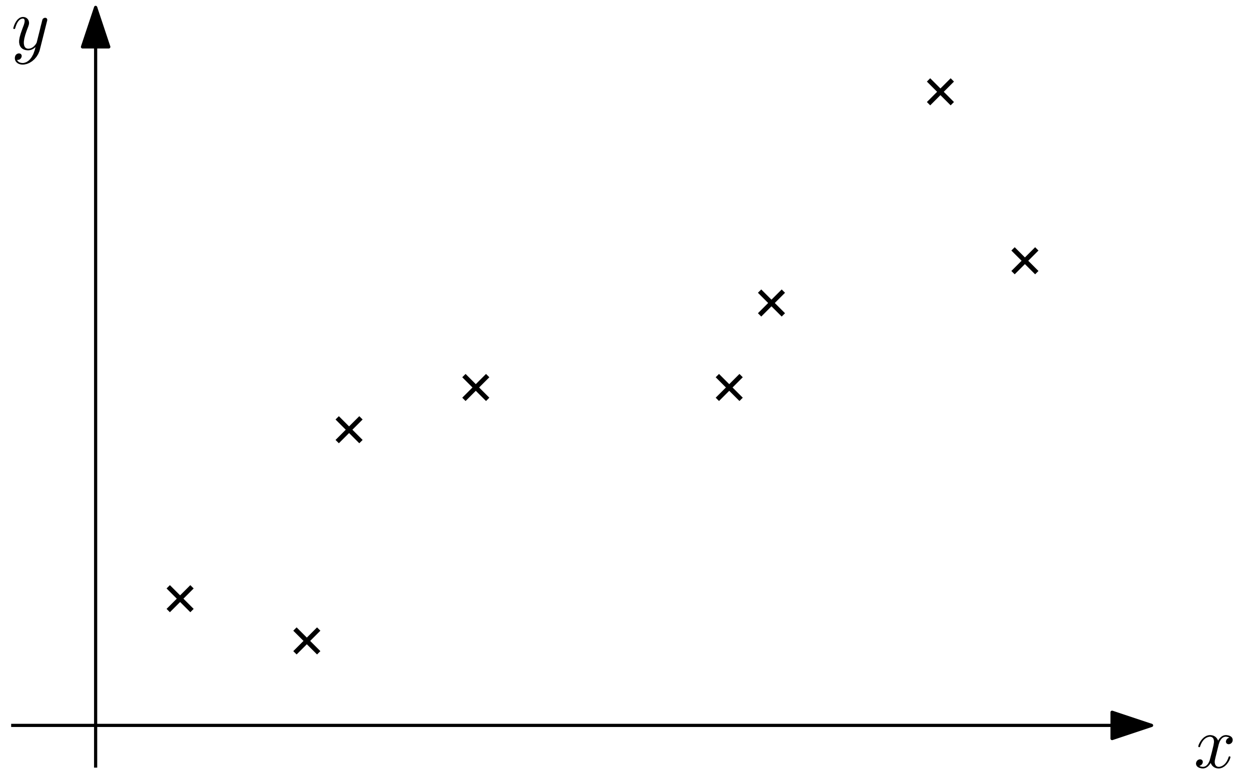
Segmented Least Squares

Linear Regression



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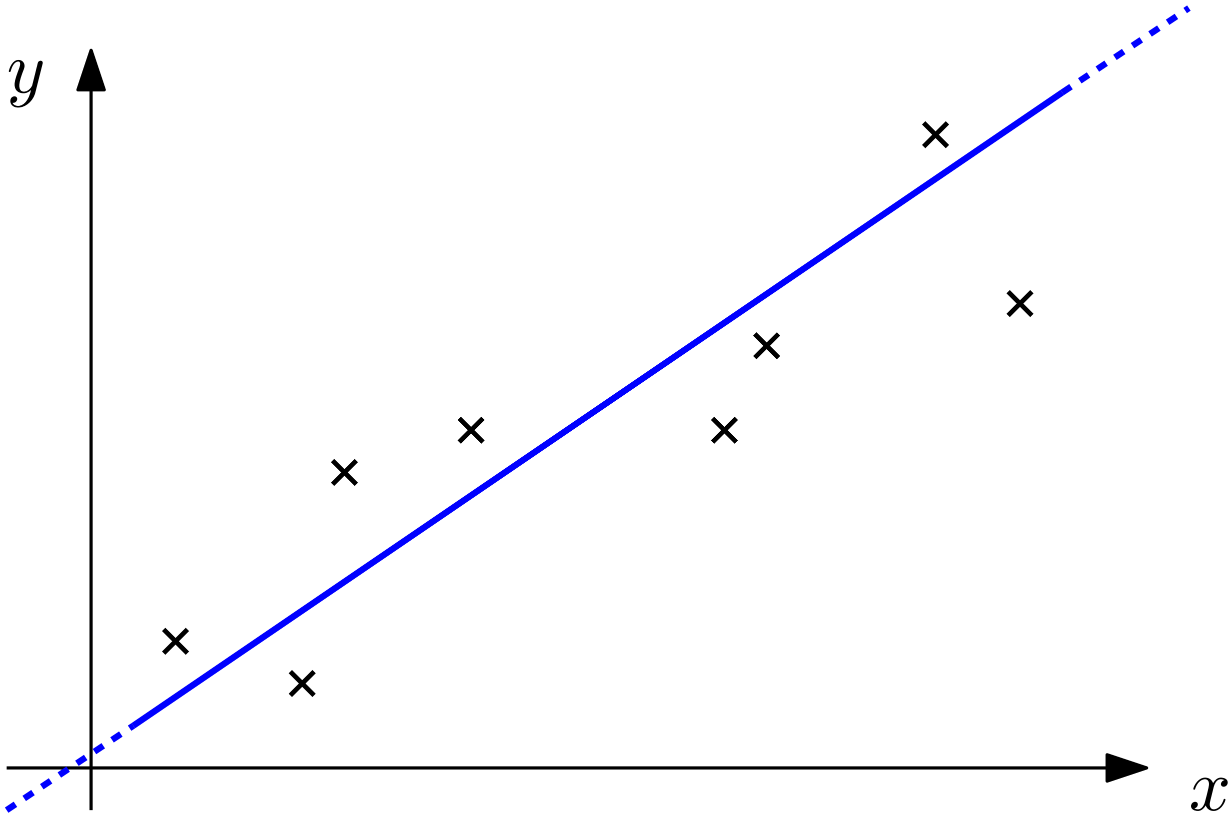

