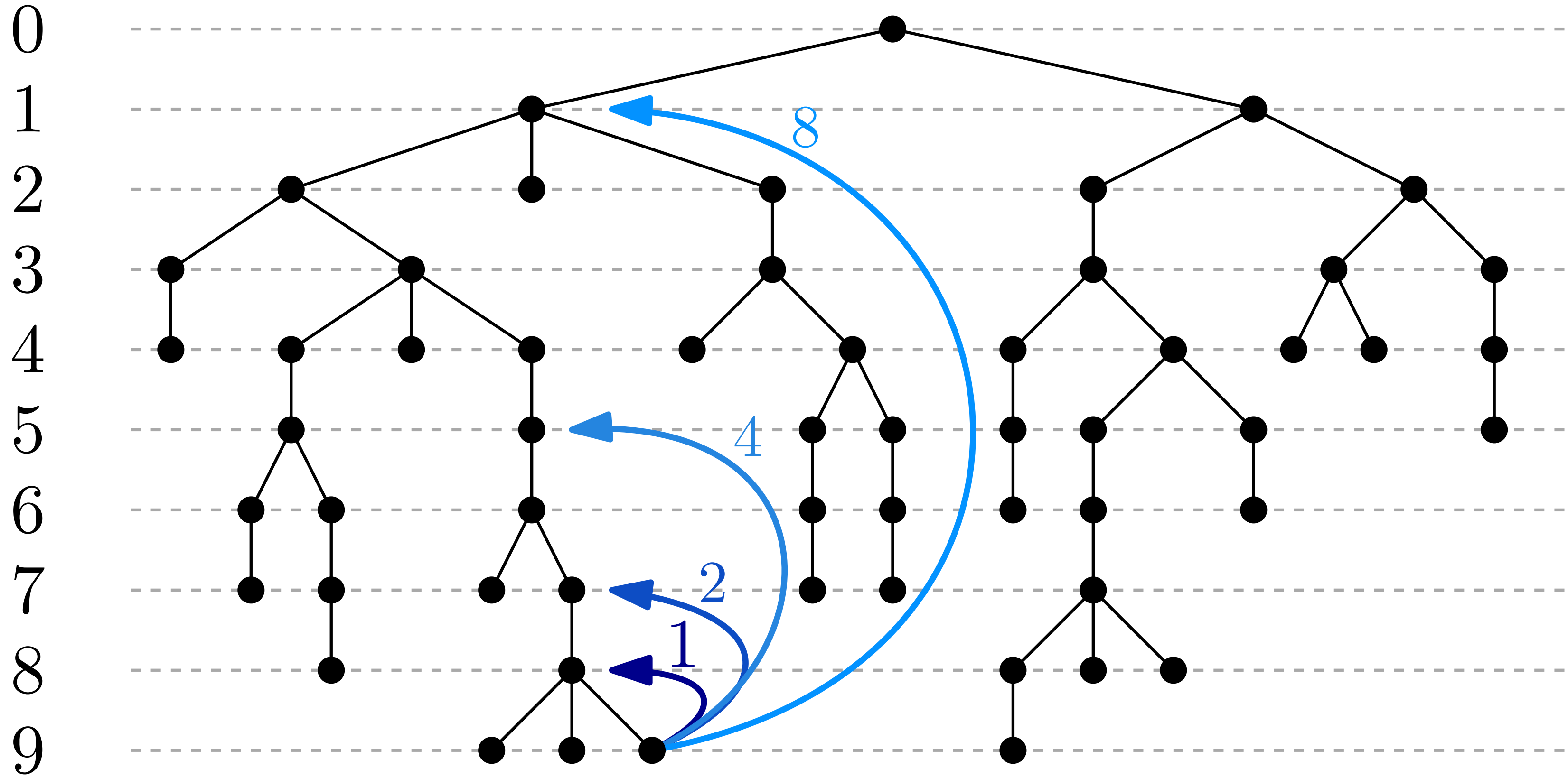

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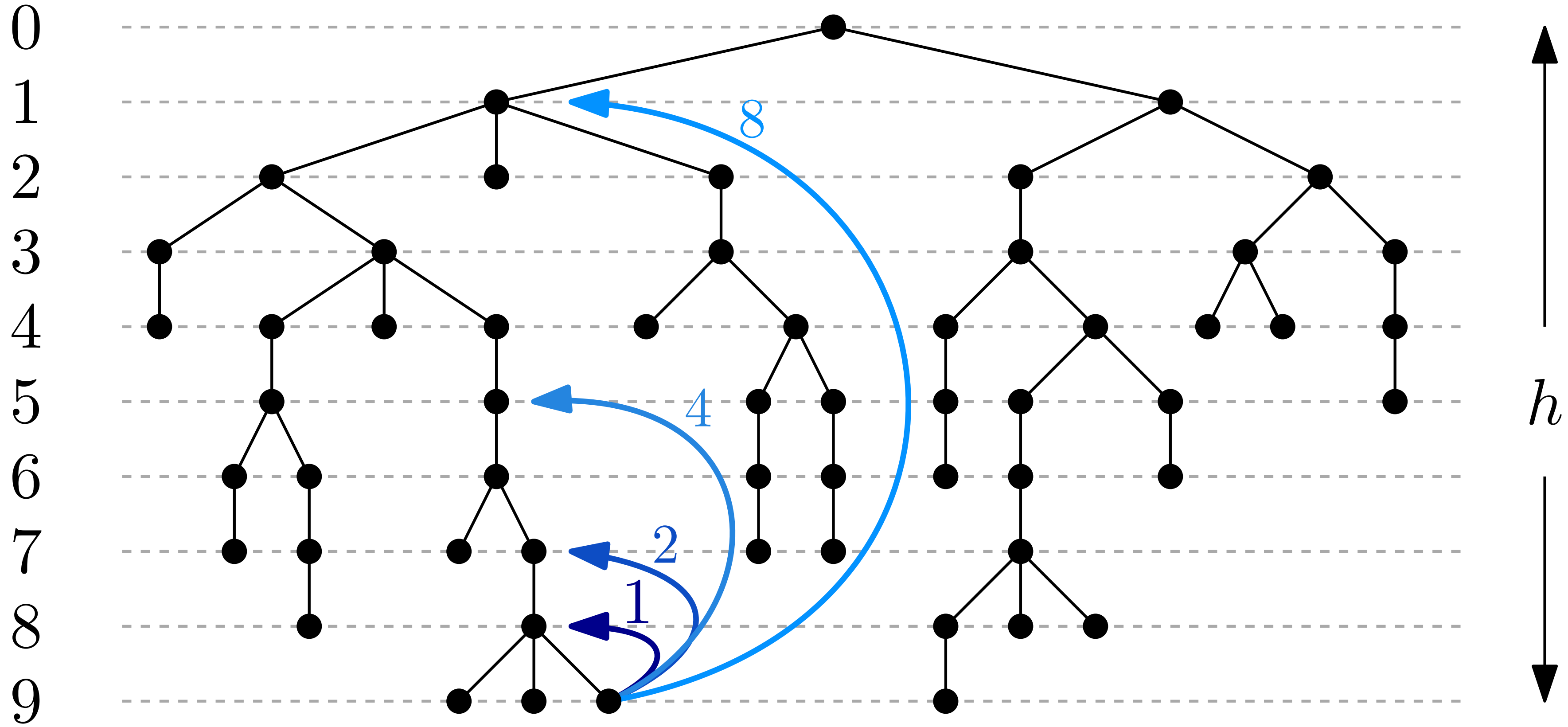

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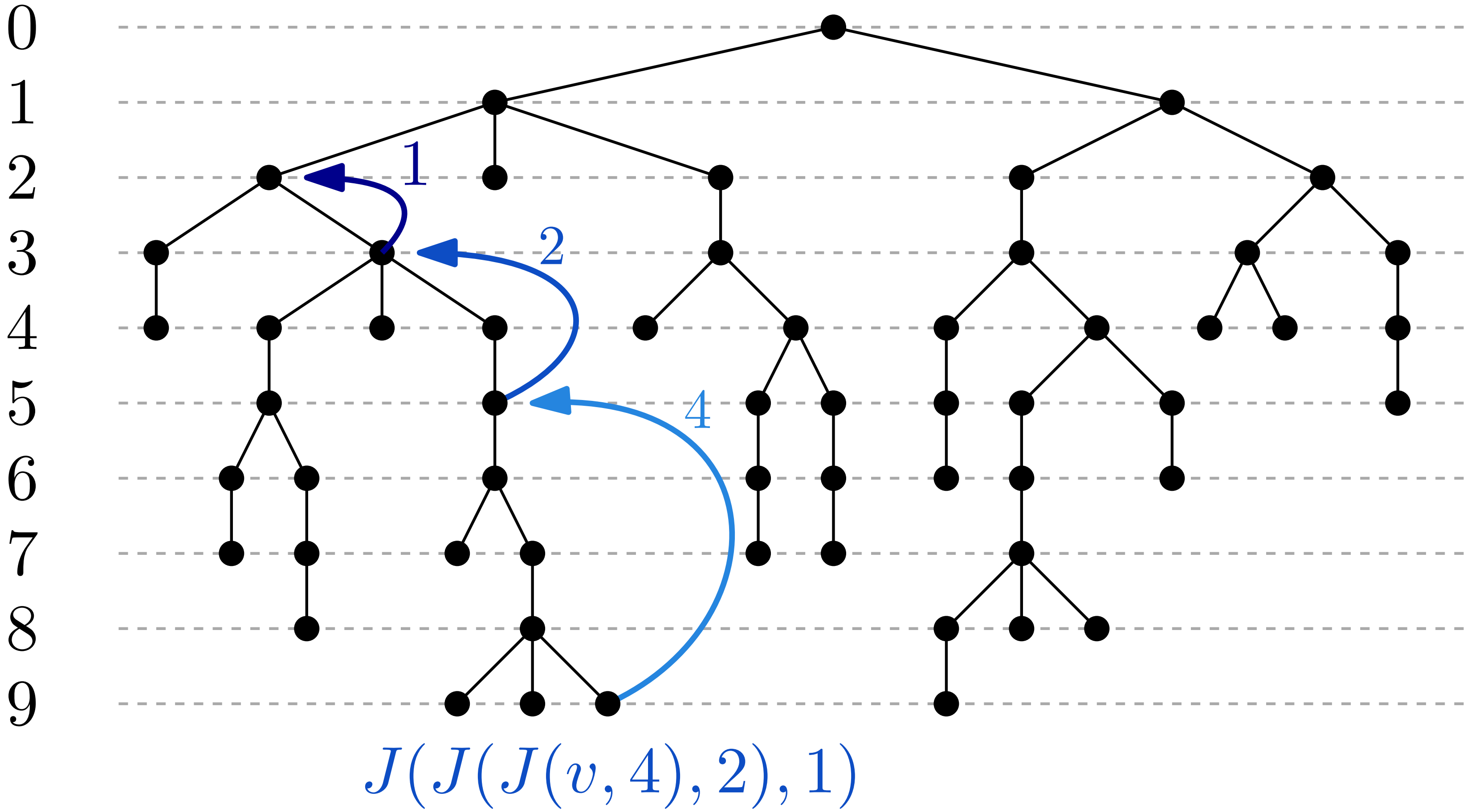
Total size: $O(n \log h) = O(n \log n)$

Jump Pointers: Query



$$0 < d_v - d = 2^{\ell_k} + 2^{\ell_{k-1}} + \cdots + 2^{\ell_1} \qquad \ell_{i+1} > \ell_i$$

$$\mathsf{LA}(v,d) = J(\ldots J(J(v,\ell_k),\ell_{k-1}),\ldots,\ell_1)$$

Number of accessed pointers: $O(\log h) = O(\log n)$

Jump Pointers: Construction

With a DFS visit of T:

- Maintain a stack S that stores all the ancestors of the current vertex v of the visit
- S can be updated in $O(1)$ per traversed edge
- When vertex v is visited, its ancestor at depth d in T is the $(d_v - d)$-th vertex from the top of the stack

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Or using dynamic programming:

$$J(v,\ell) = \begin{cases} \text{parent}(v) & \text{if } \ell = 0 \\ J(J(v,\ell-1),\ell-1) & \text{if } \ell > 0 \end{cases}$$

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Time complexity: $O(n + n \log h) = O(n \log n)$

Solutions so far

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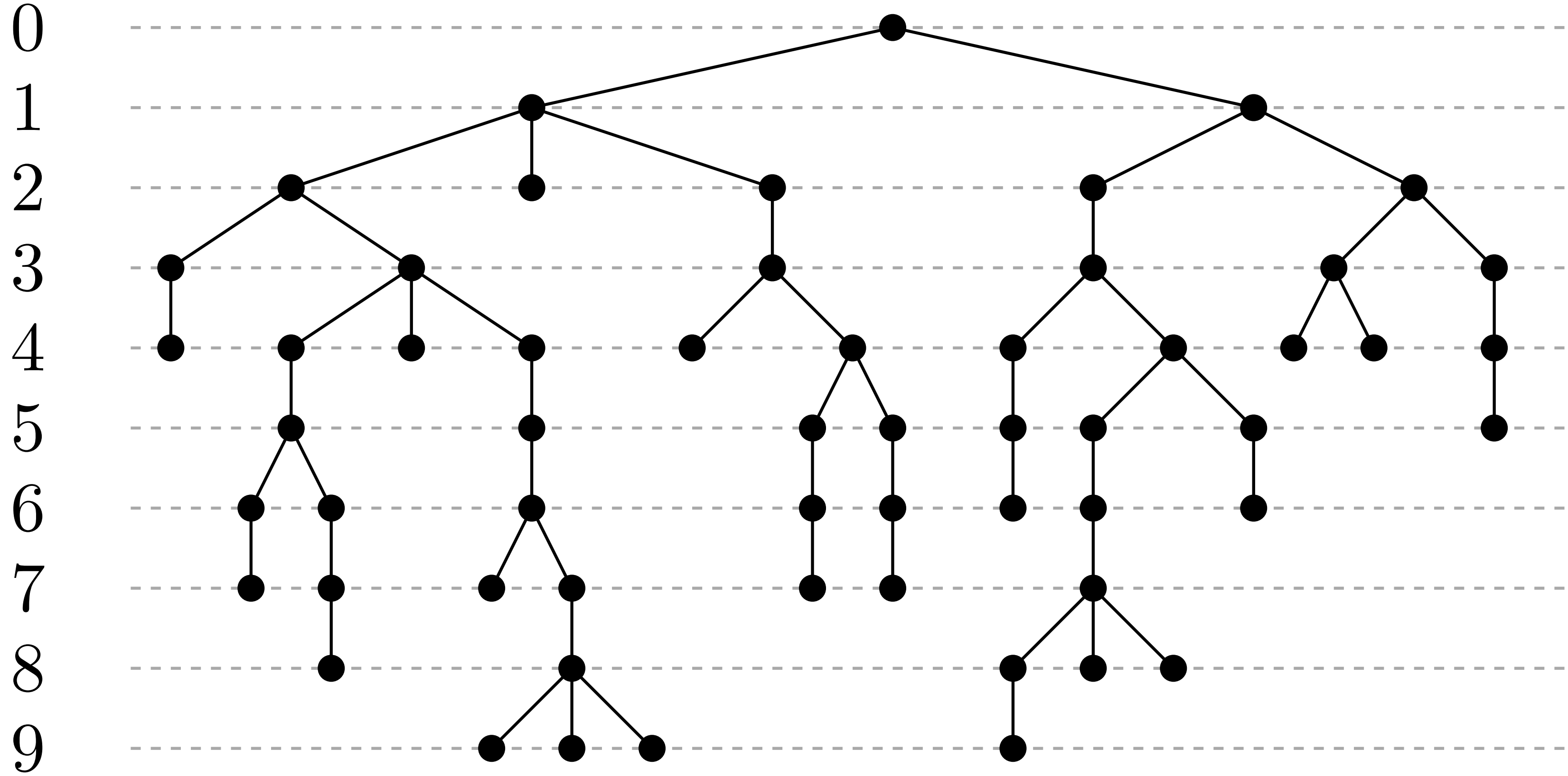
We want to get rid of the $\log n$ factors!

Long Path Decomposition



Partitions T into a collection of paths $\mathcal{D}$. Recursively defined:

- Select one of the longest root-to-leaf paths P in T
- Select paths recursively from each the tree of the forest $T \setminus P$

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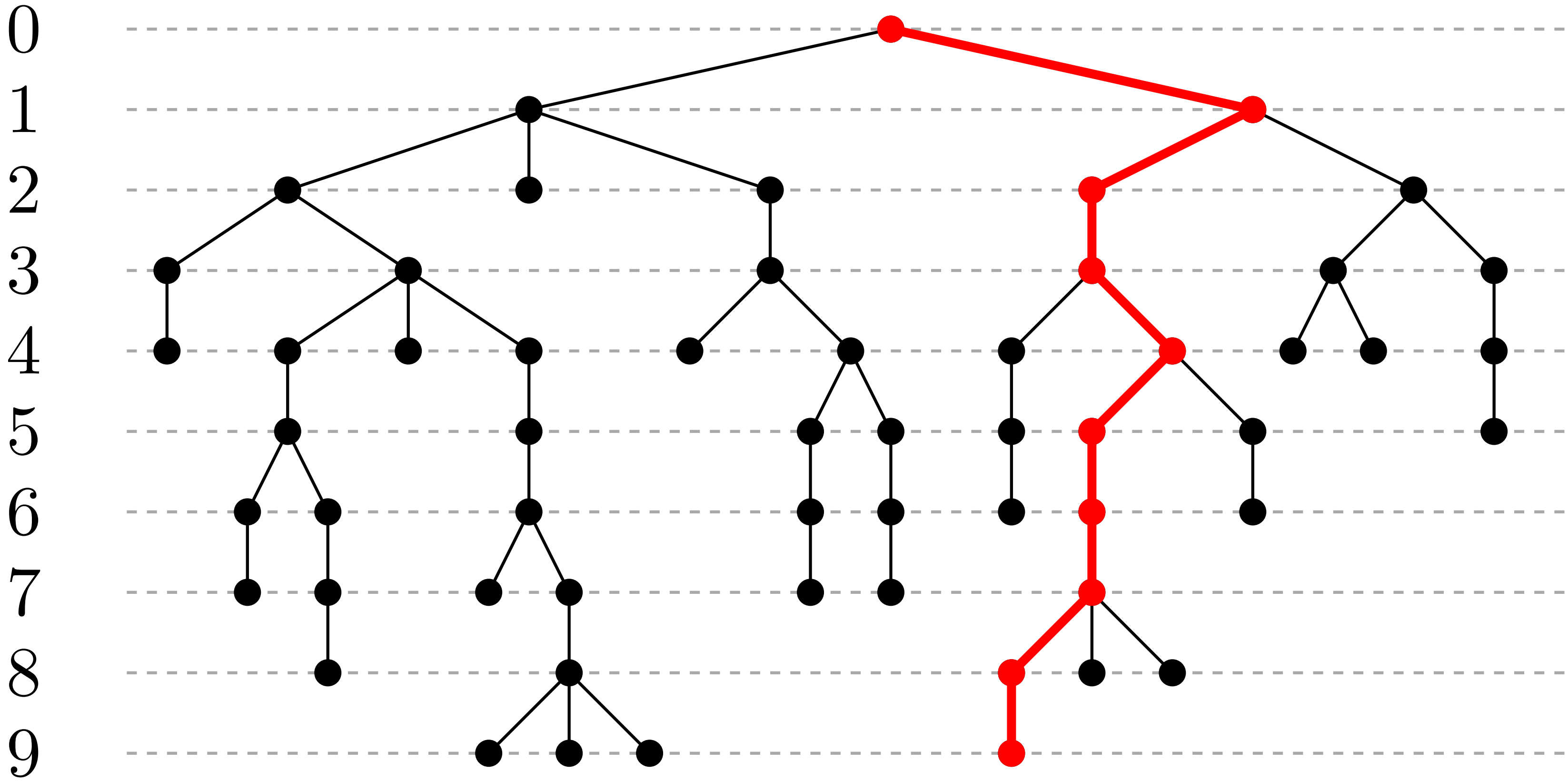

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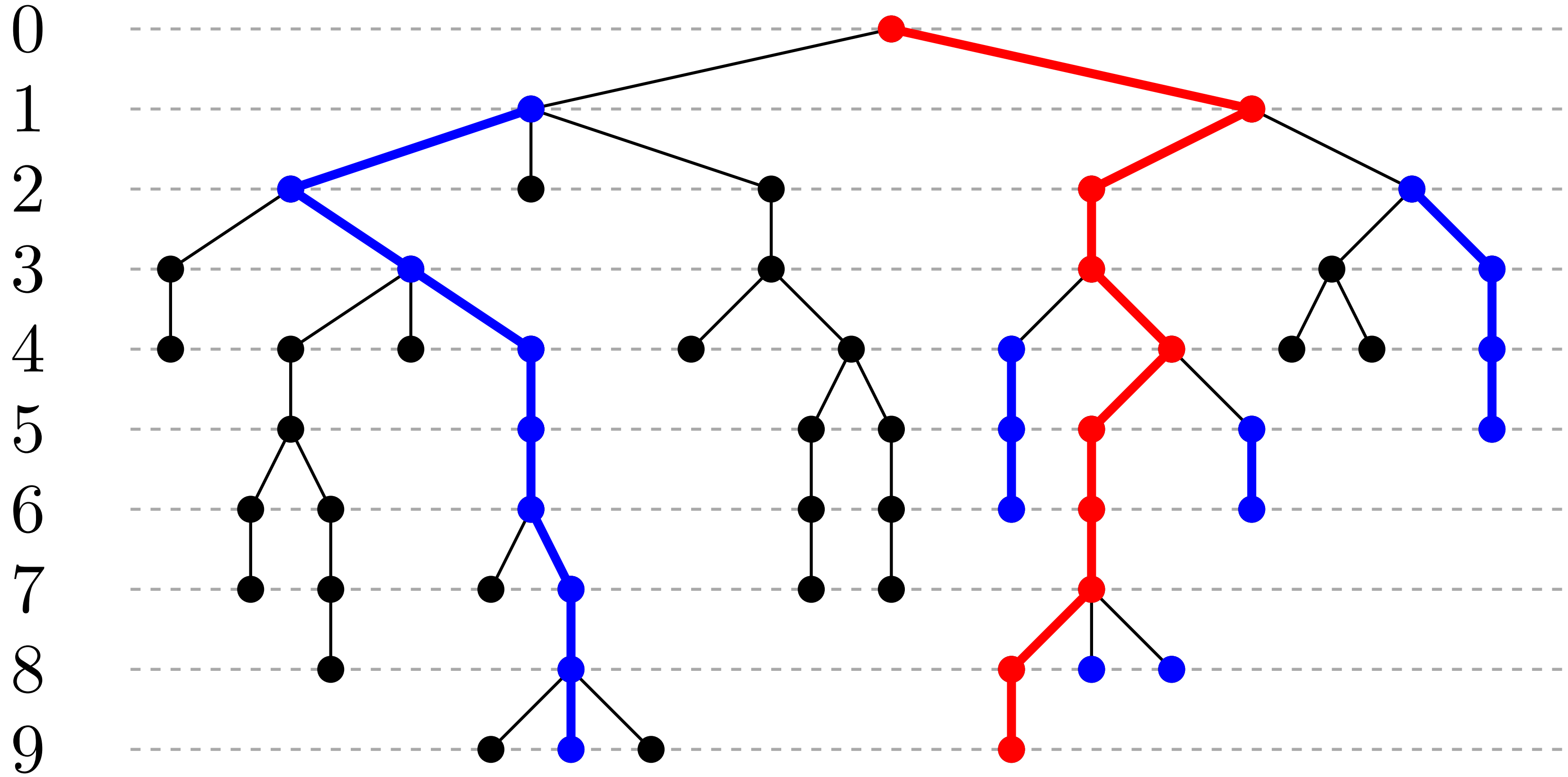

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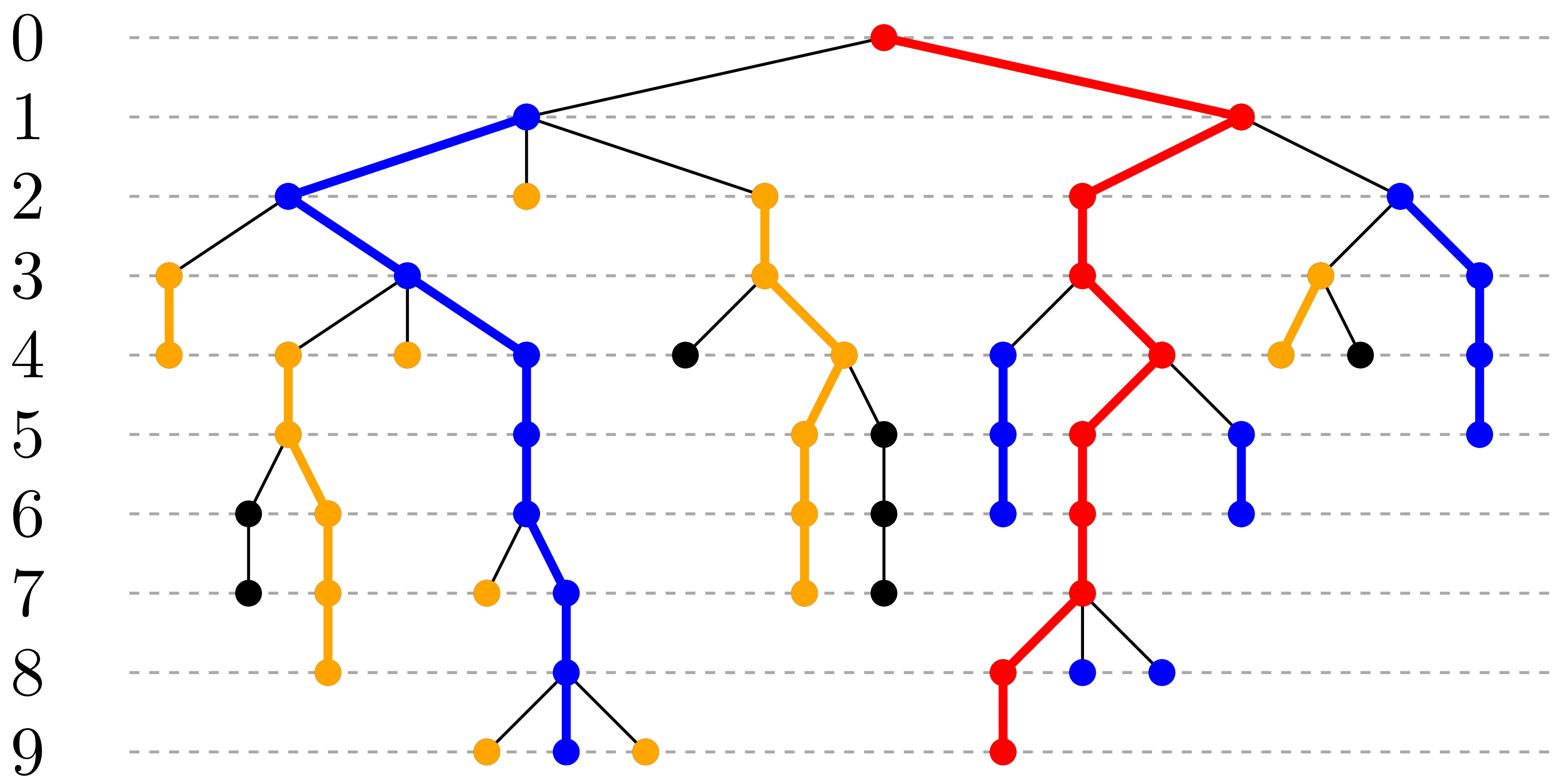

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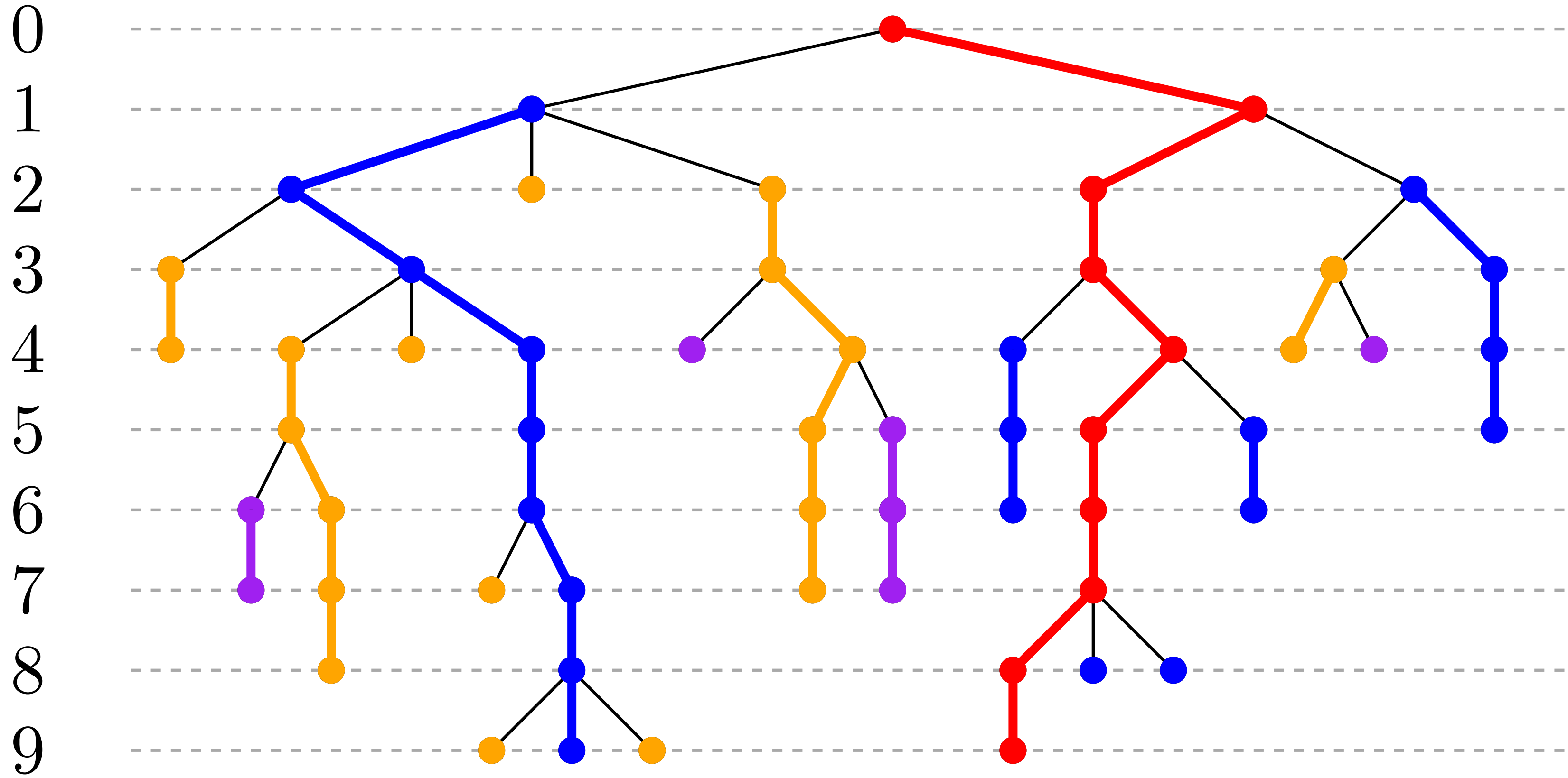

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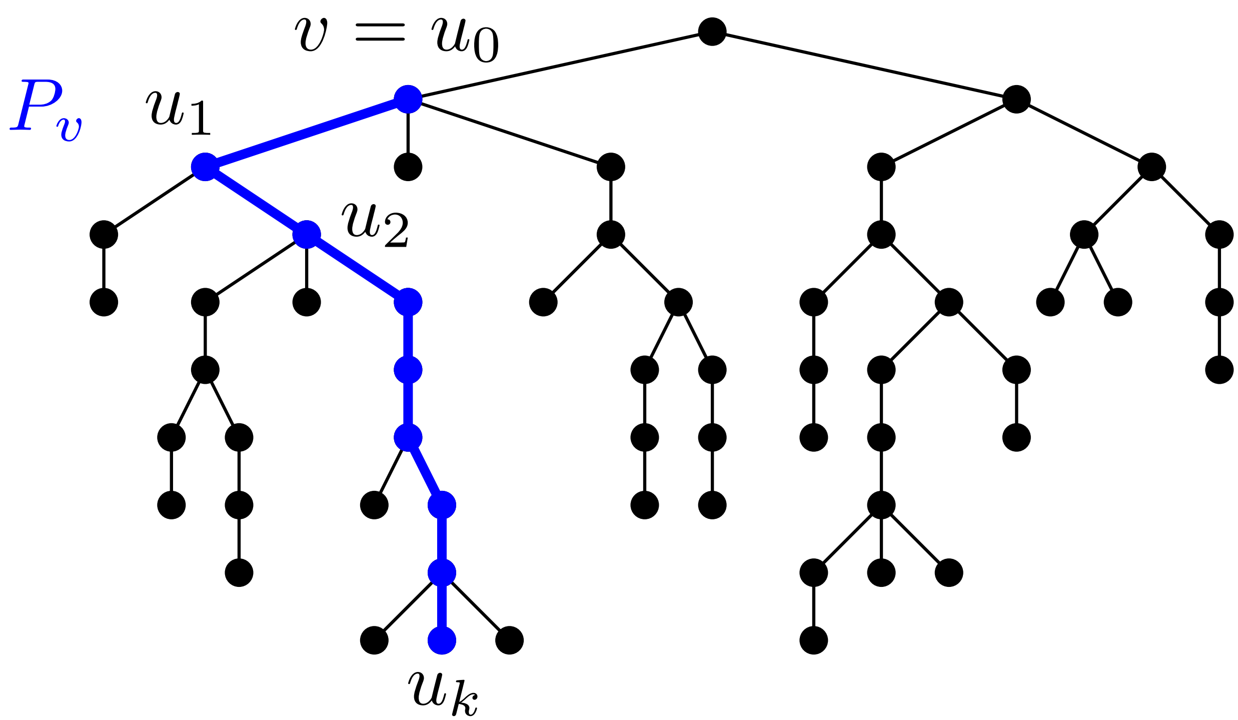

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- Store an array A_v of length $k+1$ where $A_v[i]$, $i = 0, \ldots, k$, contains (a reference to) u_i
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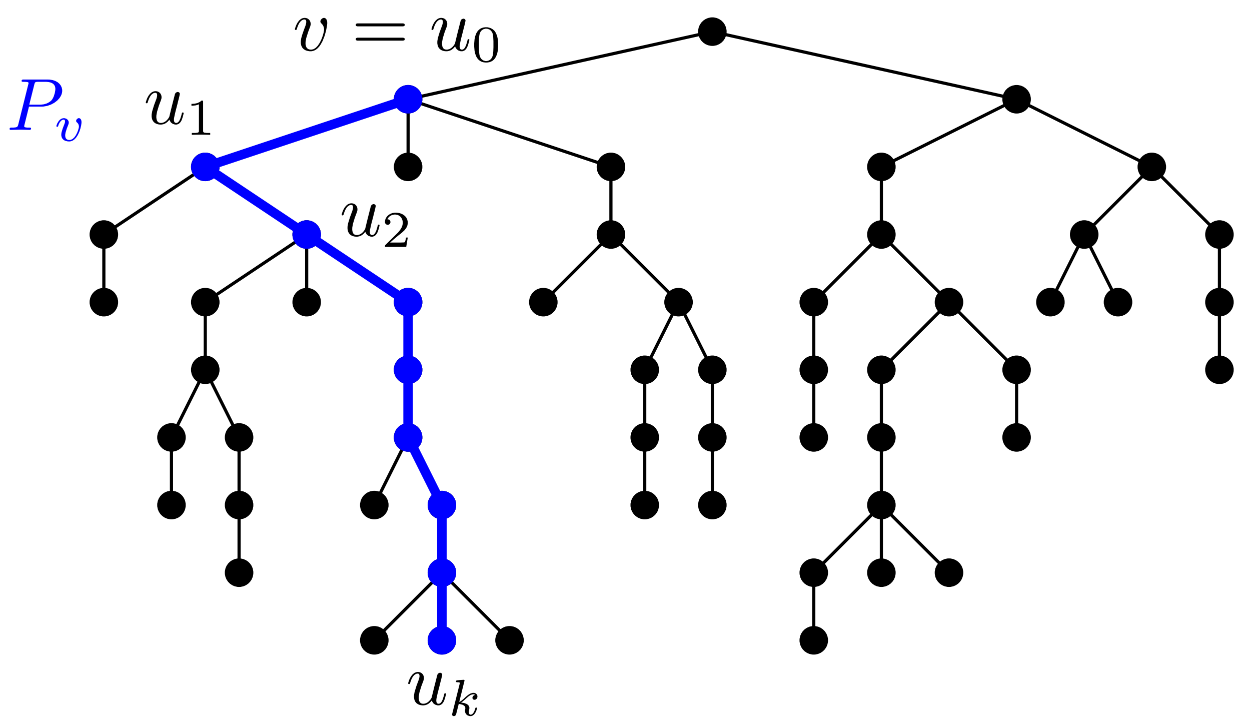

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Total space: $\sum_{P_v \in \mathcal{D}} O(1 + |P_v|) = O(n)$

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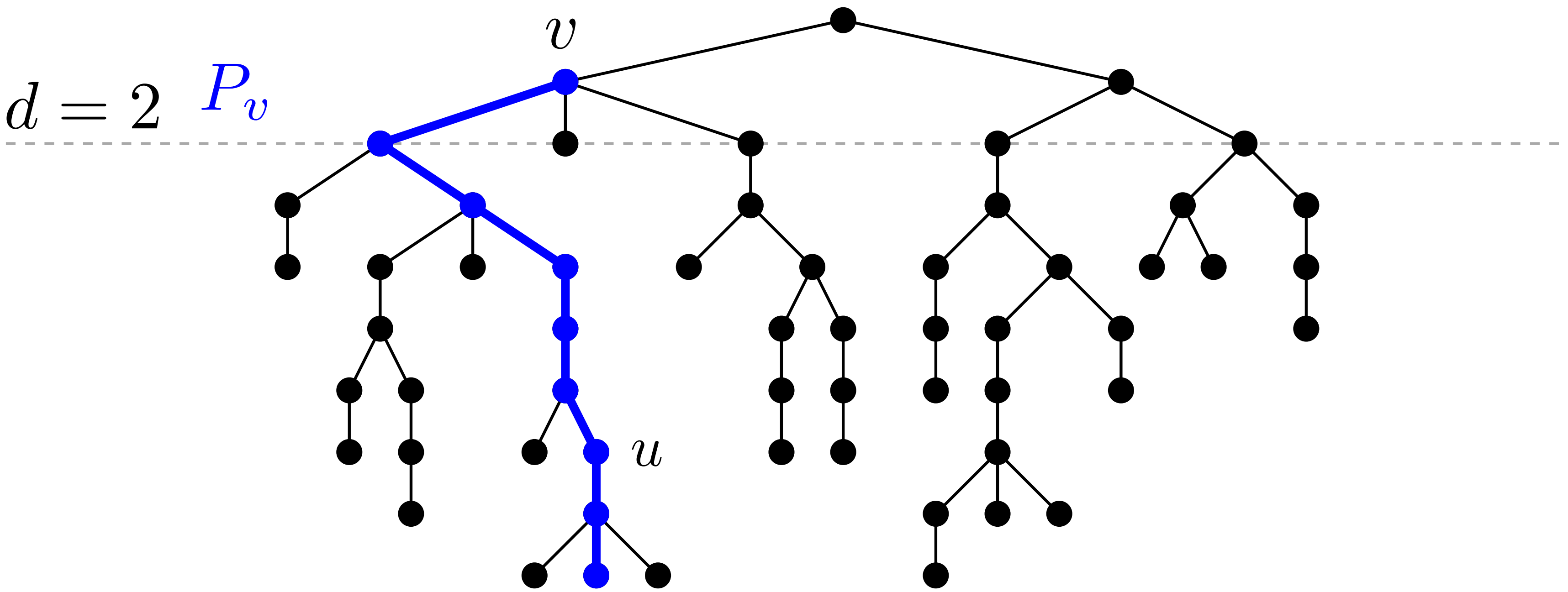