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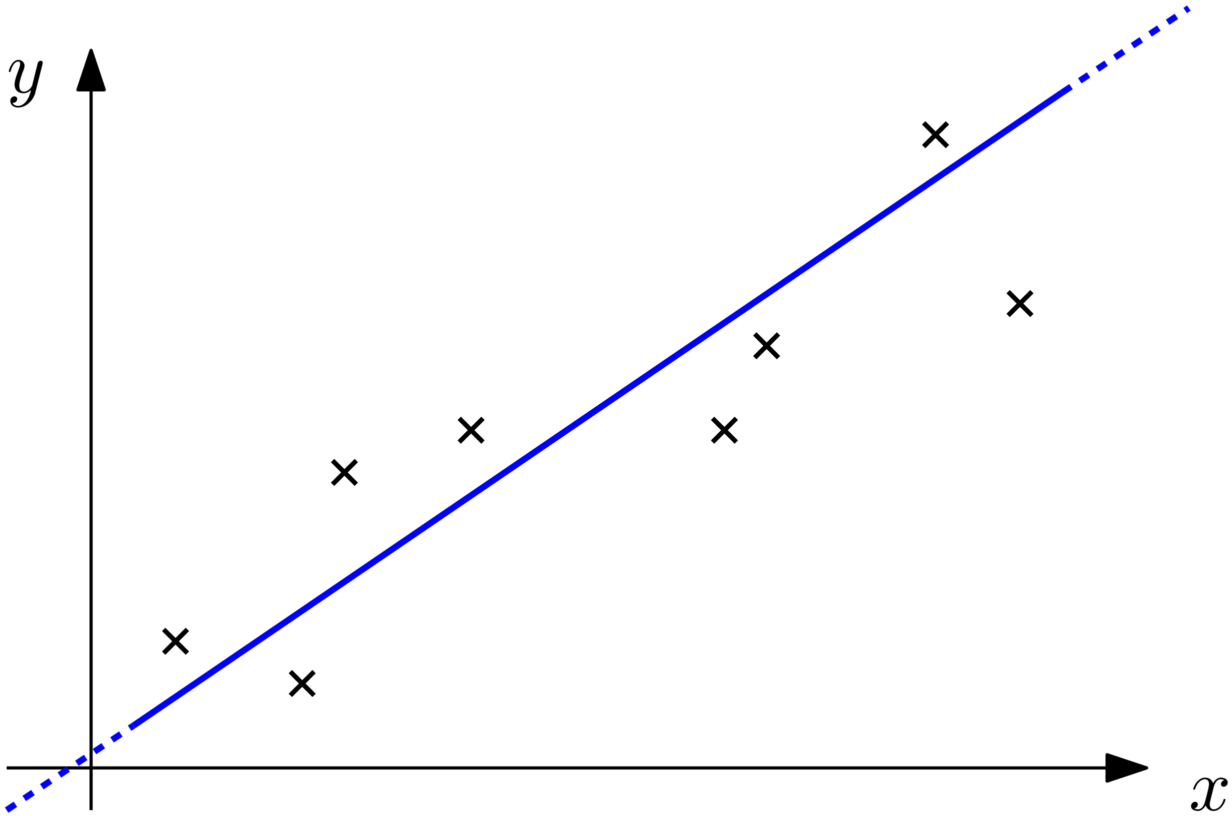

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Need to find a and b

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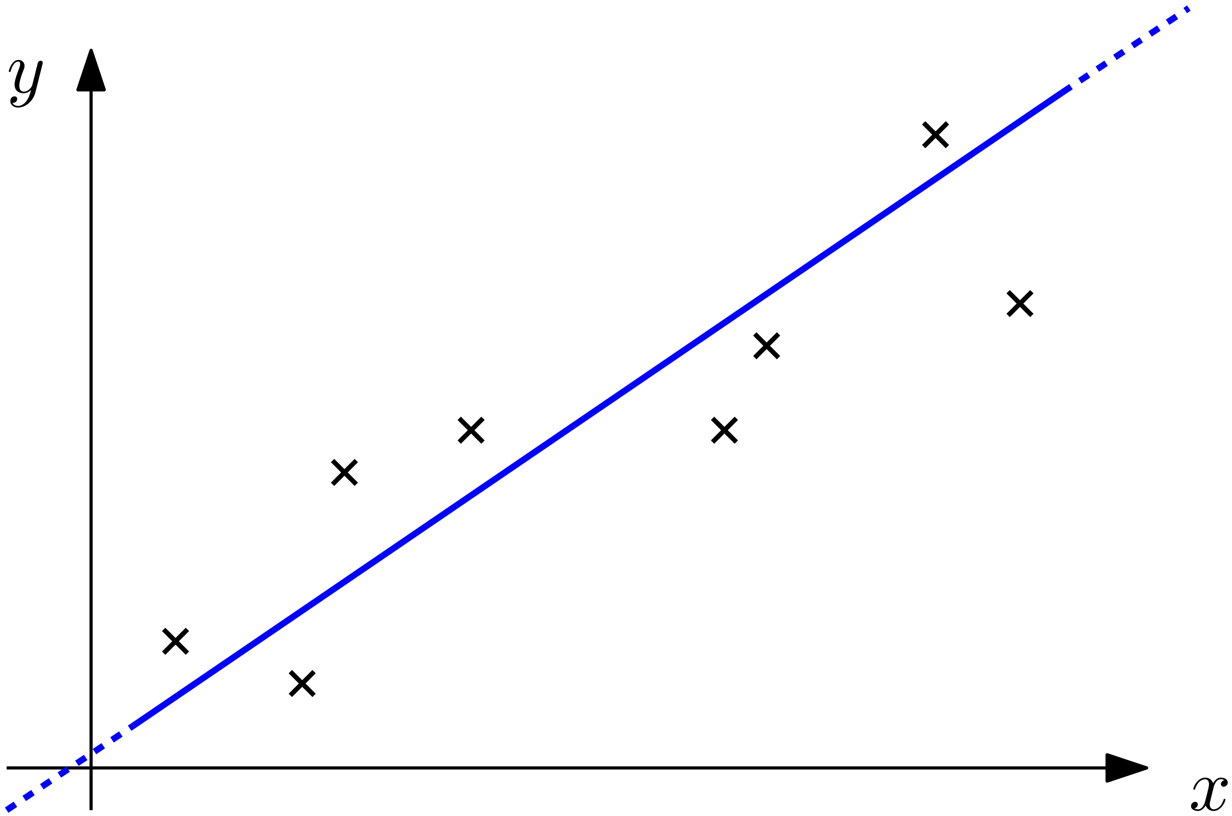

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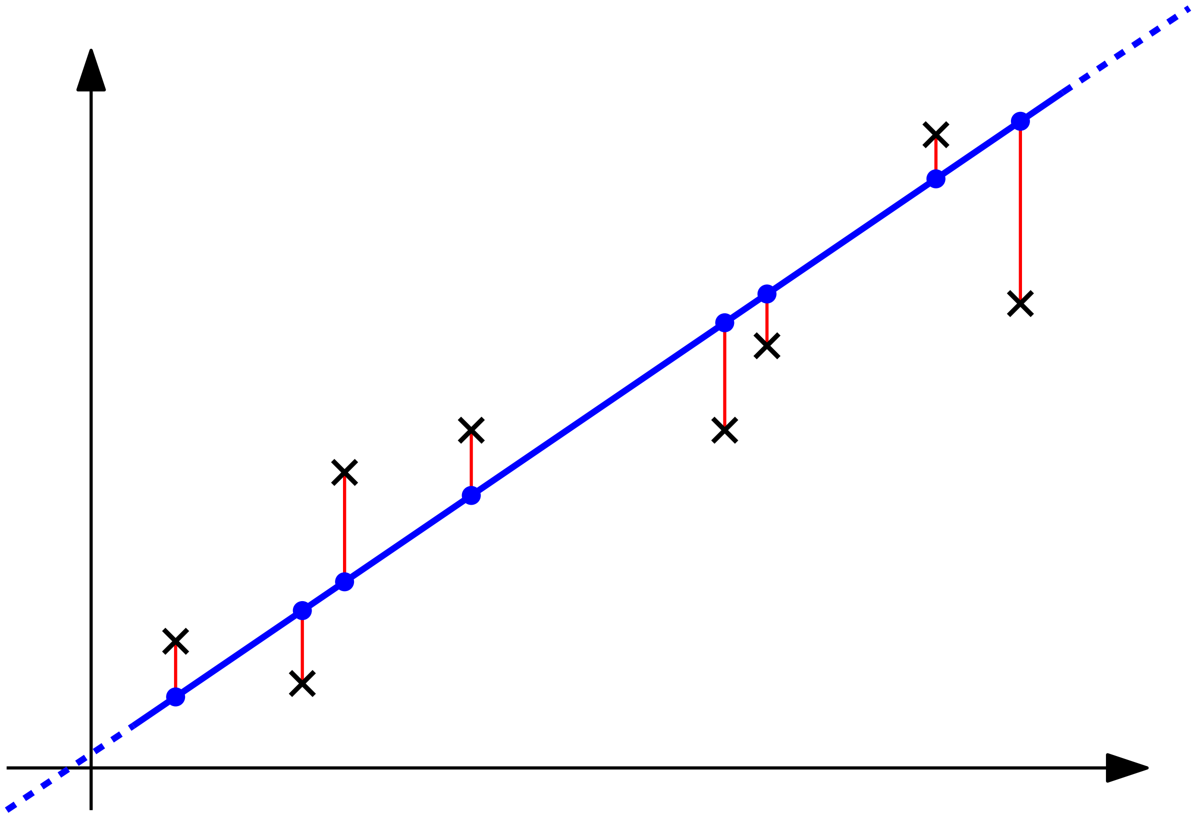
What does "best" mean? Minimizes some error measure

Least Squares Regression



Residual: difference between the y-coordinate of a point and the corresponding value of the fitted line

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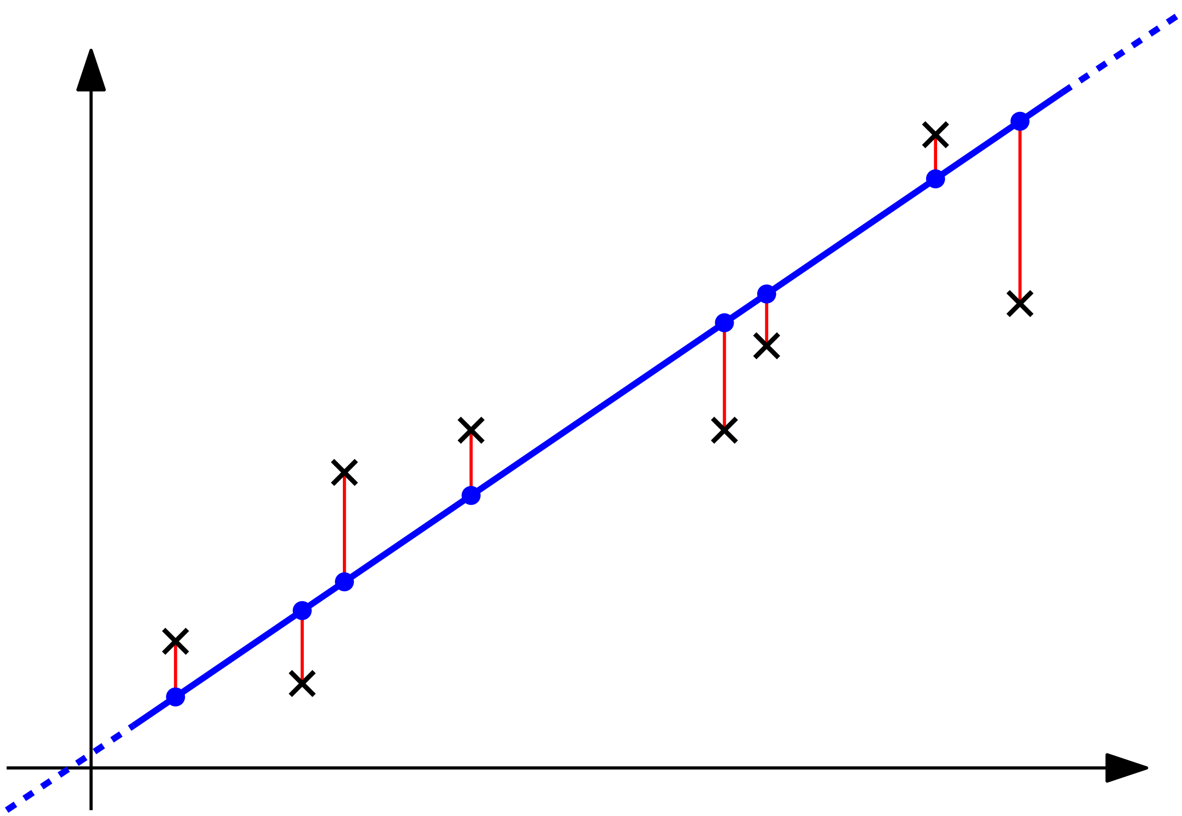
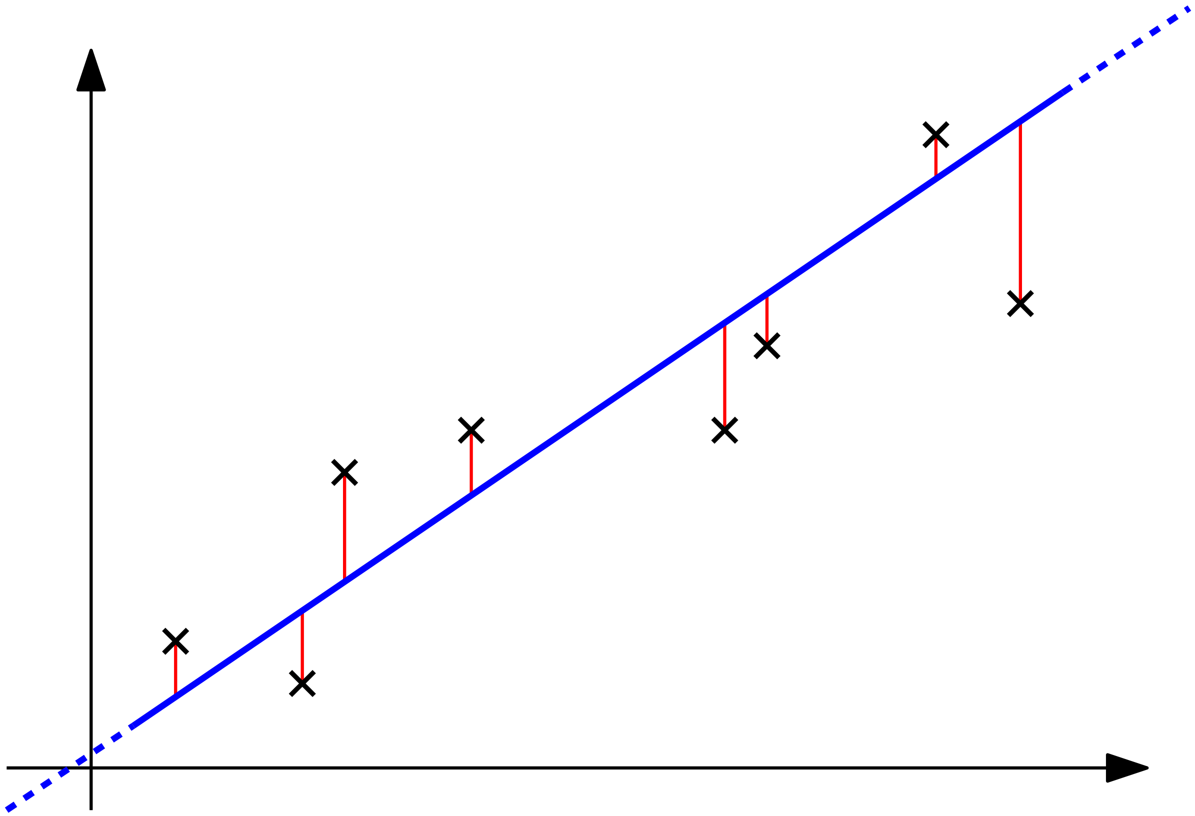