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- Let $v = \tau(u)$
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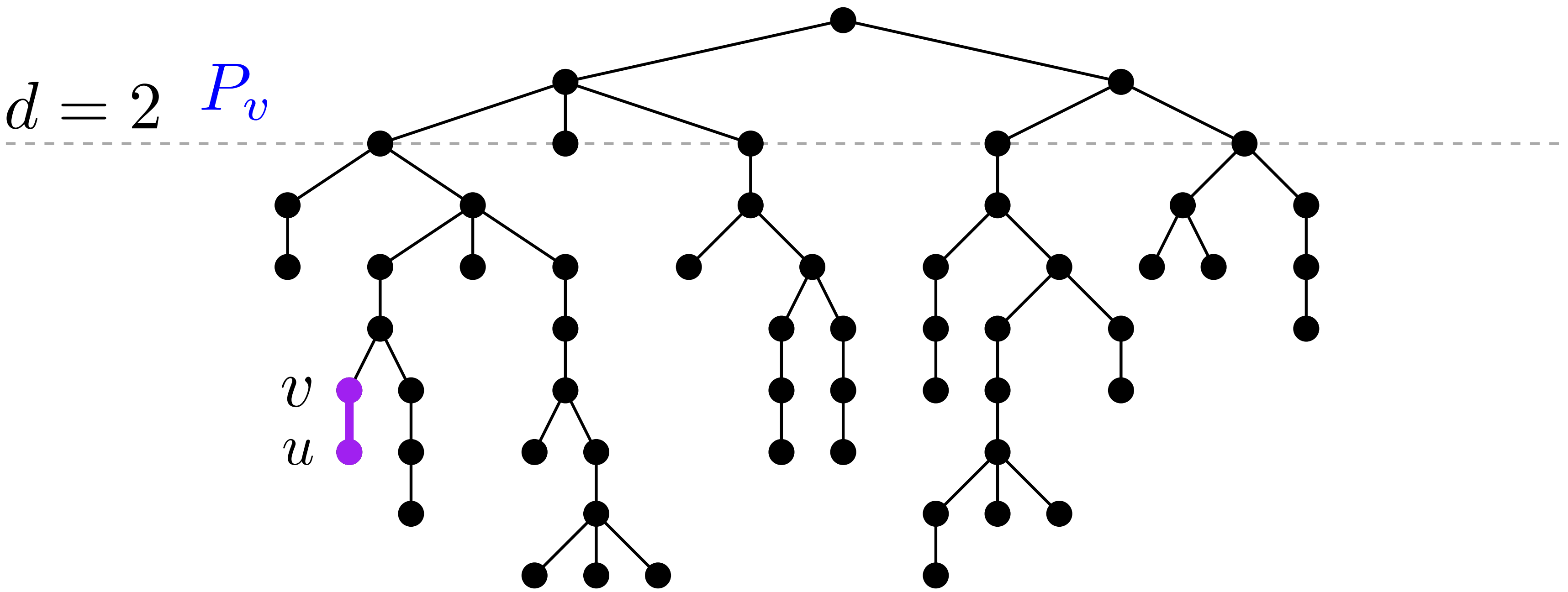

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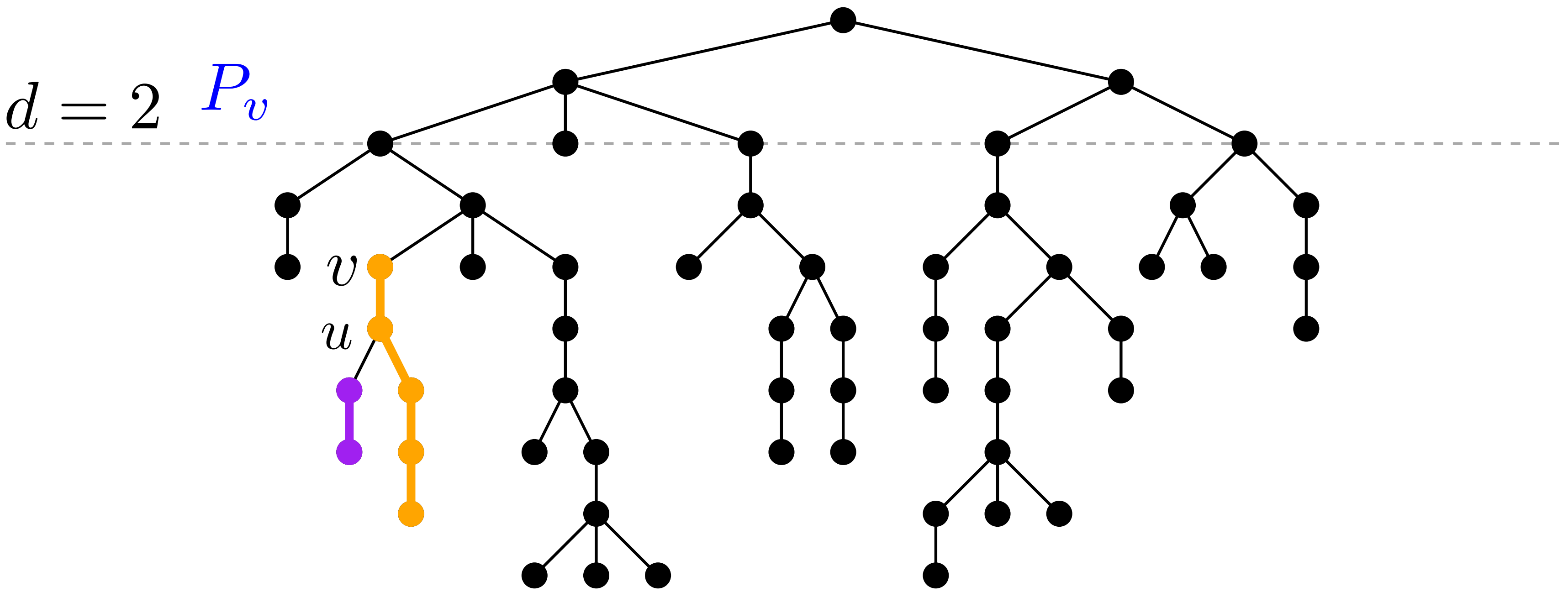

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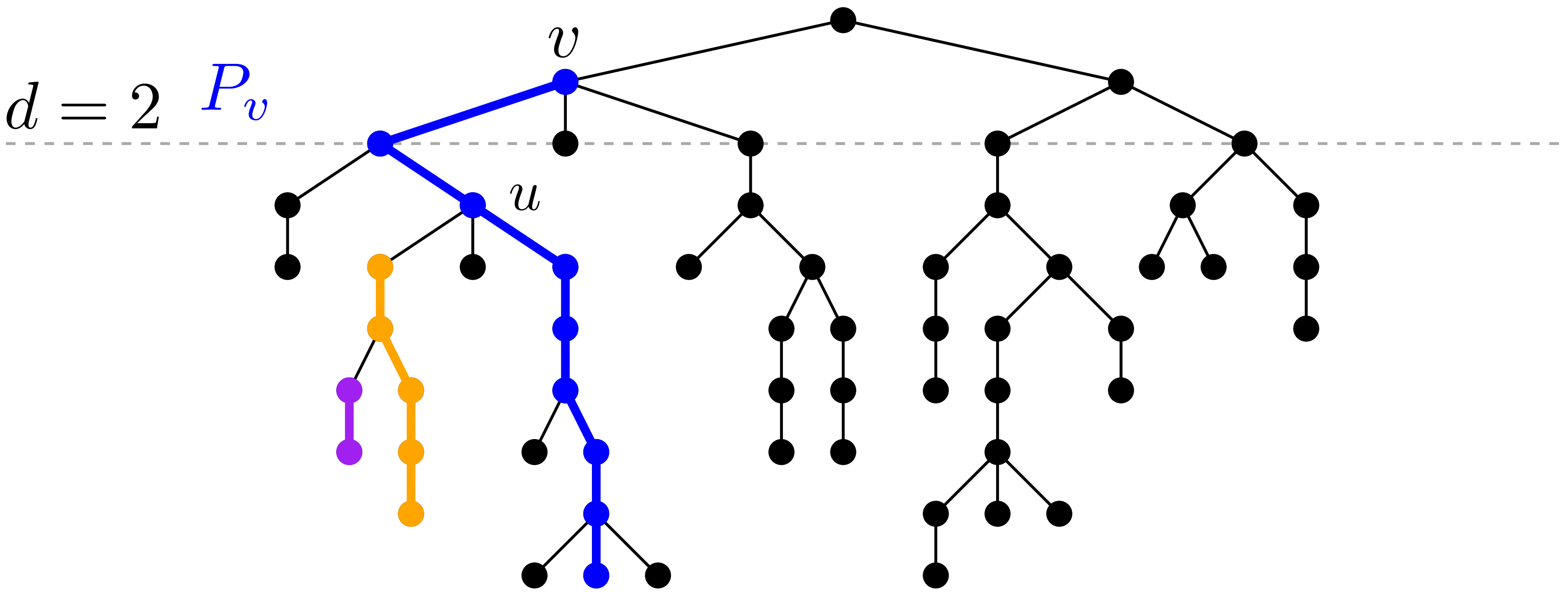

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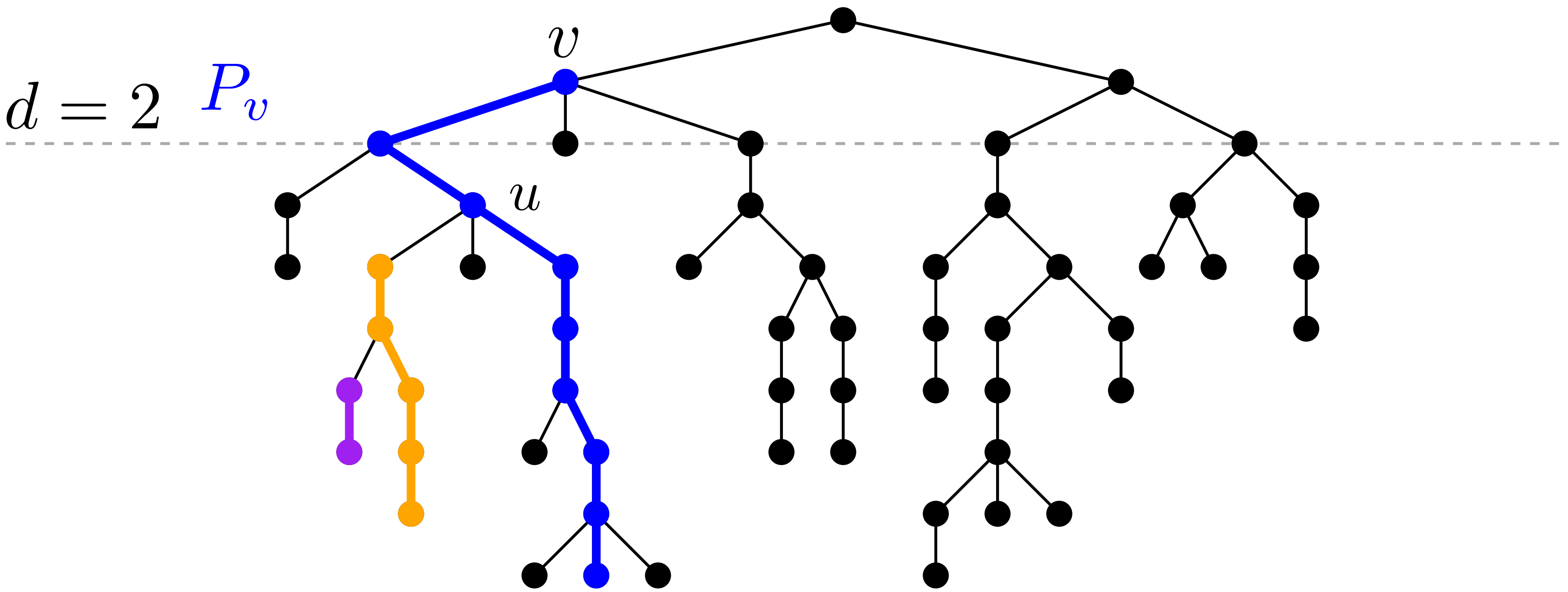

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Time: $O(\#\text{recursive calls}) = O(\#\text{paths in } \mathcal{D} \text{ from } v \text{ to the root})$.

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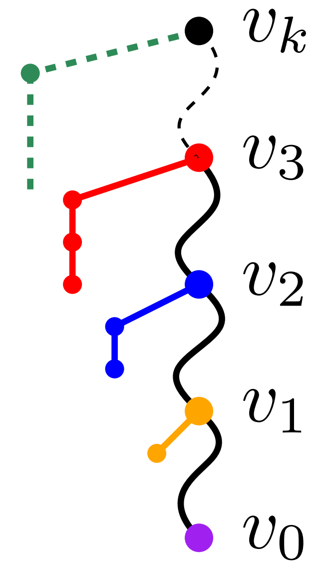

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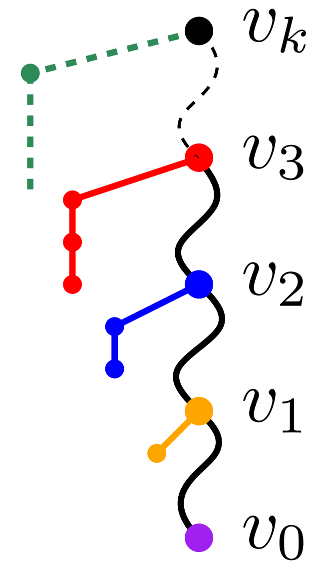

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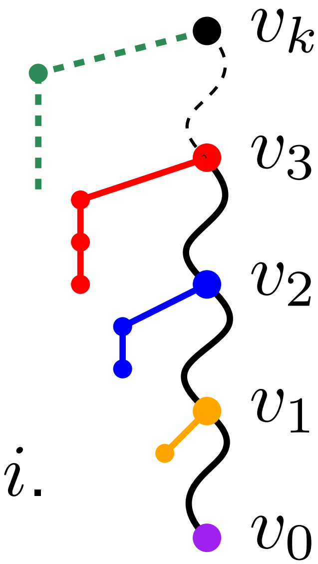
- By the long-path decomposition, $|P_i| \geq h(v_i) \geq i$.



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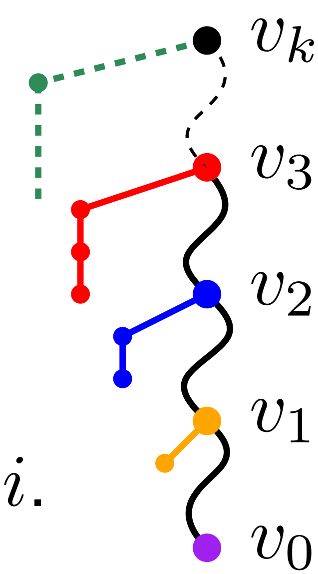

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$$n \geq \left| \bigcup_{i=1}^{k} P_i \right| \geq \sum_{i=1}^{k} i \geq \frac{k^2}{2} \implies \sqrt{2n} \geq k.$$

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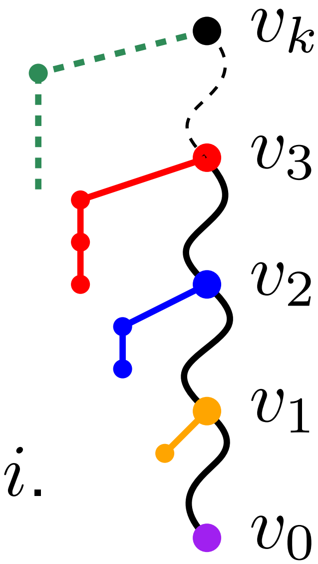

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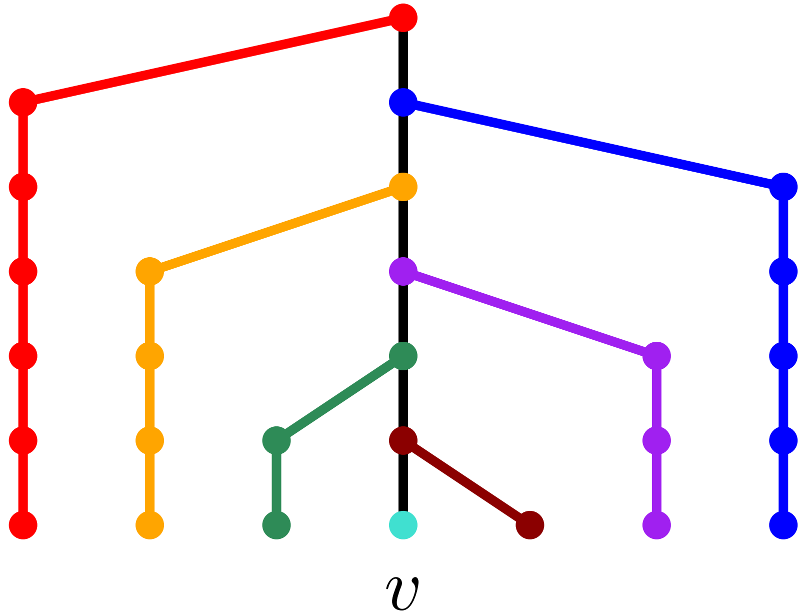
Time: $O(\sqrt{n})$

Is this tight?