Residual: difference between the y-coordinate of a point and the corresponding value of the fitted line

Error: sum of the **squares** of the residuals

$$\mathrm{Err}(\ell, P) = \sum_{i=1}^{n} \Big(\ell(x_i) - y_i \Big)^2$$

Least Squares Regression



The parameters of the line $\ell(x) = a \cdot x + b$ of best fit are:

$$a = \frac{n \sum_i x_i y_i - \left(\sum_i x_i\right)\left(\sum_i y_i\right)}{n \sum_i x_i^2 - \left(\sum_i x_i\right)^2} \qquad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

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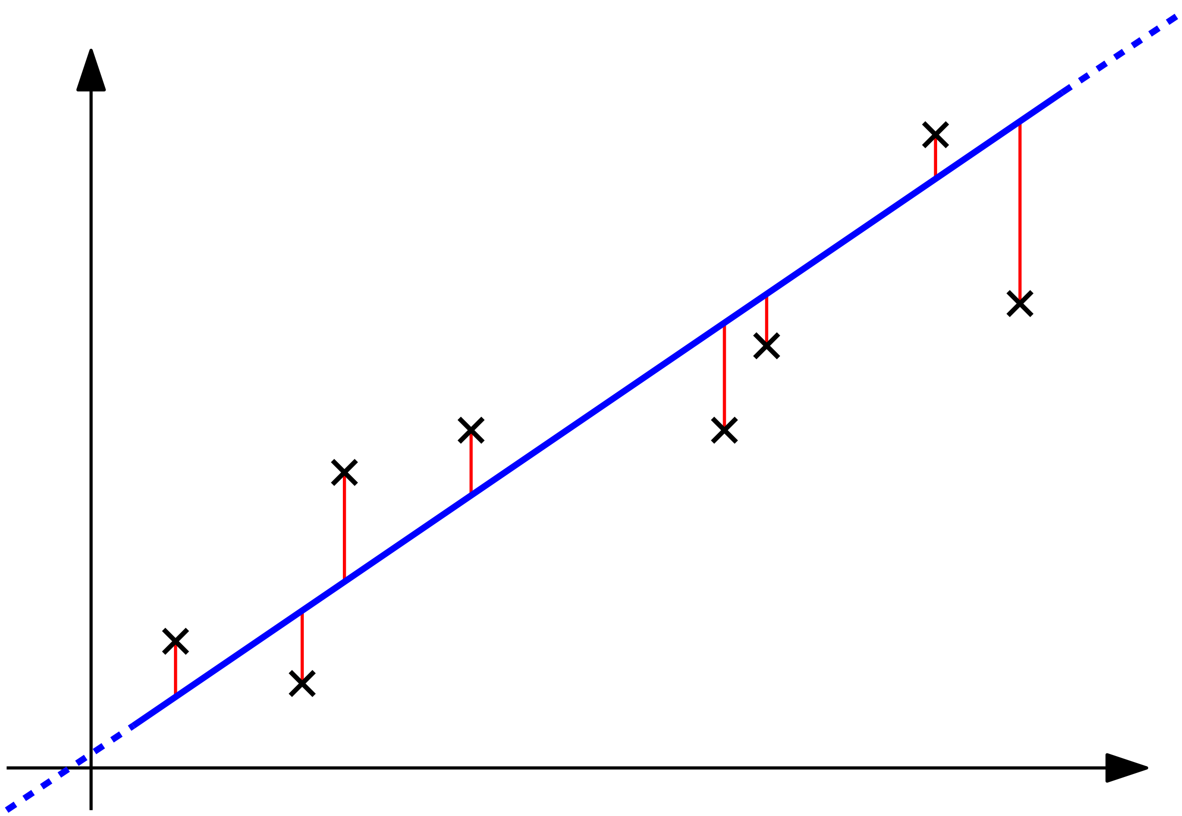

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Can be found in time $O(n)$

Segmented Least Squares Regression

What if the points look like this?

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What if the points look like this?

- No single line provides a good fit

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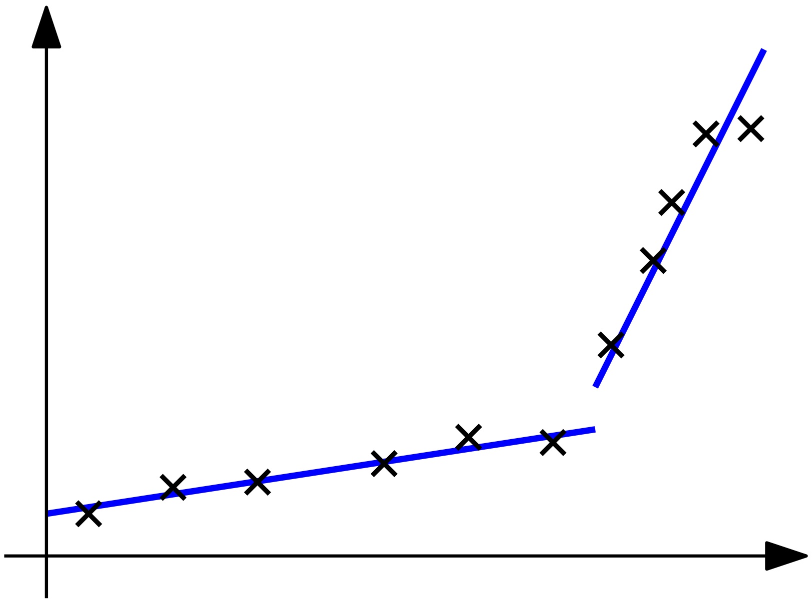

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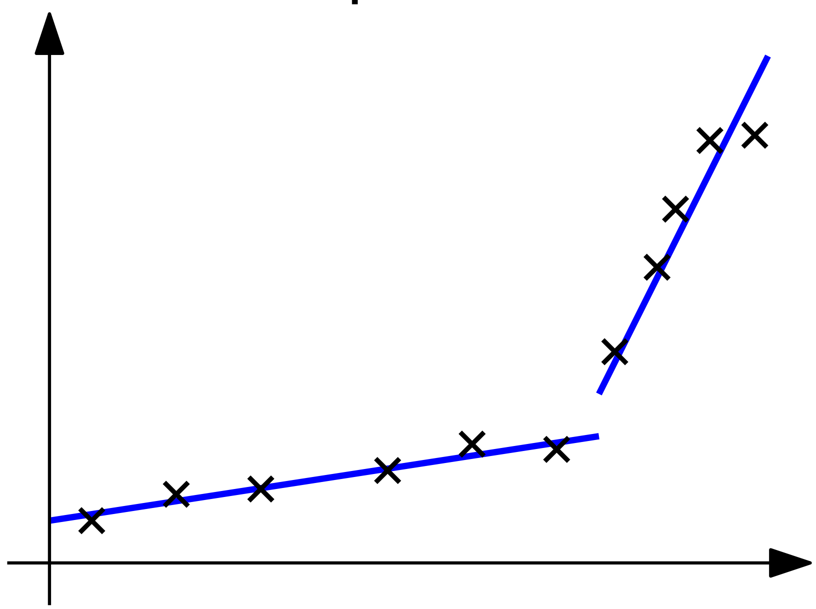

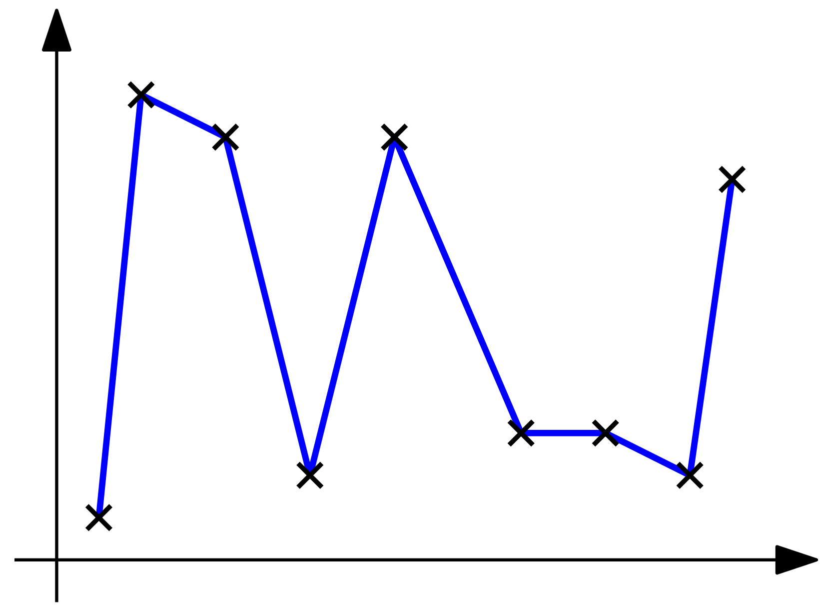

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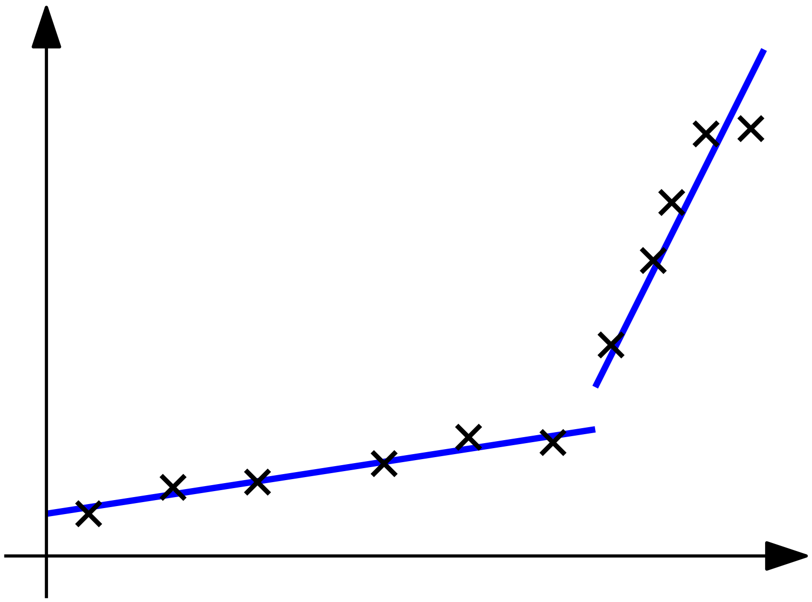

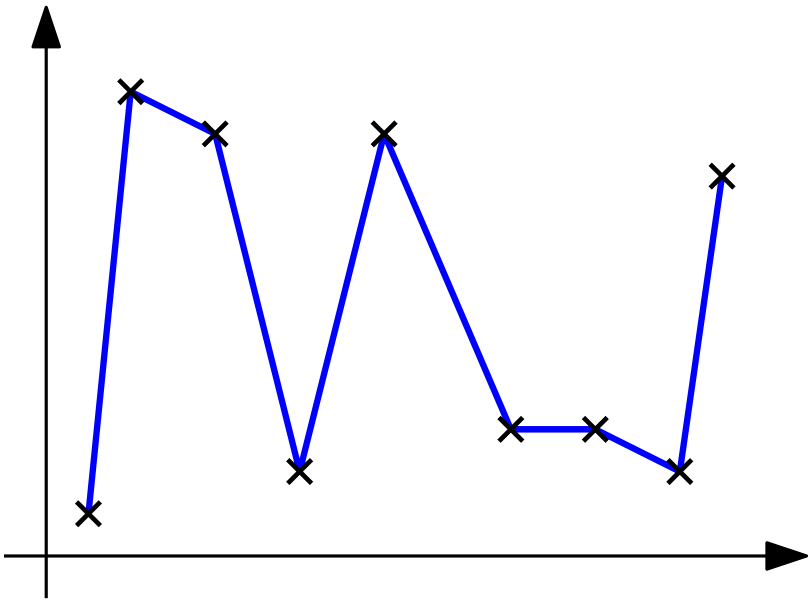

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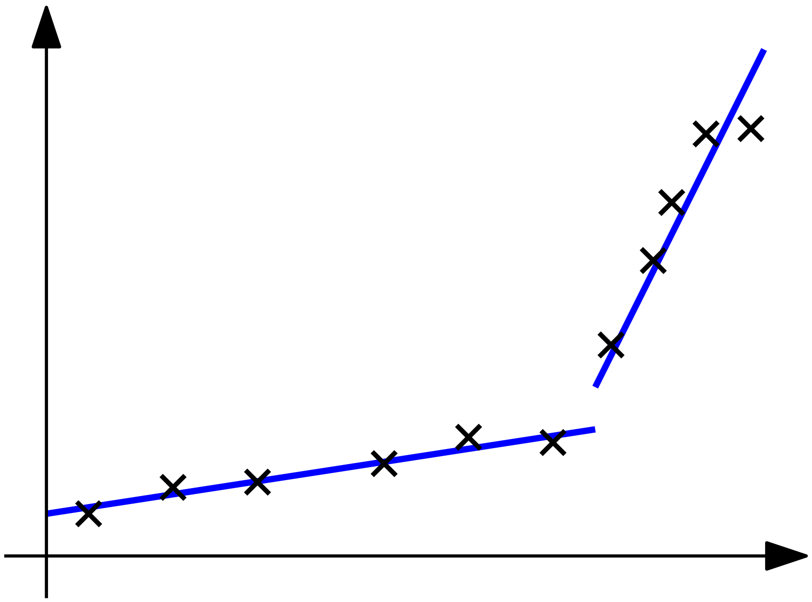