Long Path Decomposition



Time: $\Omega(\sqrt{n})$

Solutions so far

Size	Preprocessing Time	Query Time	Notes
$O(n)$	–	$O(n)$	
$O(n^2)$	$O(n^3)$	$O(1)$	
$O(n^2)$	$O(n^2)$	$O(1)$	
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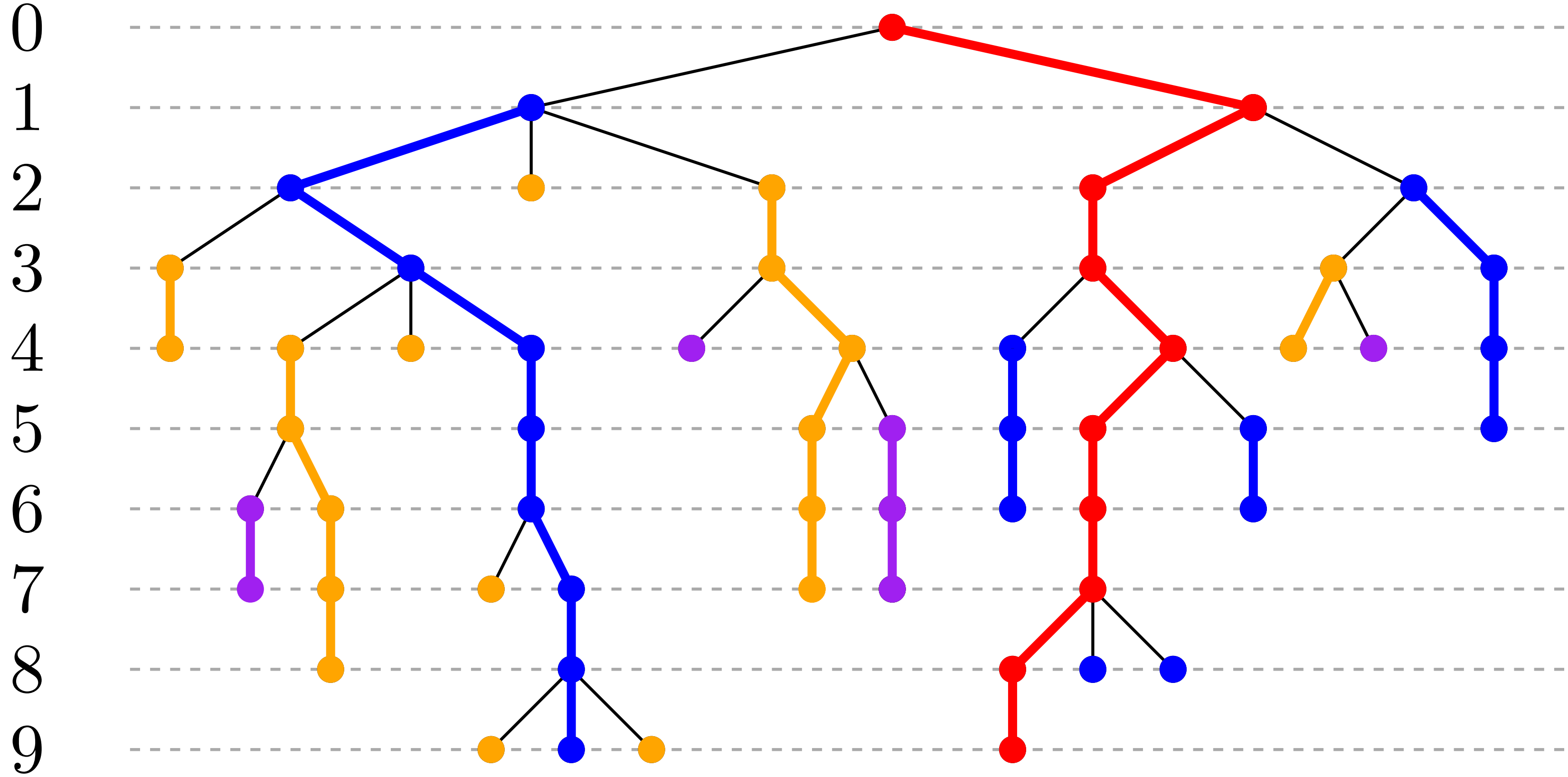
$O(n)$	$O(n)$	$O(\sqrt{n})$	Long Path Dec.

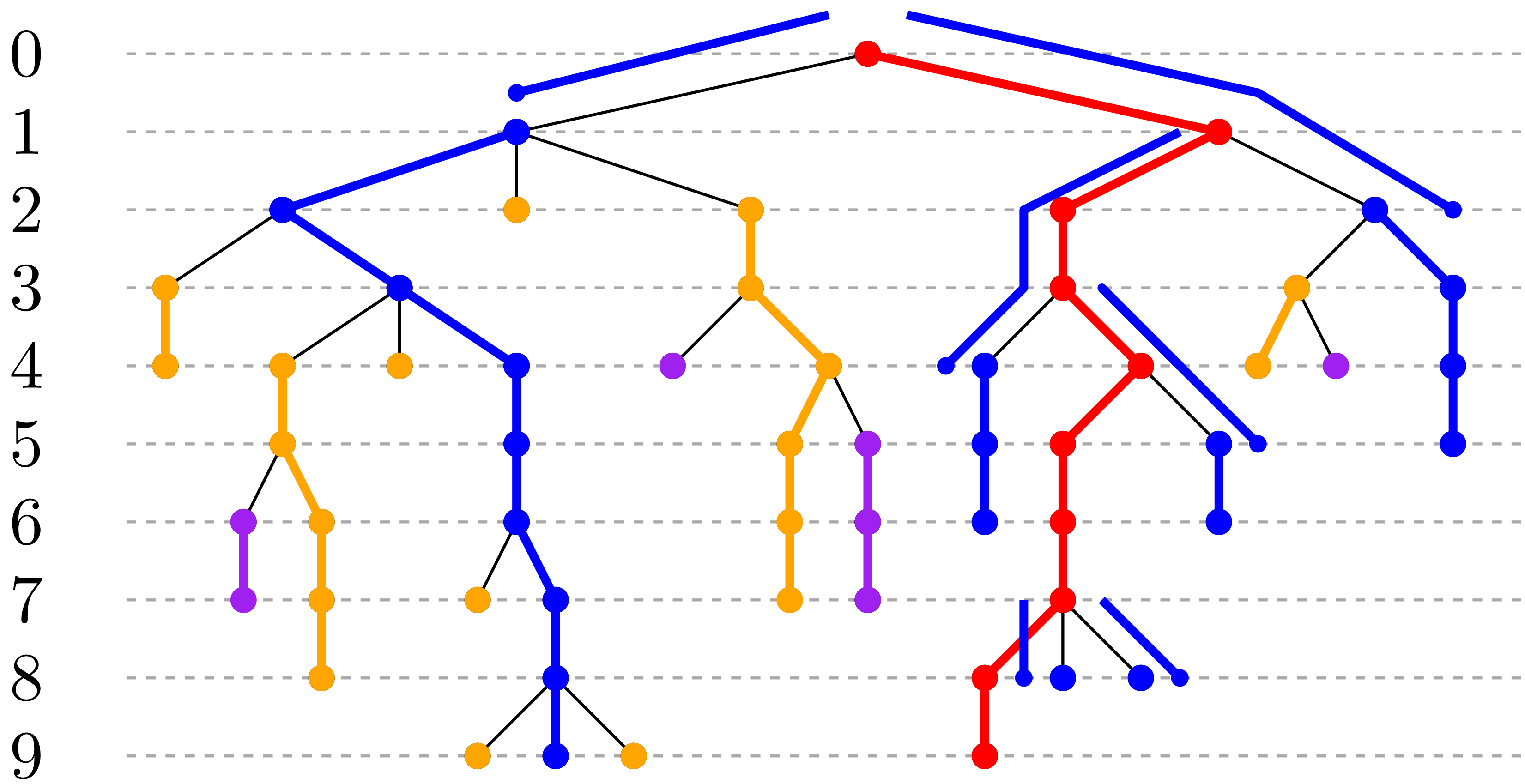
Long Path Decomposition + Ladders



Let $\eta(P_v)$ be the number of vertices of the path $P_v \in \mathcal{D}$.

Extend each path $P_v \in \mathcal{D}$ into a *ladder* L_v with $\eta(P_v)$ more vertices towards the root (if they exist).

Long Path Decomposition + Ladders



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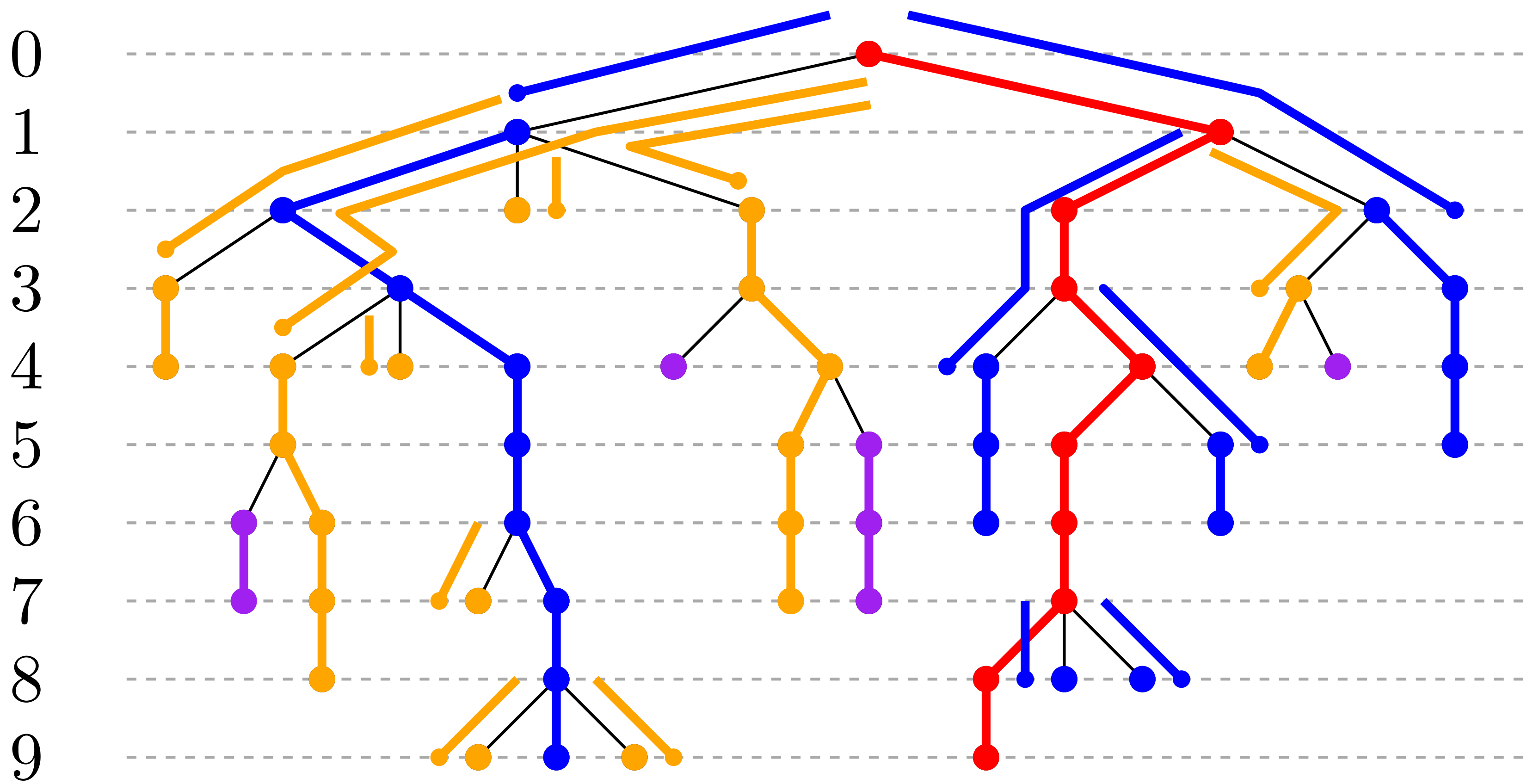

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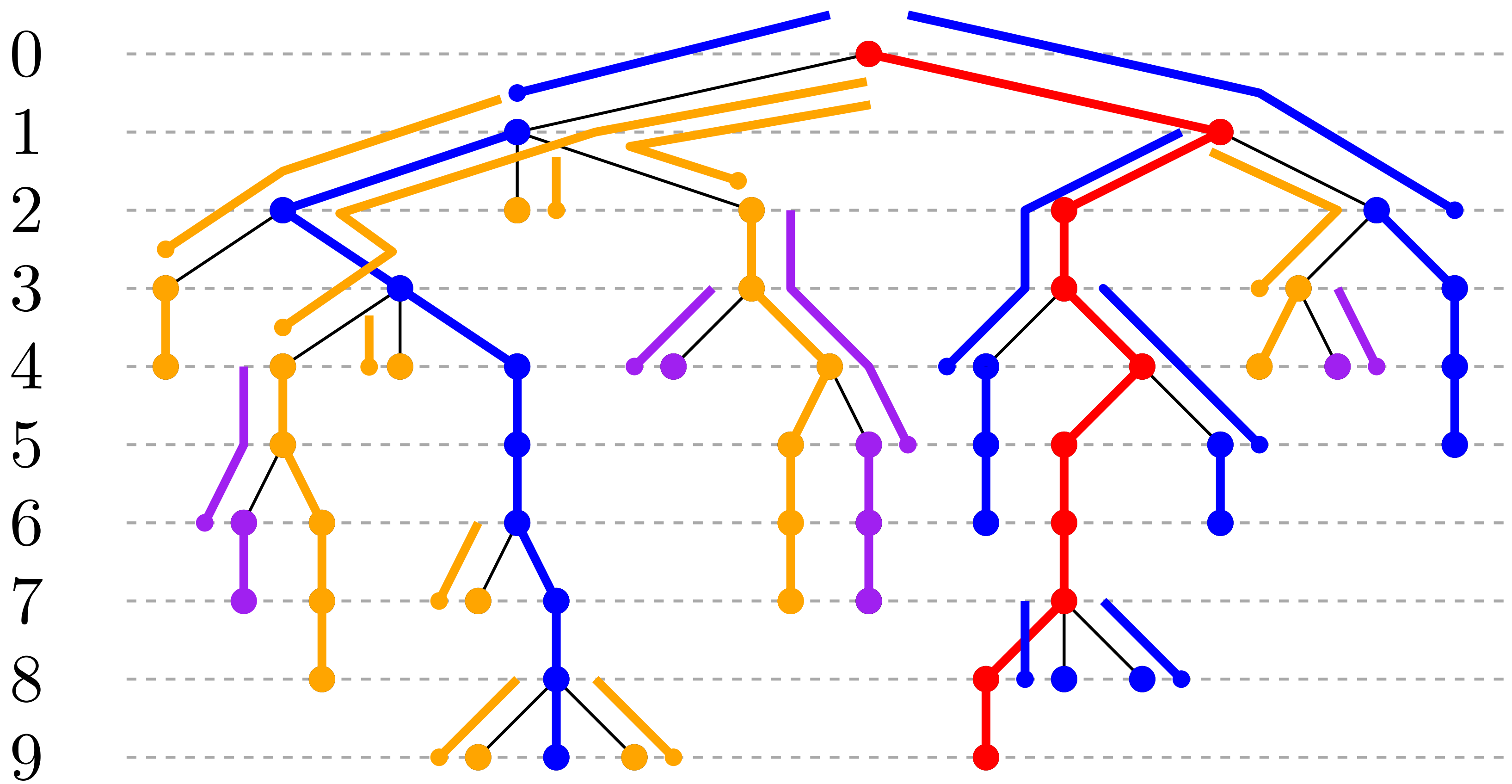

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Long Path Decomposition + Ladders

For each ladder $L_v = \langle v' = u_0, u_1, \ldots, v = u_j, \ldots, u_k \rangle$:

- Store, in v, an array B_v of length $k+1$ where $B_v[i]$ contains (a reference to) u_i
- Each u_i with $i \geq j$ stores a reference $\tau(u_i)$ to v.

The length of B_v is at most twice the length of $A_v \implies$ the total size is still $O(n)$.

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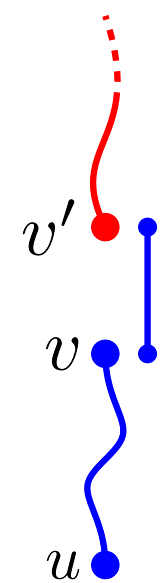
To report LA(u, d):

- Let $v = \tau(u)$ and $v' = B_v[0]$.
- If $d \geq d_{v'}$: return $B_v[d - d_{v'}]$.
- If $d < d_{v'}$: return LA(v', d). (recursively)



How many recursive calls?

Long Path Decomposition + Ladders

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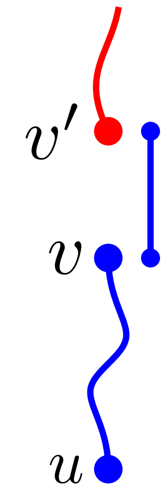

- If we recurse, L_v cannot contain the root of T.

$$|L_v| = \eta(L_v) - 1 = 2\eta(P_v) - 1$$

- Since $u \in P_v$ we have:

$$h(v') \geq |L_v| = 2\eta(P_v) - 1 \geq 2(1 + h(u)) - 1 \geq 2h(u) + 1$$

- The height of the queried vertex doubles at every iteration $\implies O(\log n)$ iterations.

Solutions so far

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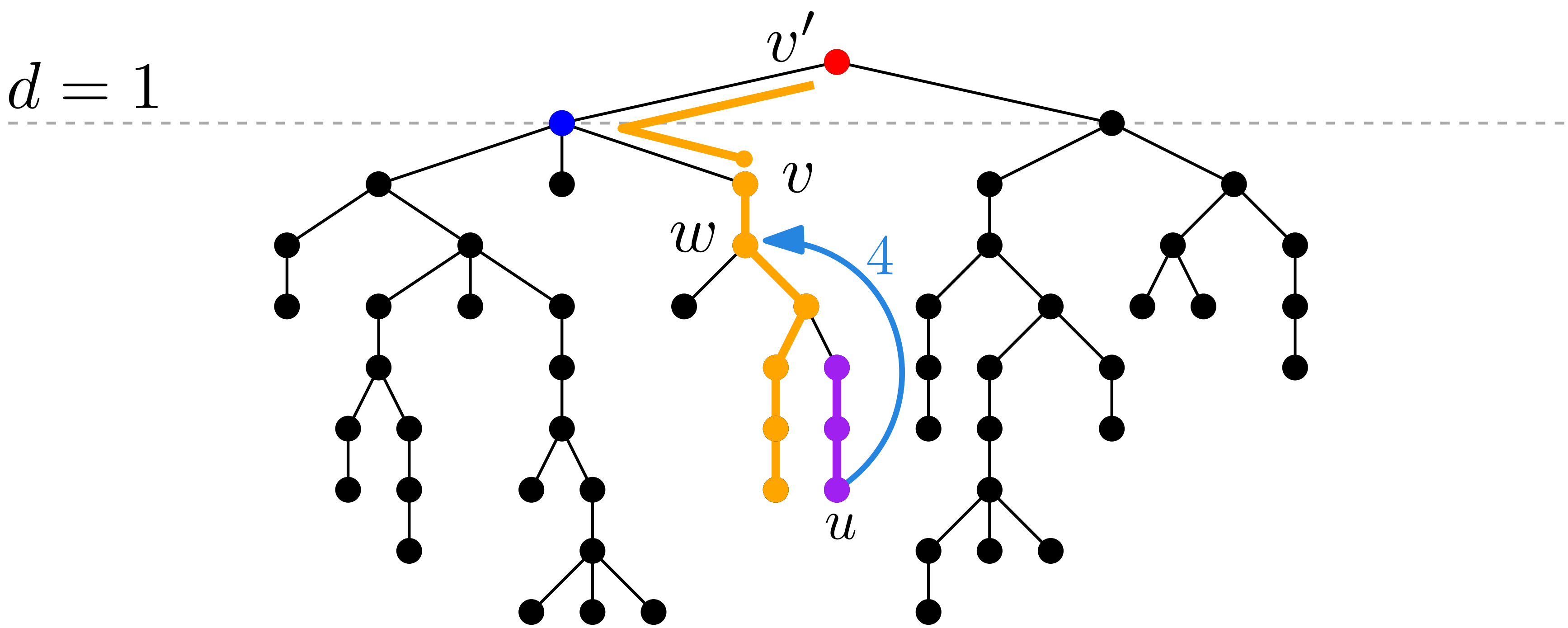
Long Path Dec. + Ladders + Jump Pointers



$$0 < d_u - d = 2^{\ell_k} + 2^{\ell_{k-1}} + \cdots + 2^{\ell_1}$$

To report LA(u, d):

- Let $w = J(u, \ell_k)$, $v = \tau(w)$ and $v' = B_v[0]$.
  - Return $B_v[d - d_{v'}]$.