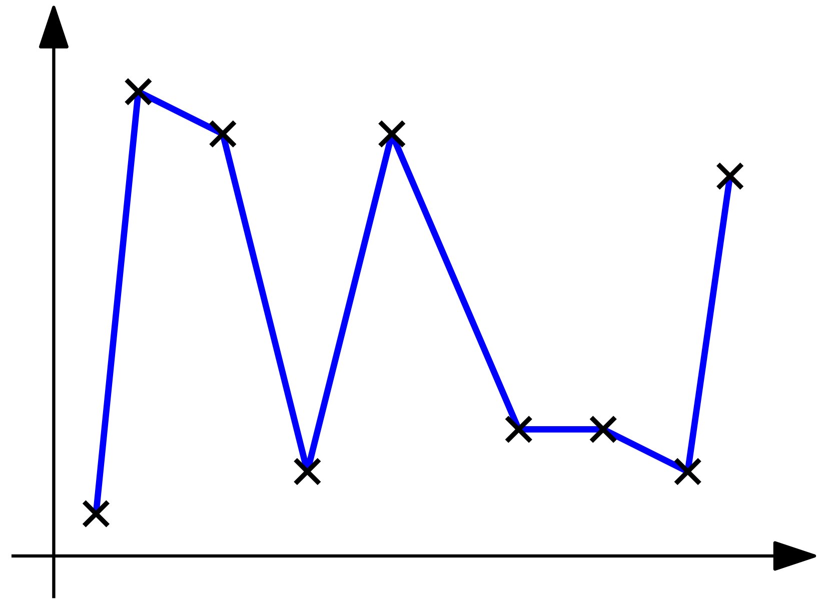

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Problem: if we use piecewise linear funct
trivially fit all points

Balances the quality of the fit with the # of used segments

Idea: each used segment incurs some cost C

The Segmented Least Squares Problem

Input:

- A collection $P = \{p_1 = (x_1, x_2), \ldots, p_n = (x_n, y_n)\}$ of points with $x_1 < x_2 < \ldots < x_n$
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Output:

- A partition of P into some number k of *segments* $S_1, \ldots, S_k$
- where a segment is a subset of P containing a contiguous interval of points, i.e., $\{p_i, p_{i+1}, \ldots, p_j\}$ for some $i \leq j$,
- that minimizes the *total penalty* $C \cdot k + \sum_{i=1}^{k} \mathrm{Err}(\ell_i, S_i)$, where ℓ_i is the best fit line for S_i

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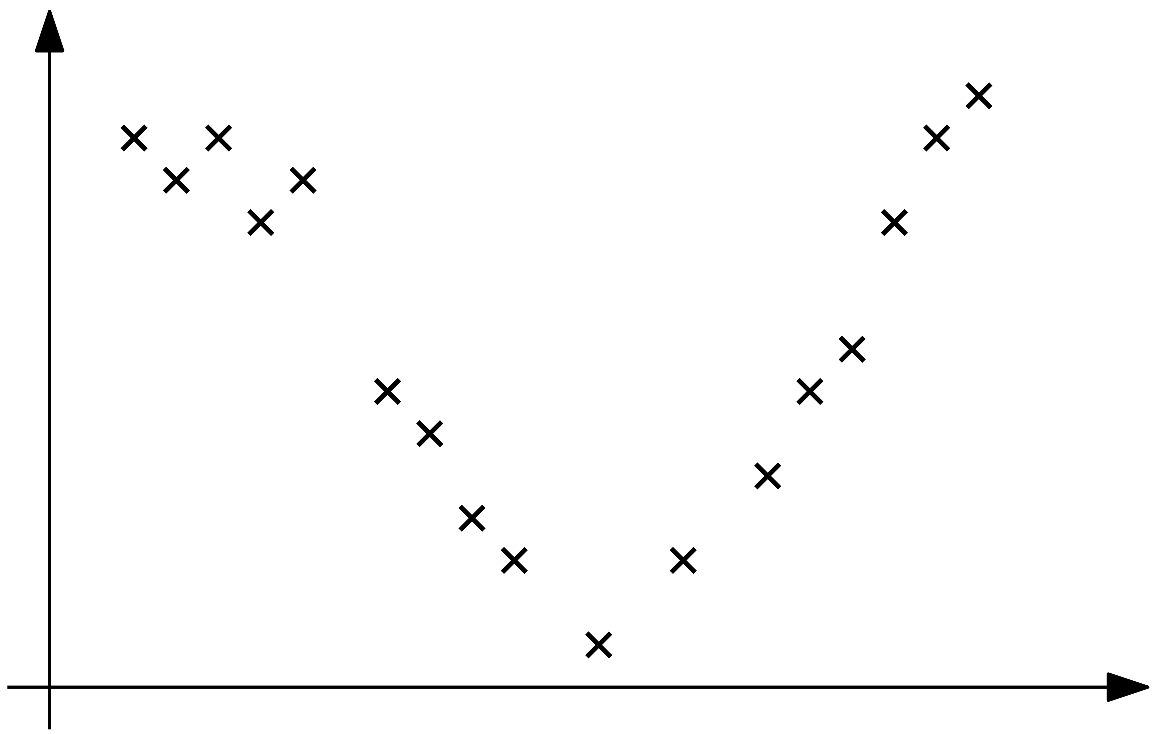
Err(S_i)

Example (qualitative)



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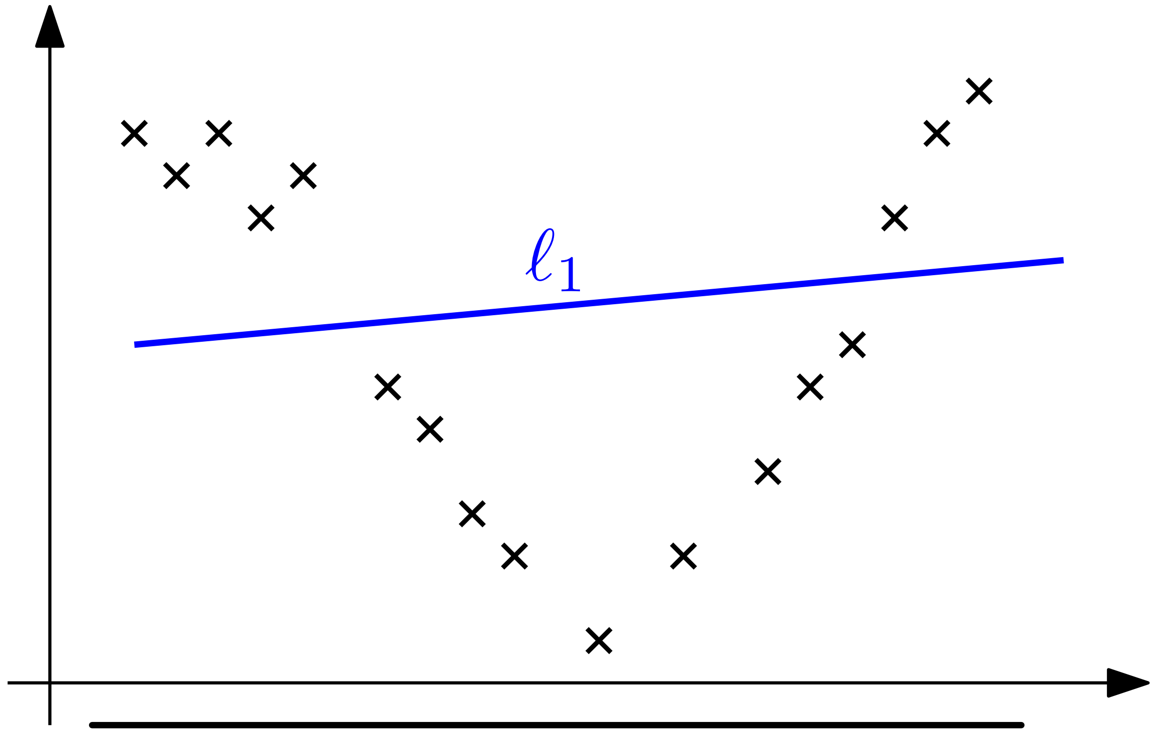

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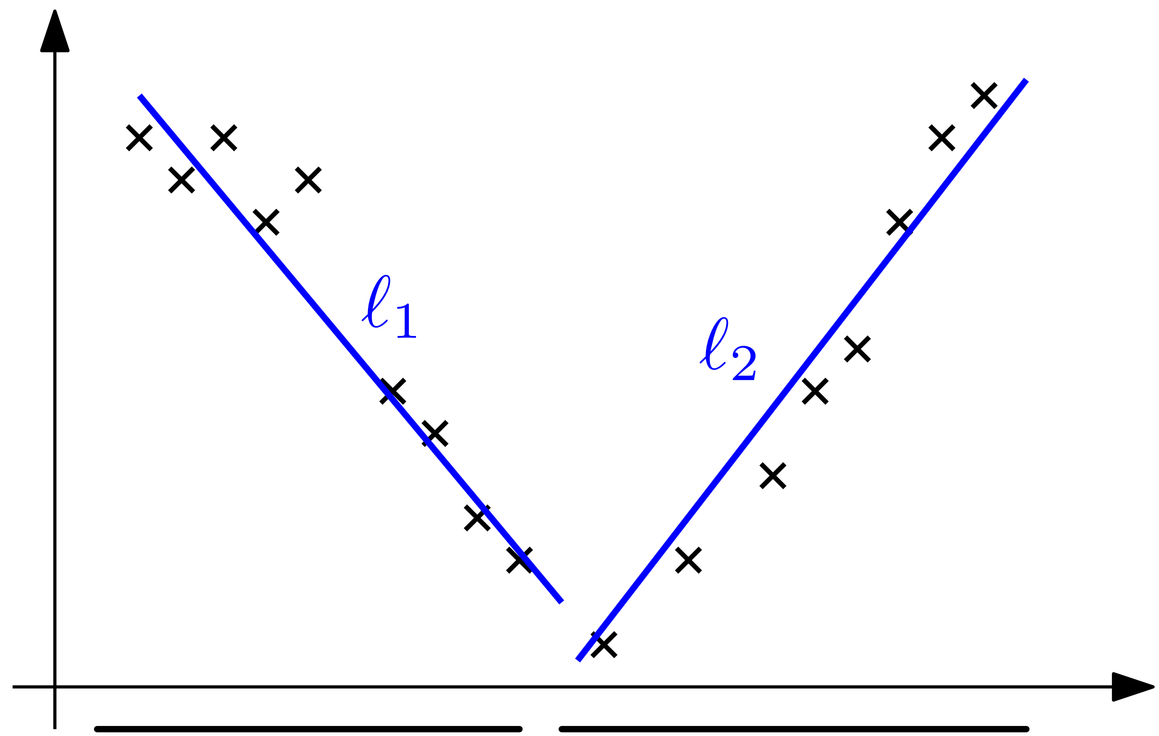

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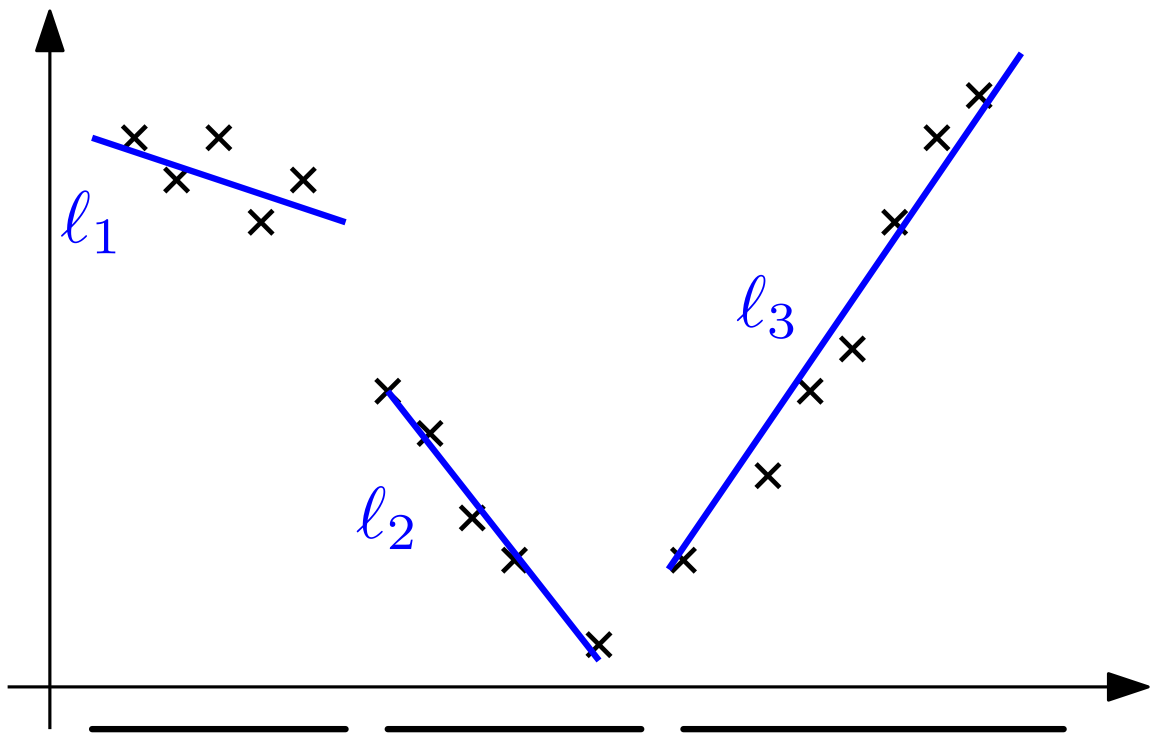