Query time: $O(1)$

Long Path Dec. + Ladders + Jump Pointers

Space usage: $O(\underbrace{n +}_{\text{Ladders}} \overbrace{n \log n}^{\text{Jump Pointers}}) = O(n \log n)$

Long Path Dec. + Ladders + Jump Pointers

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A trick to reduce space:

- Only store jump pointers $J(v, \ell)$ in the leaves v of T.
- For each node u of T, store a reference to a leaf λ_u in the subtree of T rooted at u.
- $\mathsf{LA}(u, d) = \mathsf{LA}(\lambda_u, d)$

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Space usage: $O(n + L \log n)$, where $L = \#$leaves of T.

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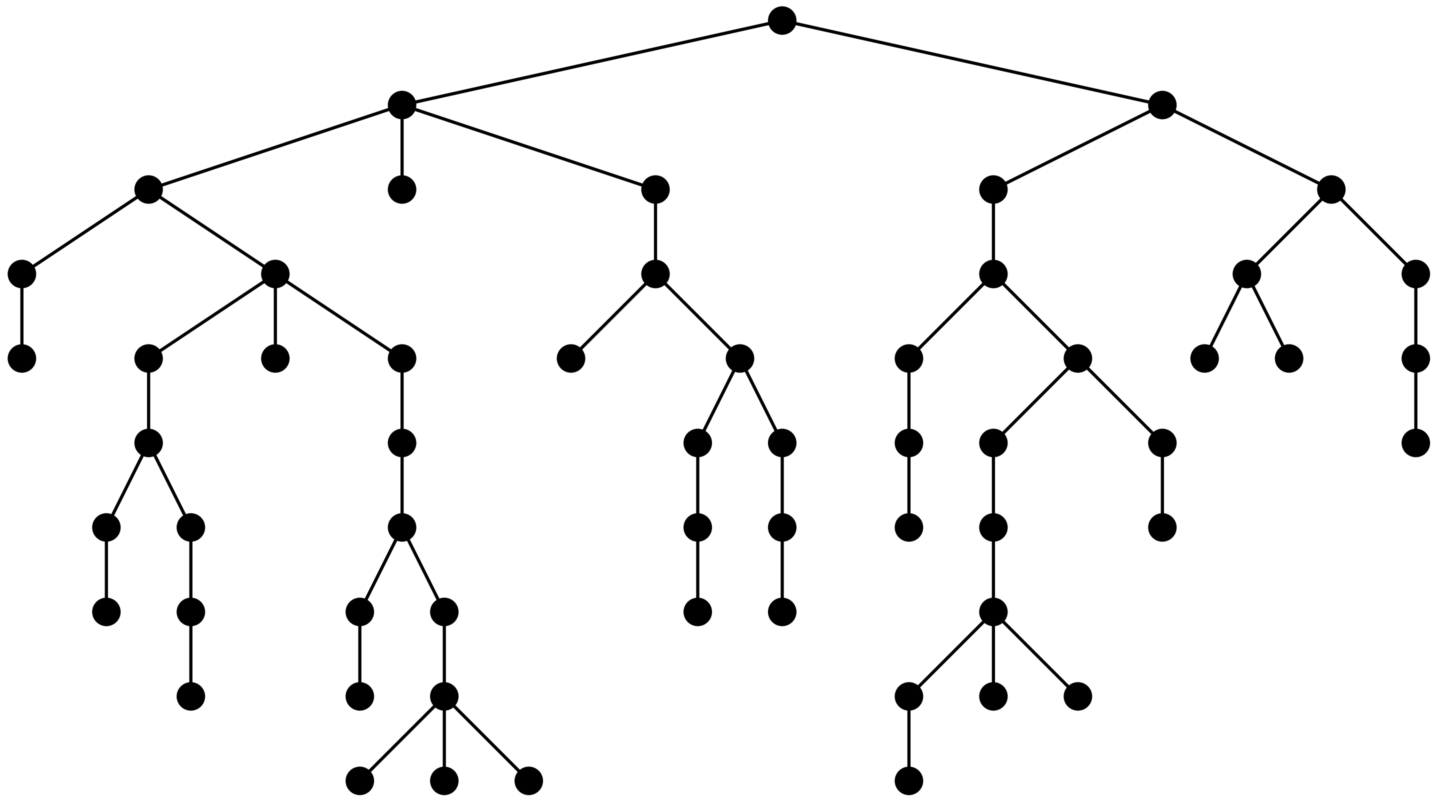
If only we had $O(\frac{n}{\log n})$ leaves...

Macro-Micro trees



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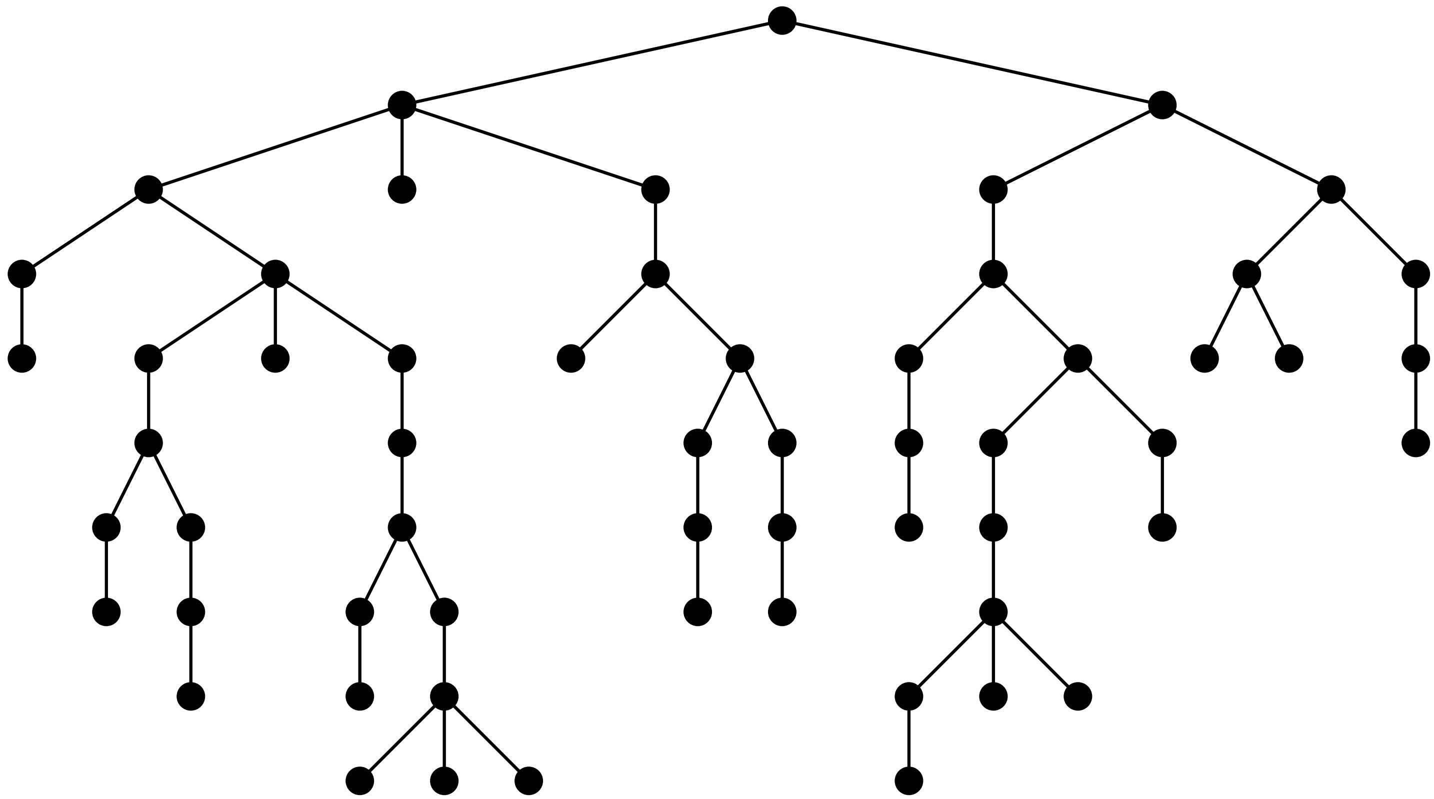
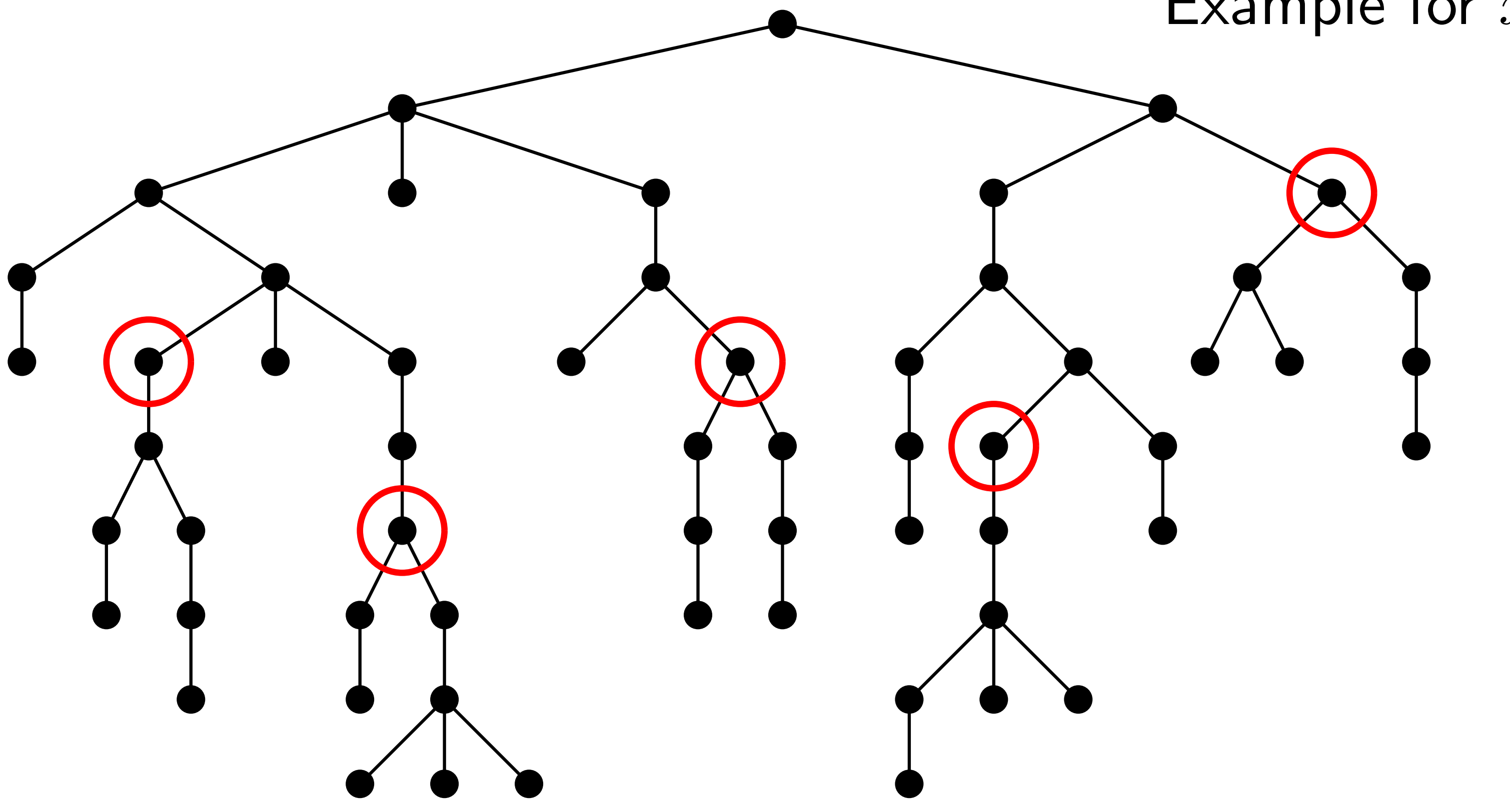


Find the set M of all maximally deep vertices with at least $x = \frac{1}{4} \log n$ descendants.

Macro-Micro trees



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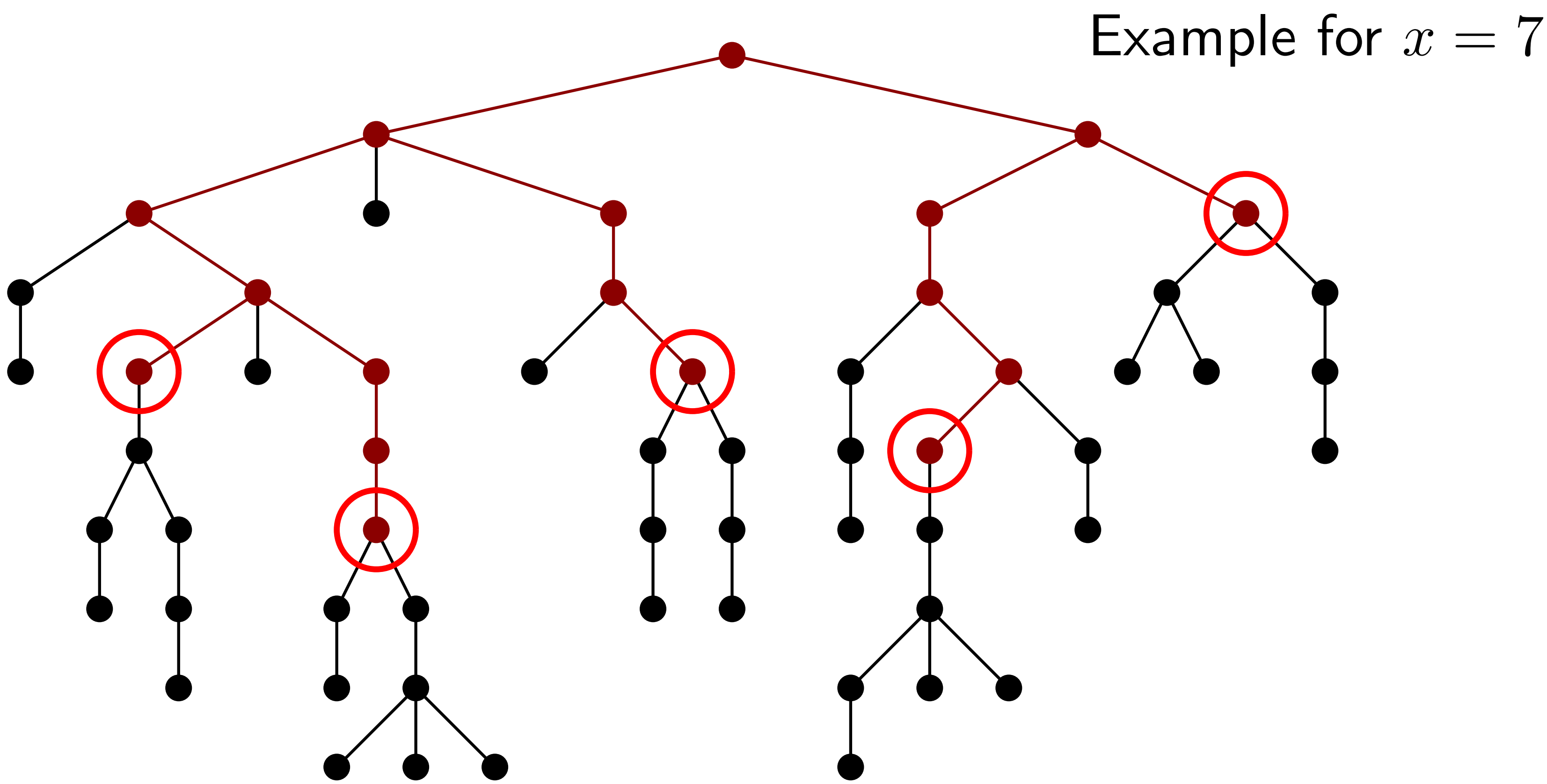

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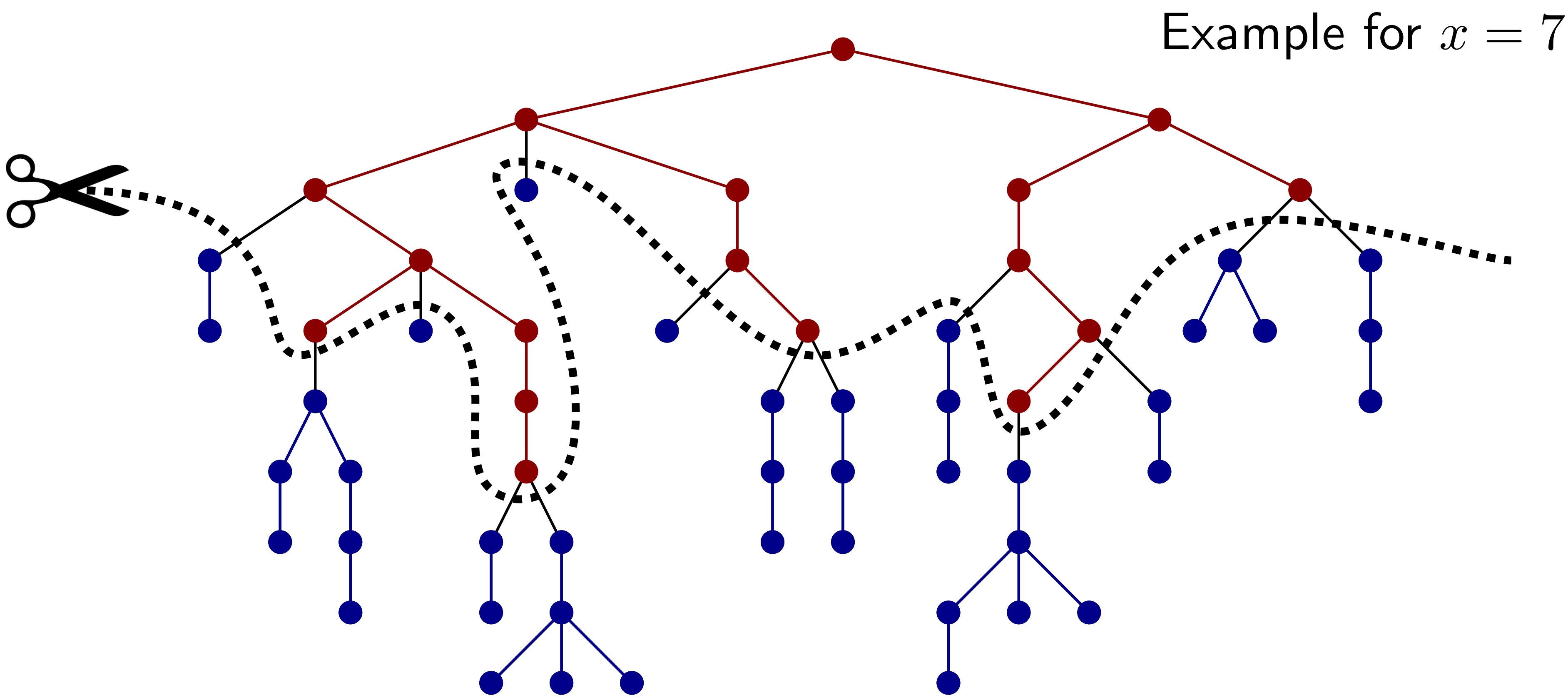

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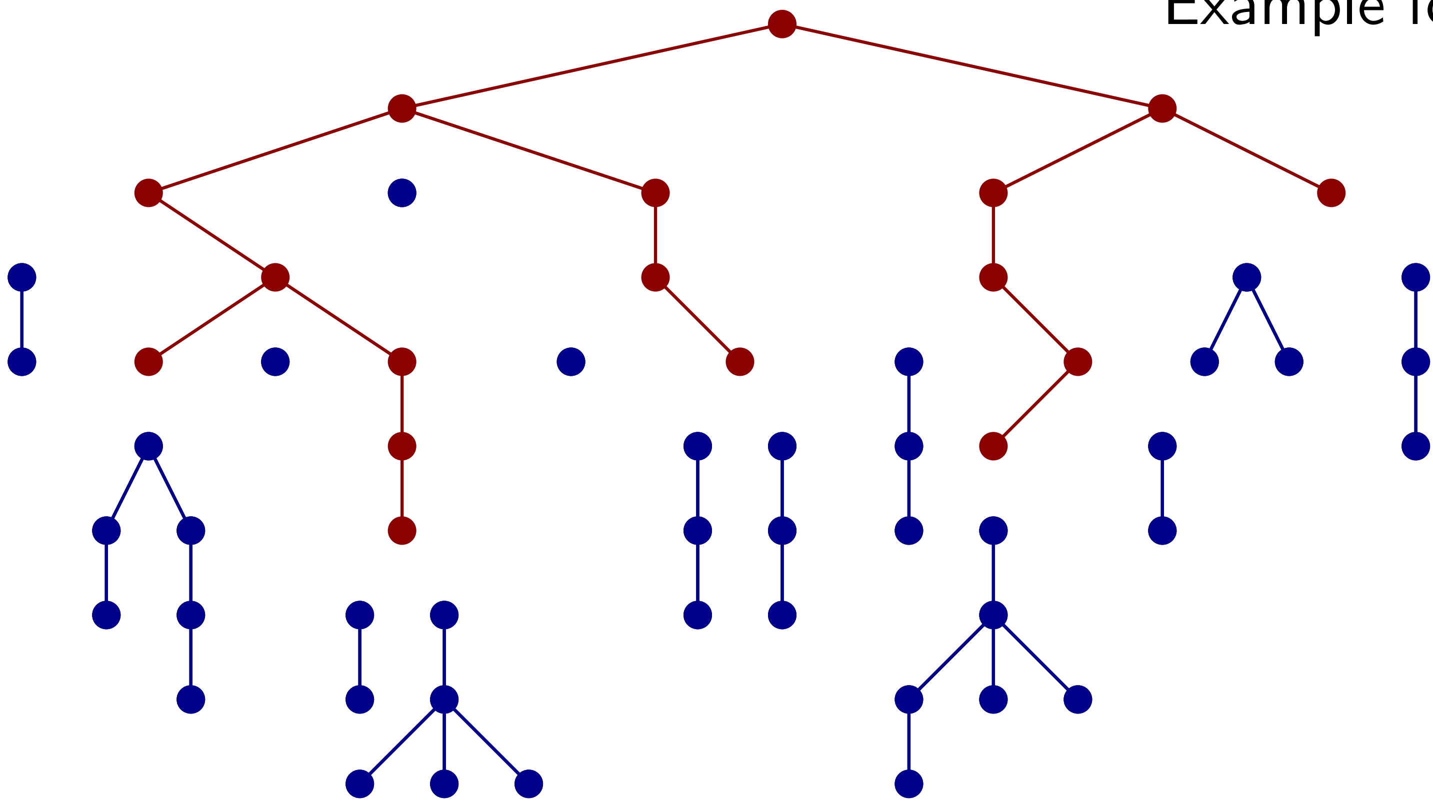
Macro-Micro trees

Example for $x = 7$



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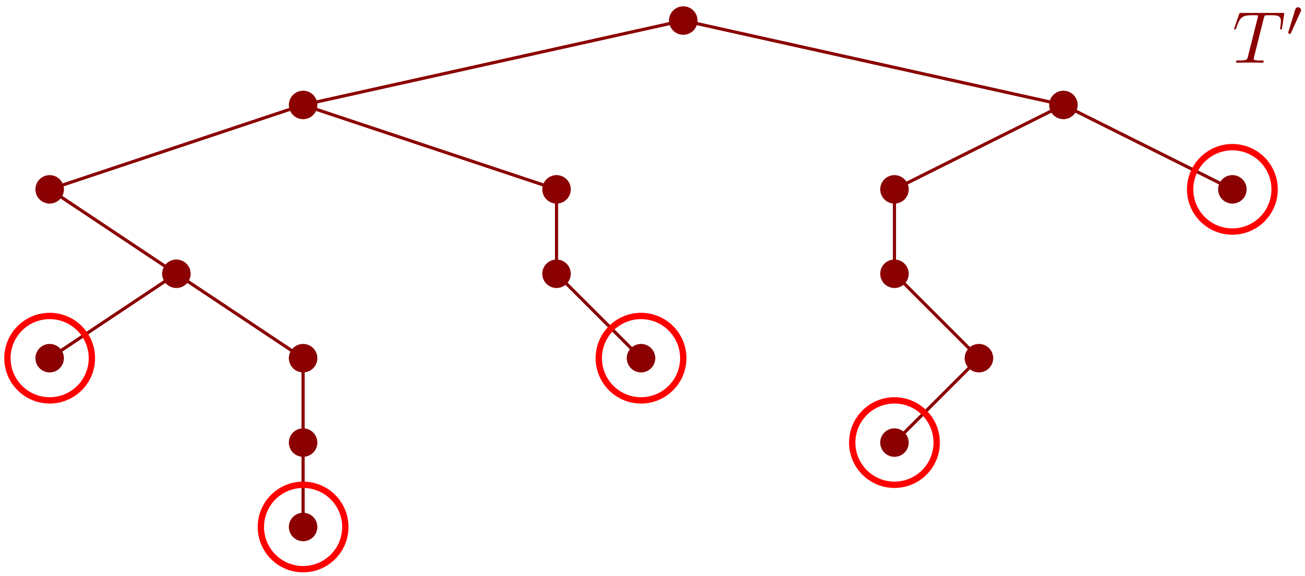
Handling the Macro-tree



How many leaves in T'? The leaves of T' are the vertices in M.

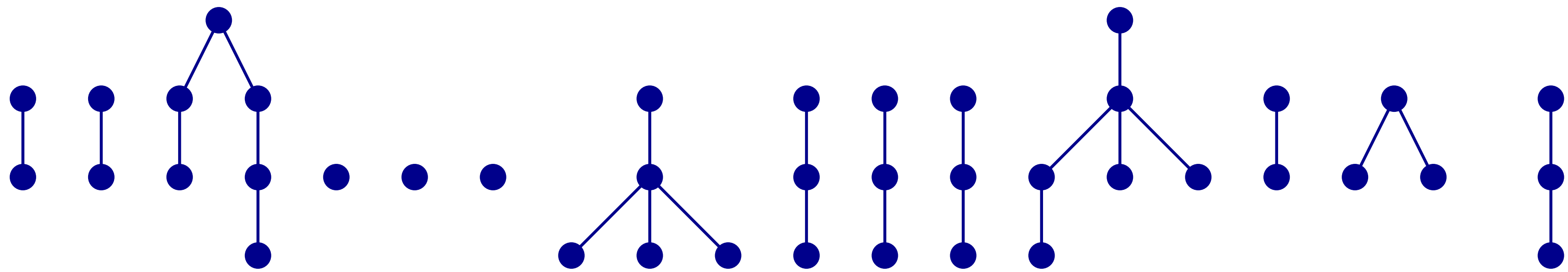
Each vertex in M has at least $\frac{1}{4}\log n$ descendants in T.

$|M| \cdot \frac{1}{4}\log n \leq n \implies |M| = O(\frac{n}{\log n})$.

Build the previous LA oracle $\mathcal{O}'$ on T'.

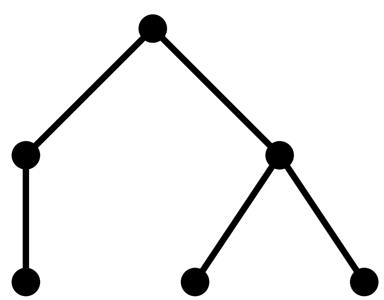
Size/build time: $O(n+|M|\log n)=O(n)$. **Query time:** $O(1)$.

Handling the Micro-trees



How many different *types* of micro-trees?

A rooted tree on $\leq m$ vertices can be uniquely represented by an array of $2(m-1)$ bits.



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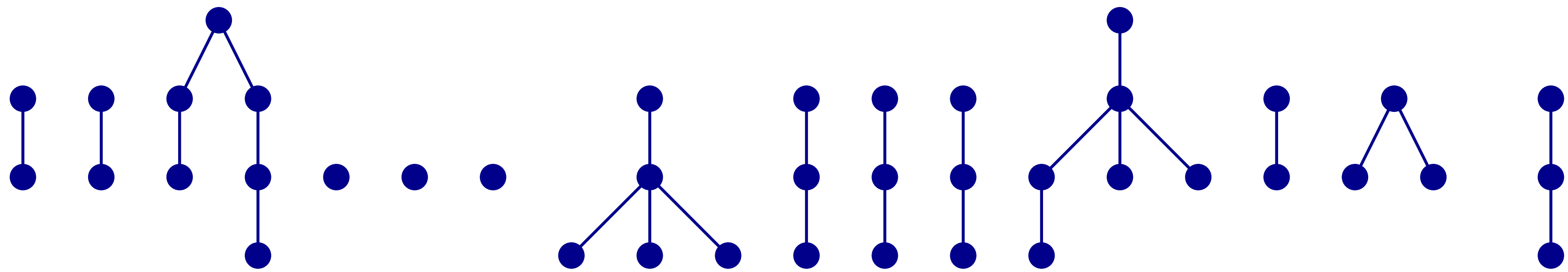

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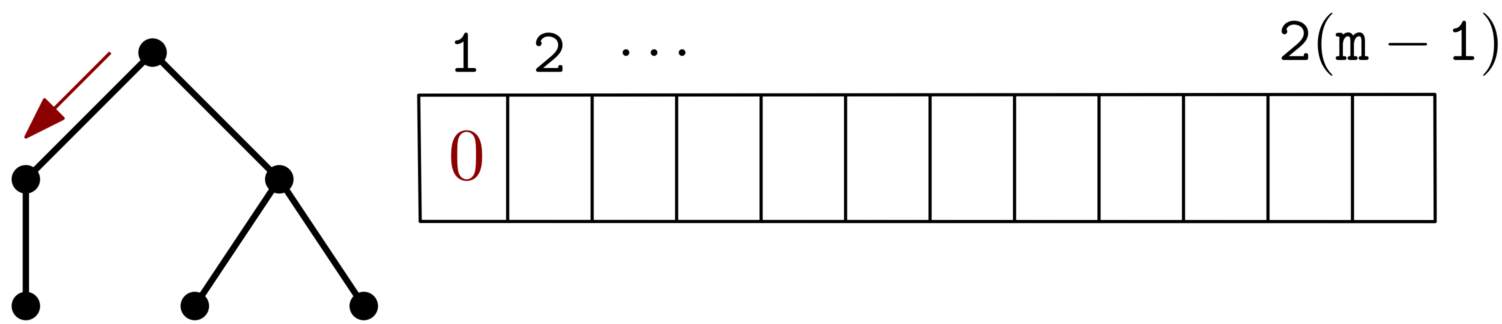

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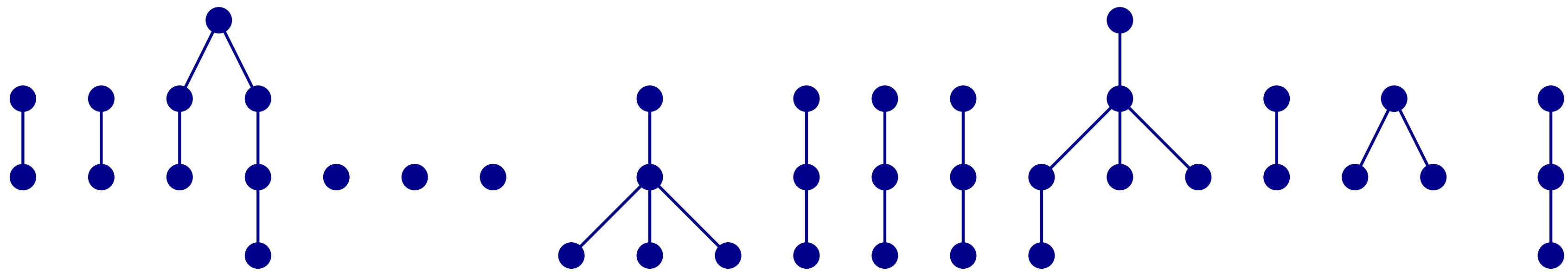

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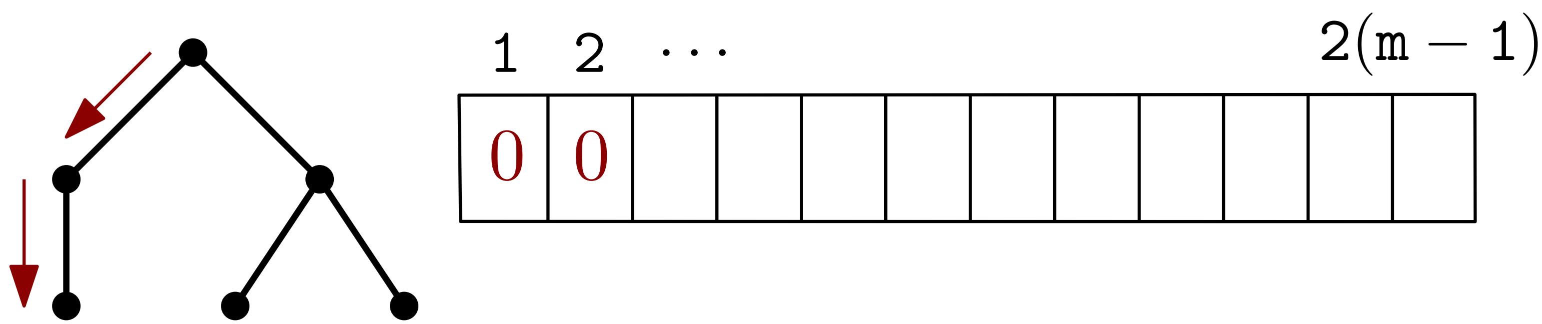

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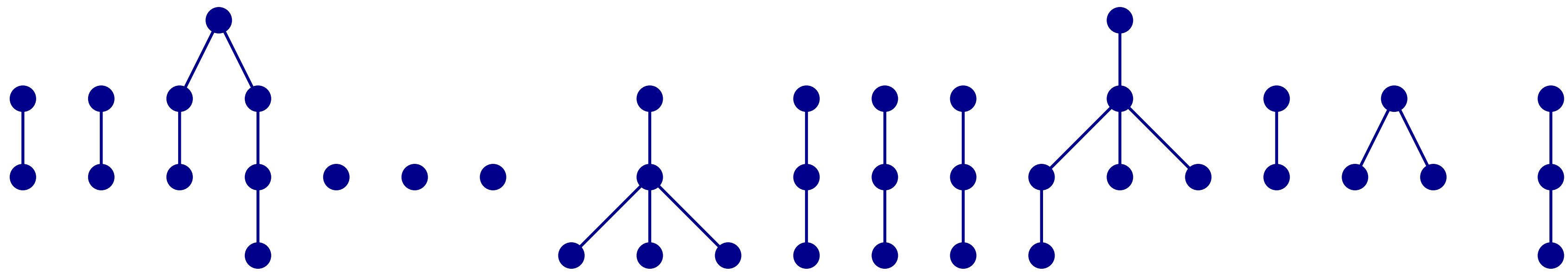

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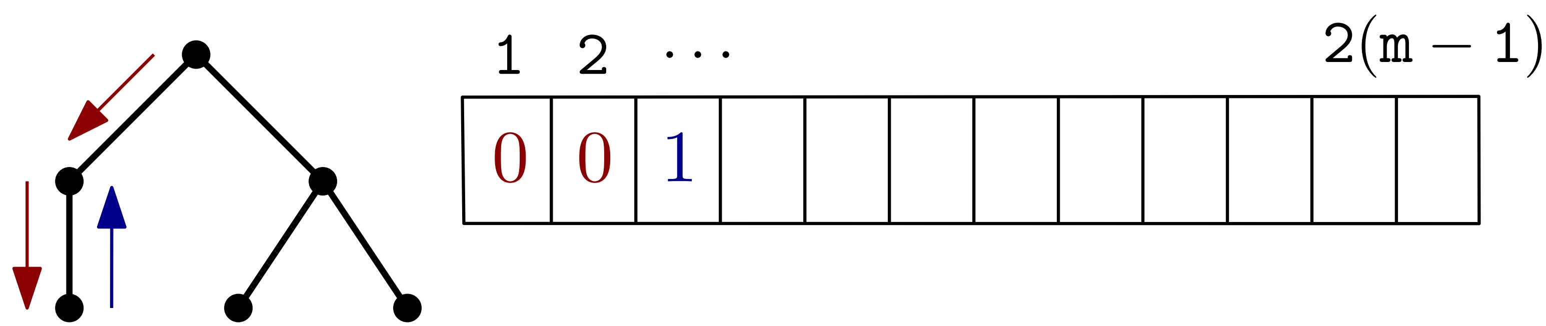

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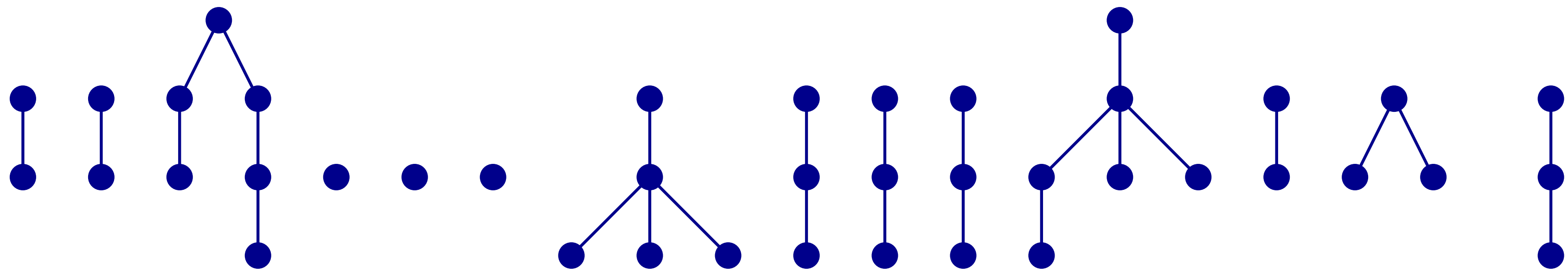

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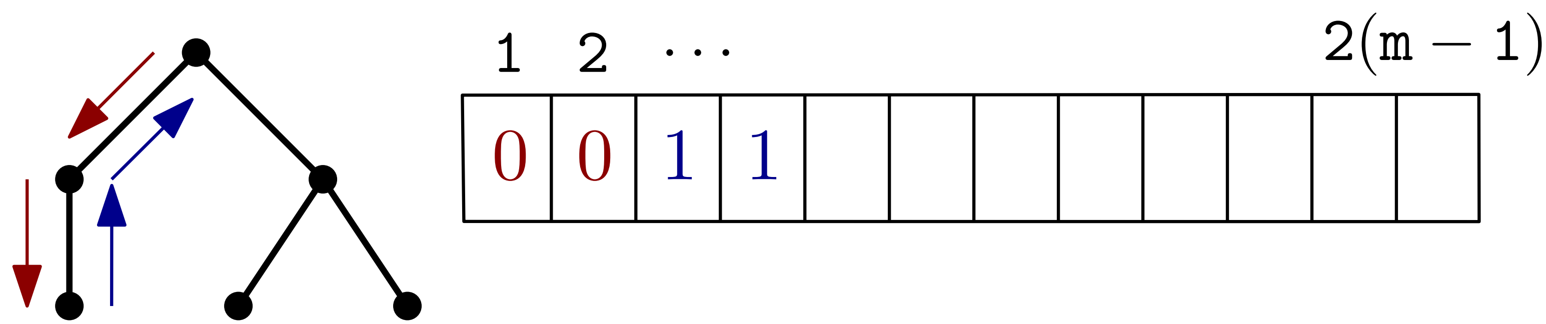

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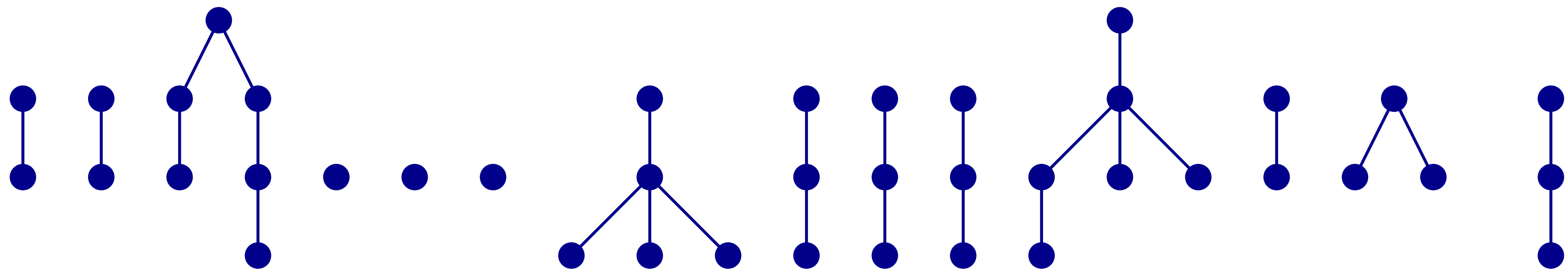

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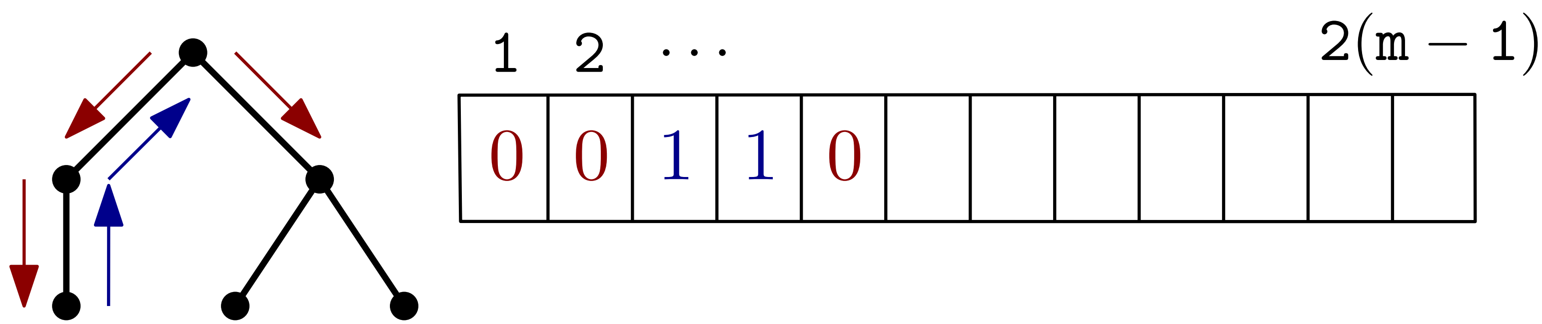

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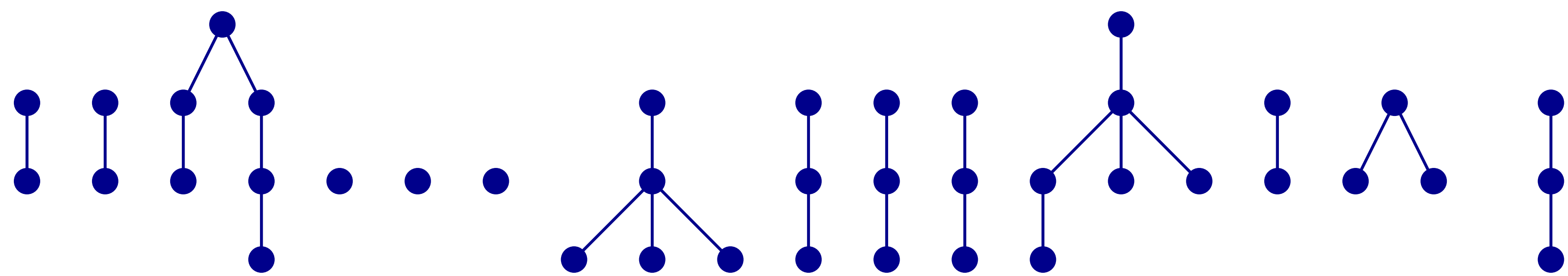

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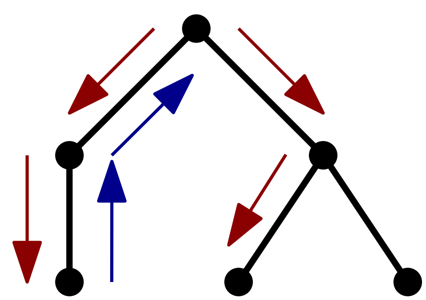
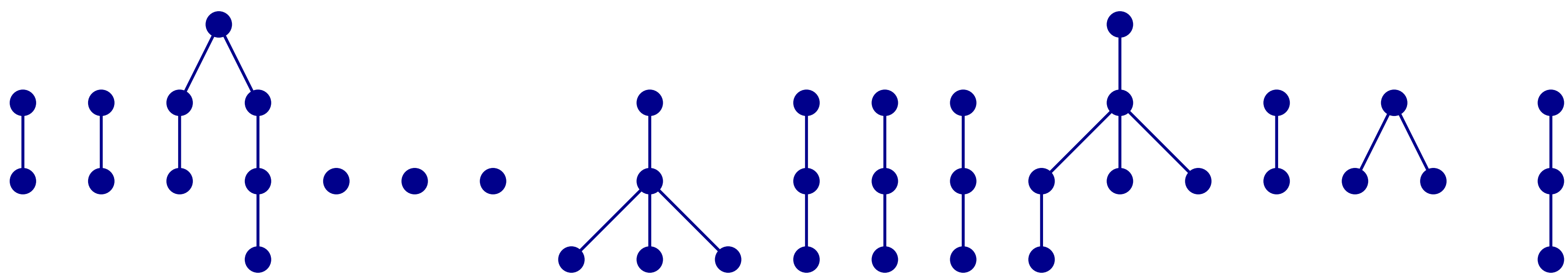


1	2	$\cdots$									$2(m-1)$
0	0	1	1	0	0						

Handling the Micro-trees



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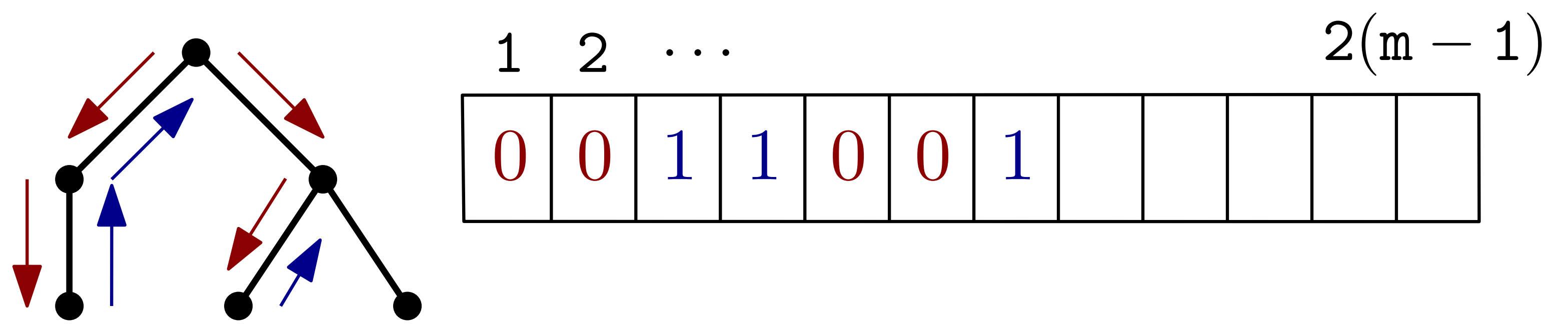

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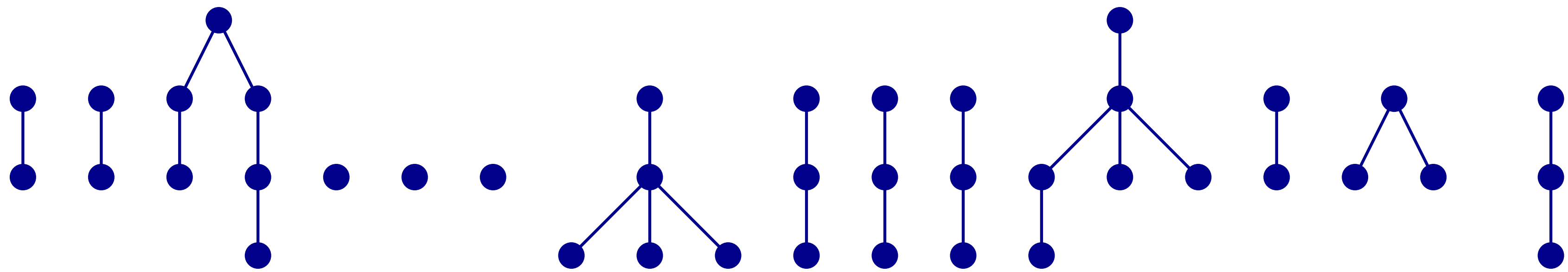

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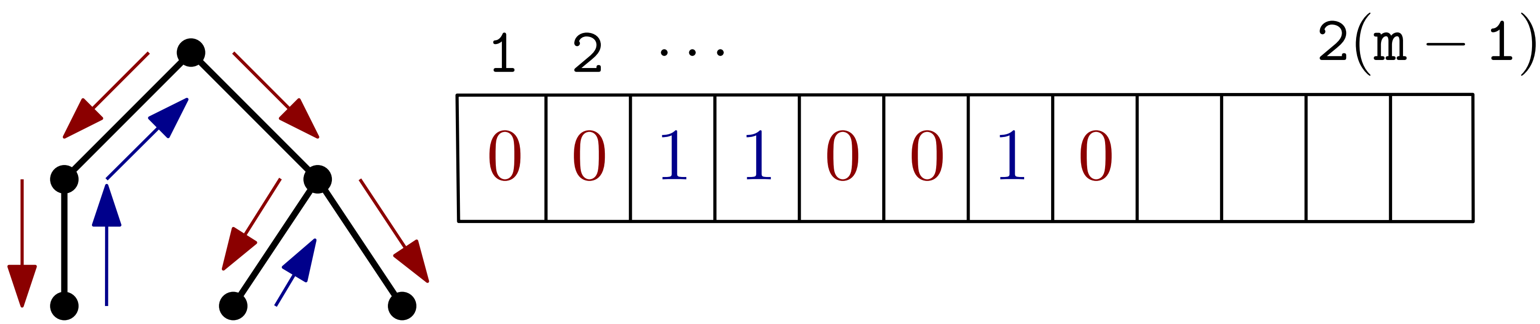

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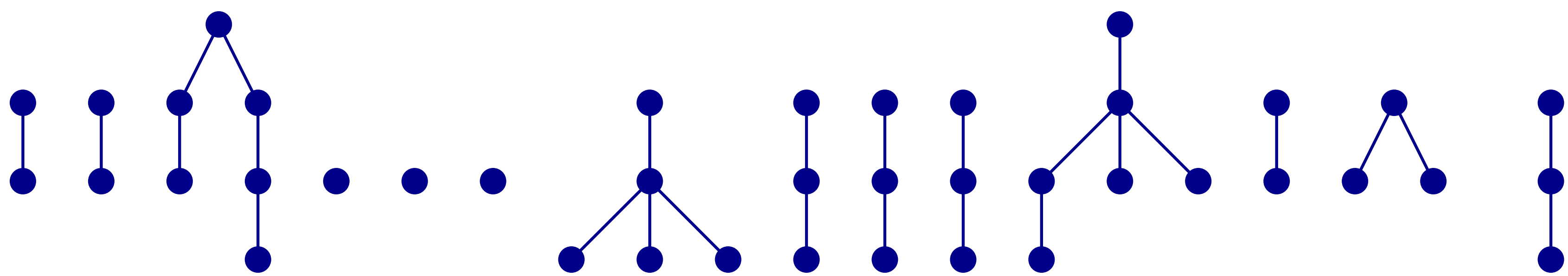

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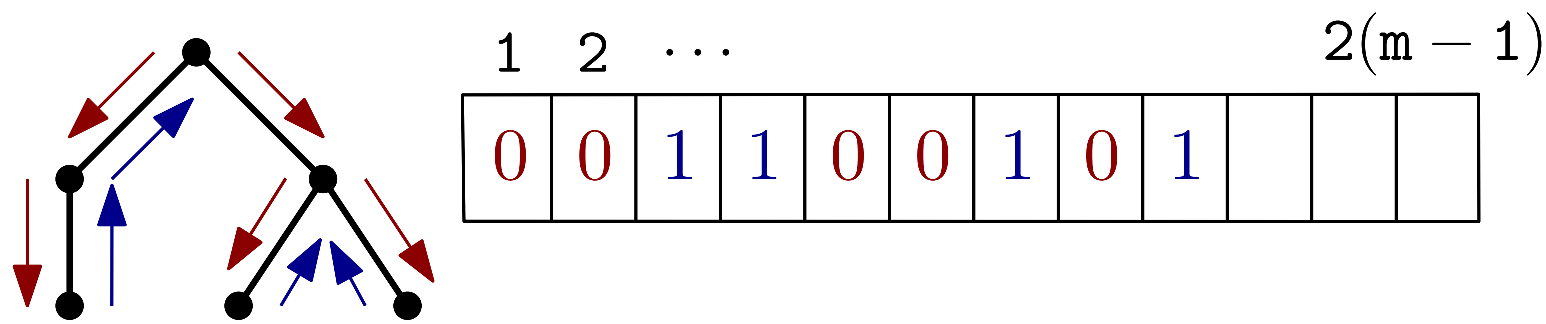

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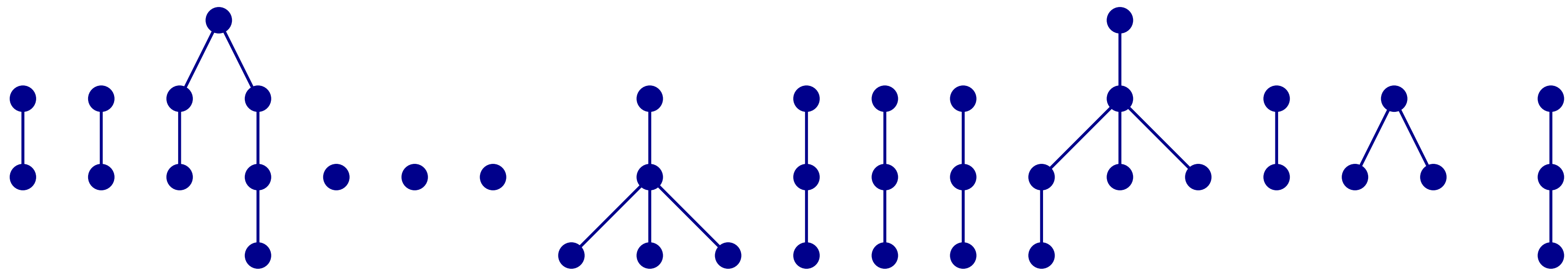

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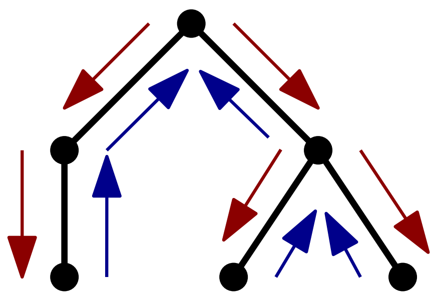

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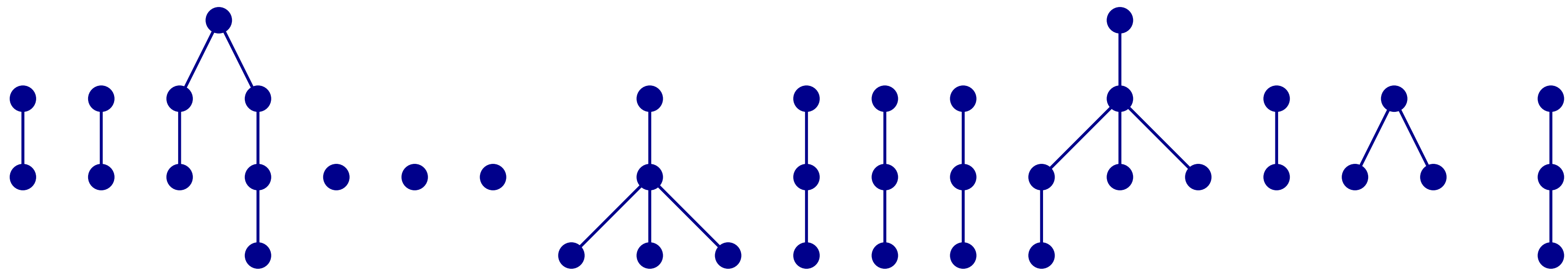
0	0	1	1	0	0	1	0	1	1		

Handling the Micro-trees



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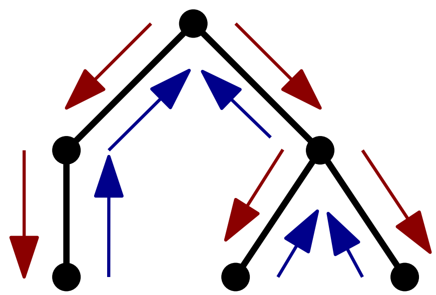
Pad with 1s.



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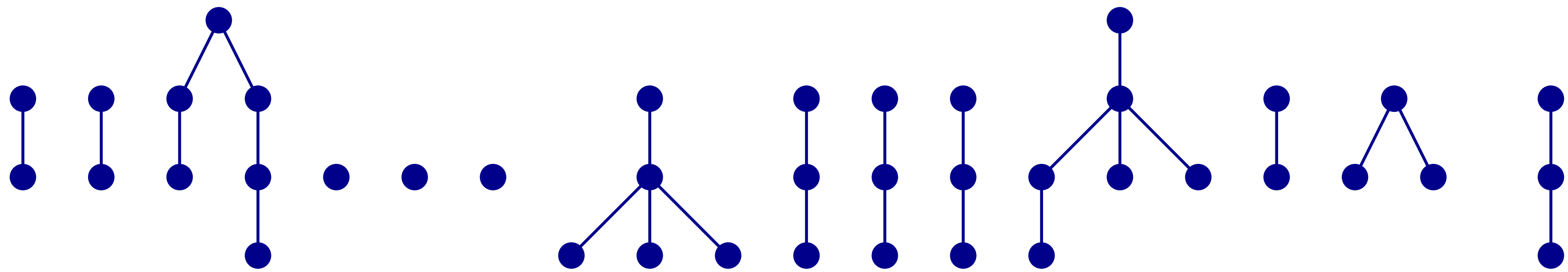
Handling the Micro-trees

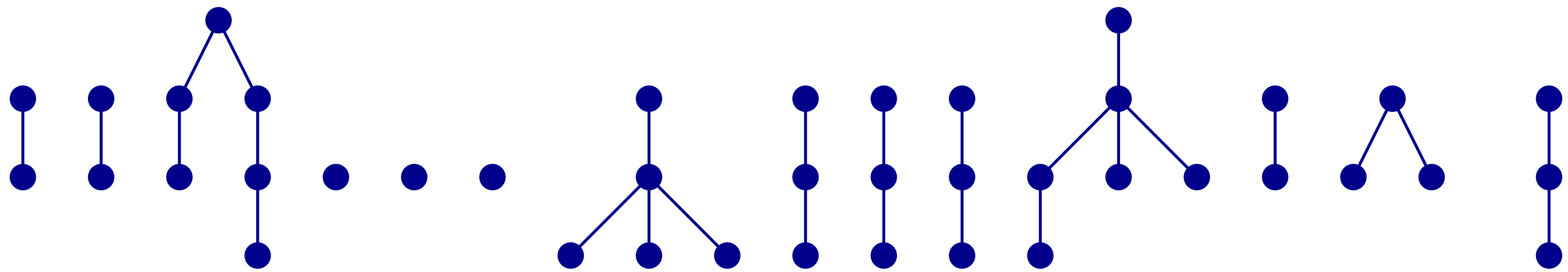


How many different *types* of micro-trees?

A rooted tree on $\leq m$ vertices can be uniquely represented by an array of $2(m-1)$ bits.

At most $2^{2(m-1)} < 2^{2m}$ trees with up to m vertices
$\implies O(2^{2\frac{1}{4}\log n}) = O(\sqrt{n})$ micro-tree types.

Handling the Micro-trees



For each of the $O(\sqrt{n})$ distinct micro-trees types T_i

- Build the trival oracle $\mathcal{O}_i$ with size/preprocessing time $O(|T_i|^2)$ and query time $O(1)$.

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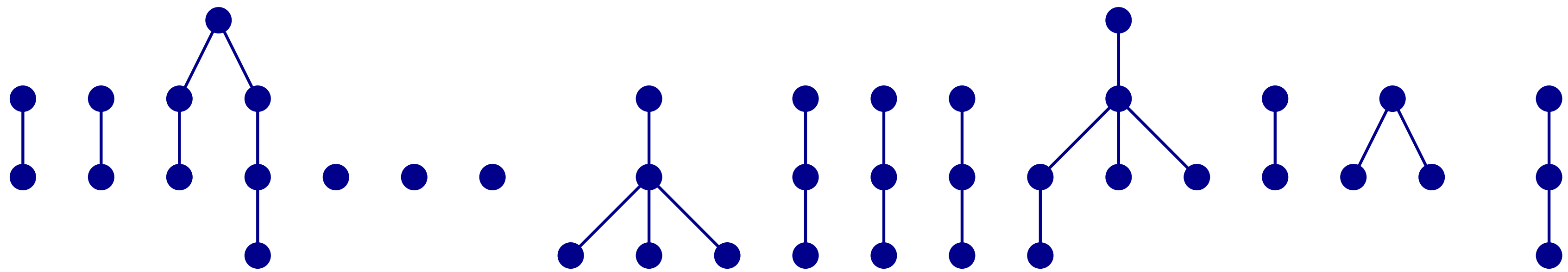

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For each vertex u of T that belongs to a micro tree:

- Store, in u, the index i of the *type* of its micro-tree.
- Store, in u, the vertex $\mu(u)$ in T_i corresponding to u.
- Store, in u, the root $\rho(u)$ of its micro-tree.

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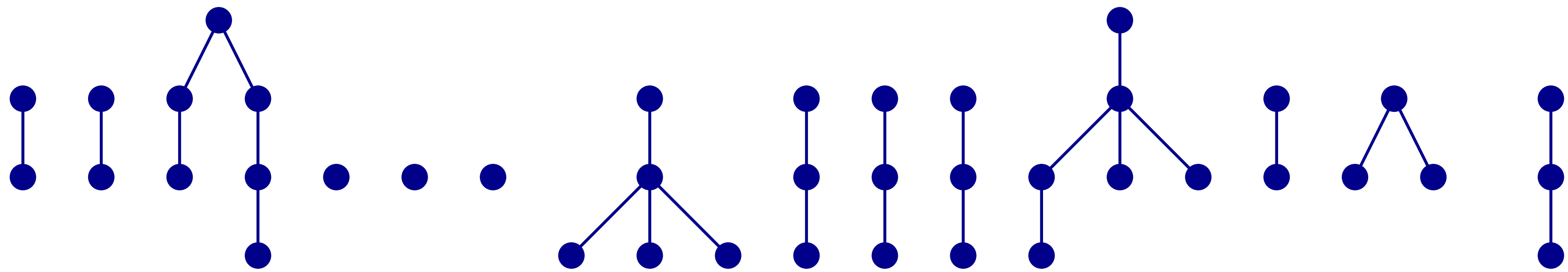

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Total size/time: $O(\sqrt{n}) \cdot O(\log^2 n) + O(n) = O(n)$.

Answering a Query

To answer LA(u, d):

- If u is in the macro tree T': query $\mathcal{O}'$ for LA(u, d).
- If u is in a micro-tree T'':
  - If $d < d_{\rho(u)}$: query $\mathcal{O}'$ for LA$(\text{parent}(\rho(u)), d)$.
  - Otherwise:
    - Let i be the type of the micro-tree containing u.
    - Query O_i for LA$(\mu(u), d - d_{\rho(u)})$.

      (and map it back to a vertex in T)

Query time: $O(1)$.

Solutions so far

Size	Preprocessing Time	Query Time	Notes
$O(n)$	–	$O(n)$	
$O(n^2)$	$O(n^3)$	$O(1)$	
$O(n^2)$	$O(n^2)$	$O(1)$	
$O(n \log n)$	$O(n \log n)$	$O(\log n)$	Jump Pointers
$O(n)$	$O(n)$	$O(\sqrt{n})$	Long Path Dec.
$O(n)$	$O(n)$	$O(\log n)$	+ Ladders
$O(n + L \log n)$	$O(n + L \log n)$	$O(1)$	+ Ladders, JP

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$O(n)$	$O(n)$	$O(1)$	$+$ Macro-Micro trees

Weighted Level Ancestors

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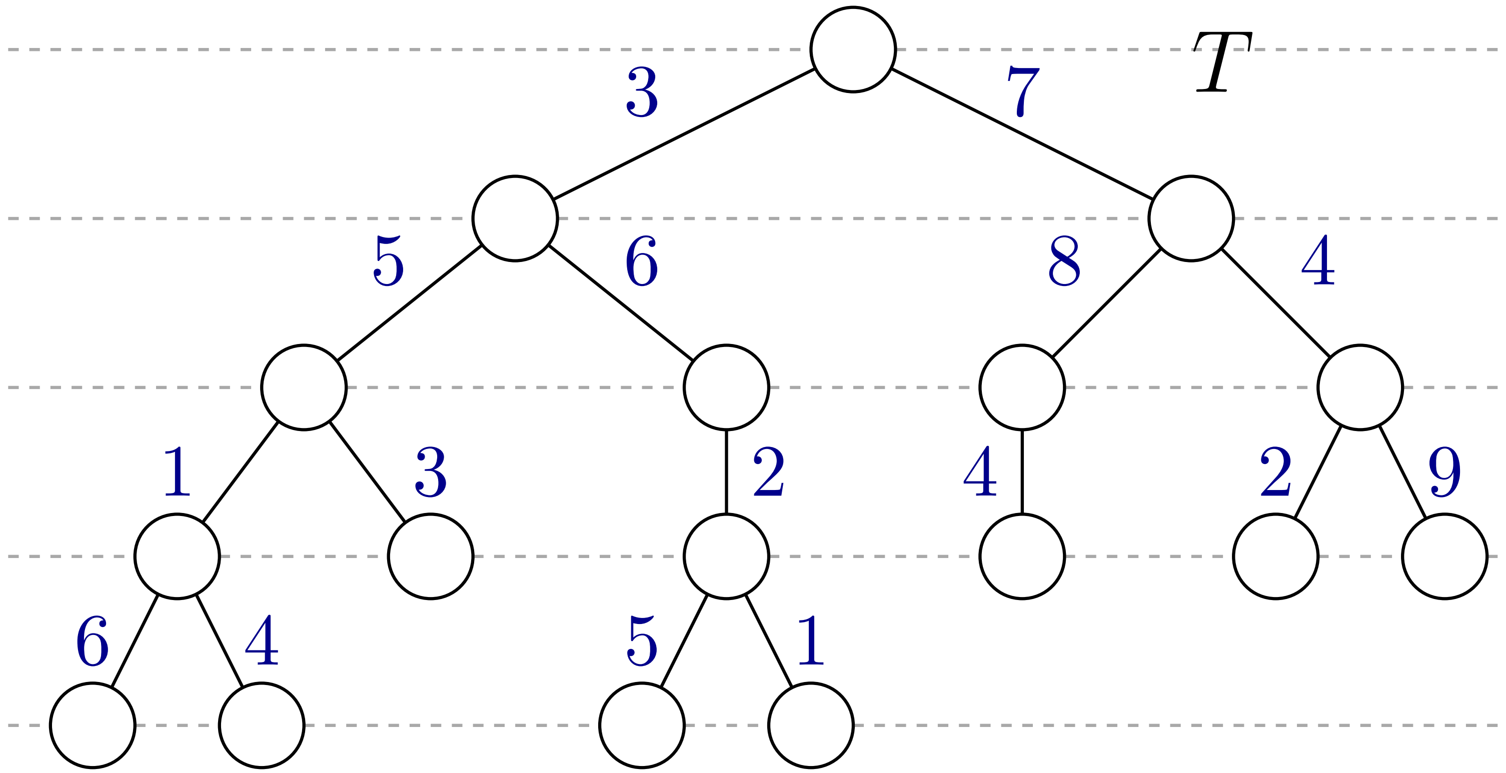

Each edge has a positive weight

Query: Given a vertex v and $d \in \mathbb{N}$, report the deepest ancestor u of v such that the distance from u to v is at least d.

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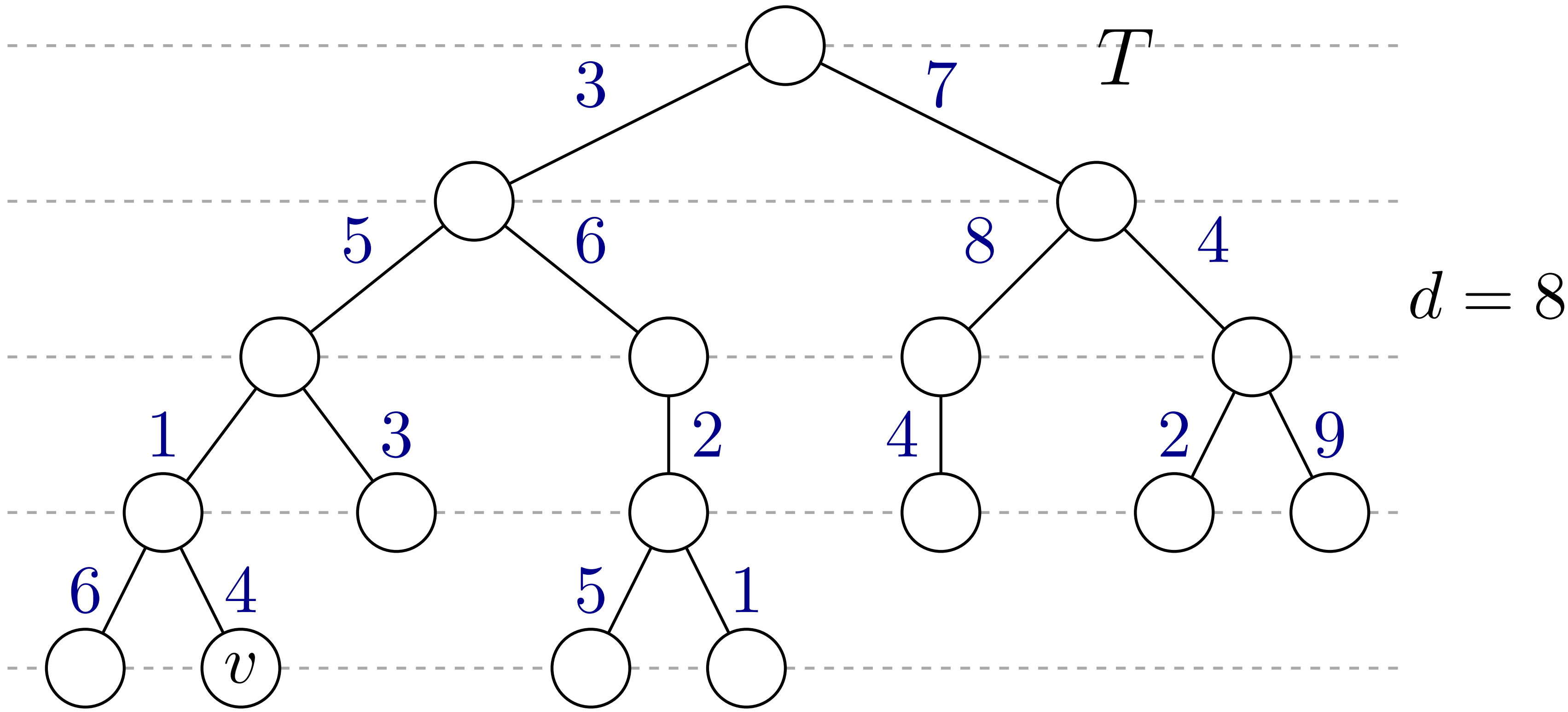

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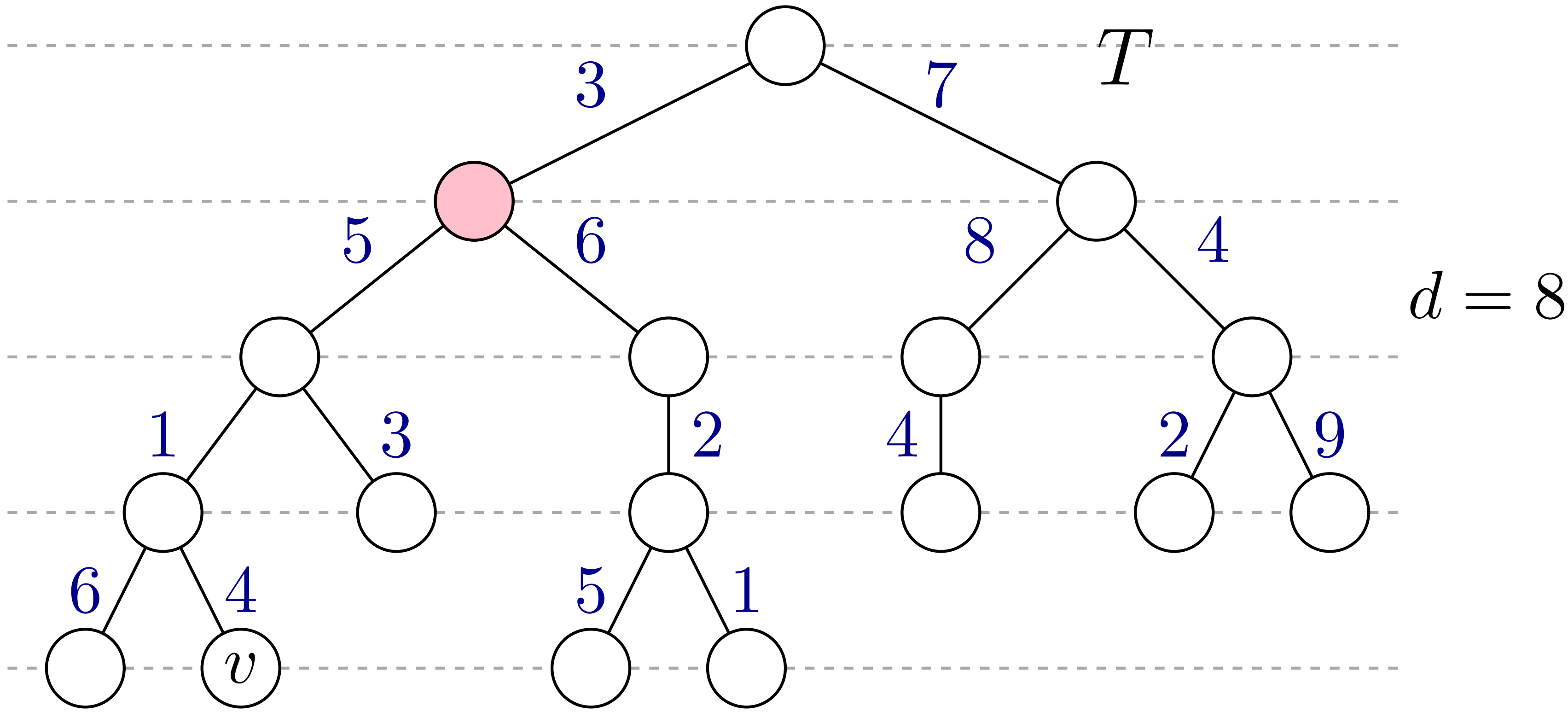

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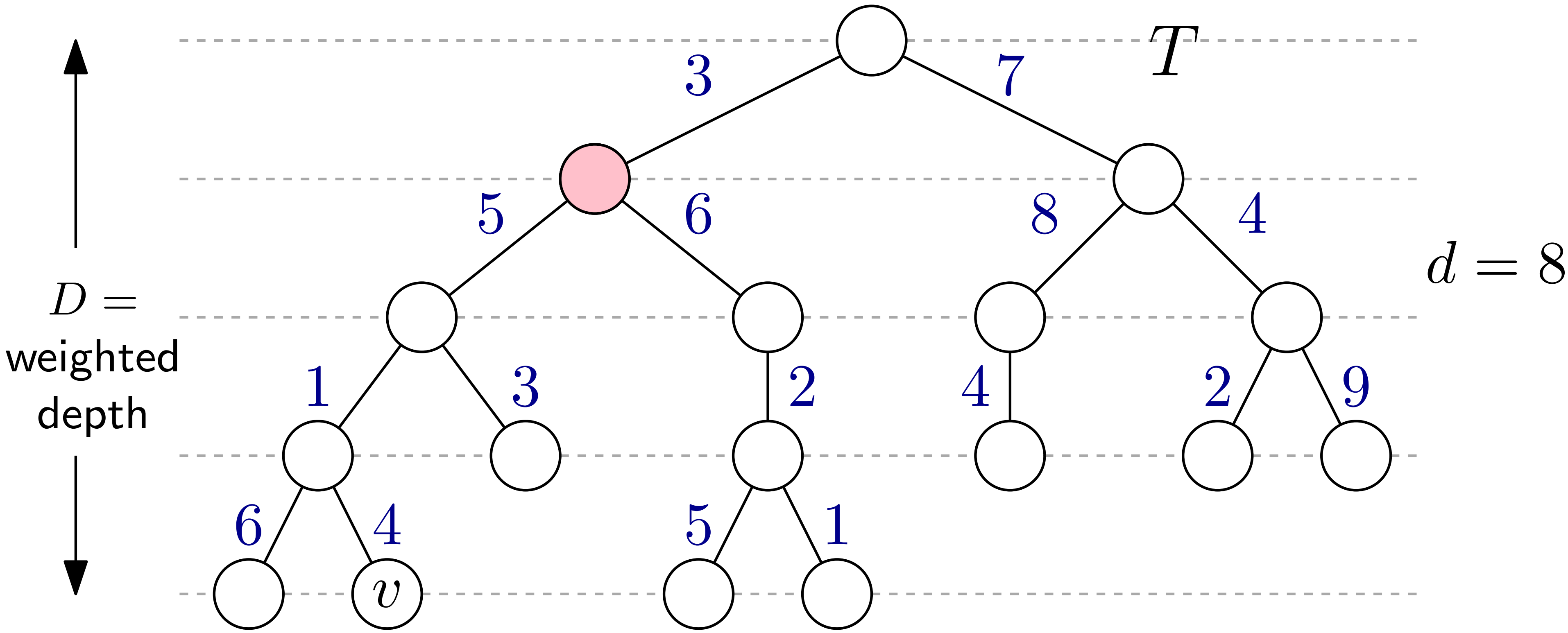

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Can be solved with $O(n)$ preprocessing, $O(n)$ space, and $O(\log \log D)$ time [Amit et. al., Dynamic Text and Static Pattern Matching]