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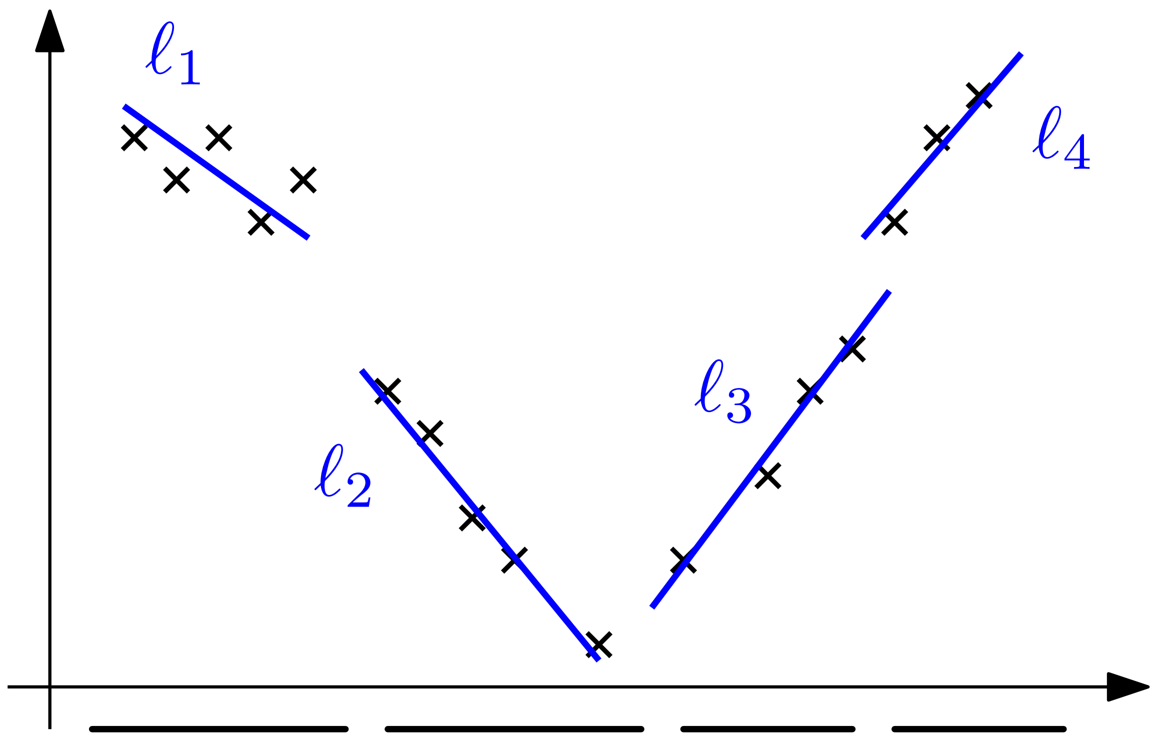

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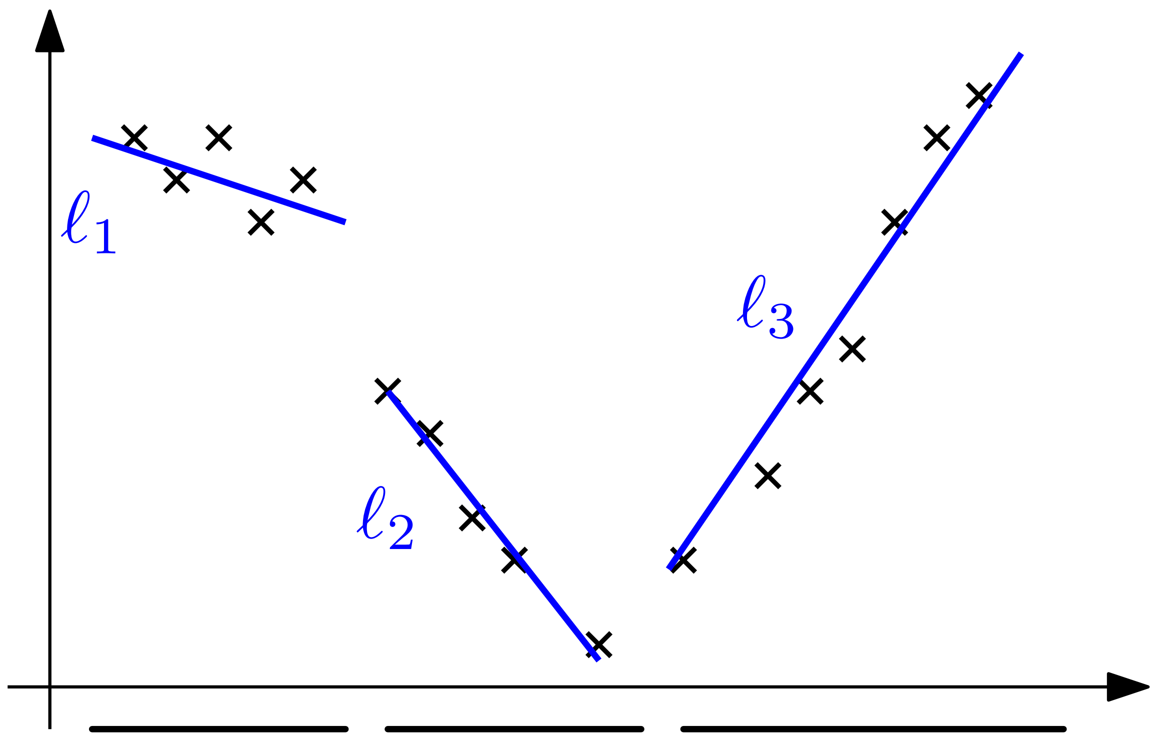

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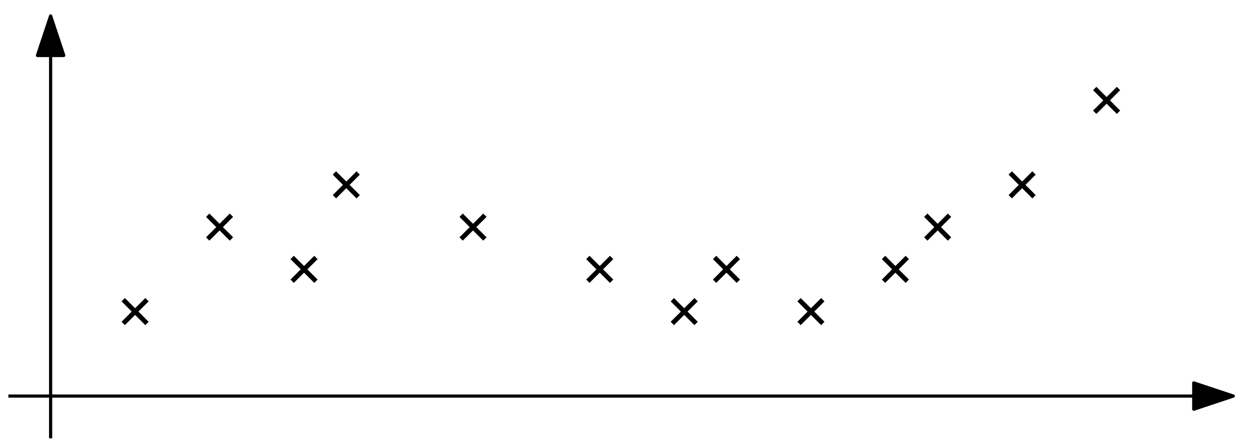

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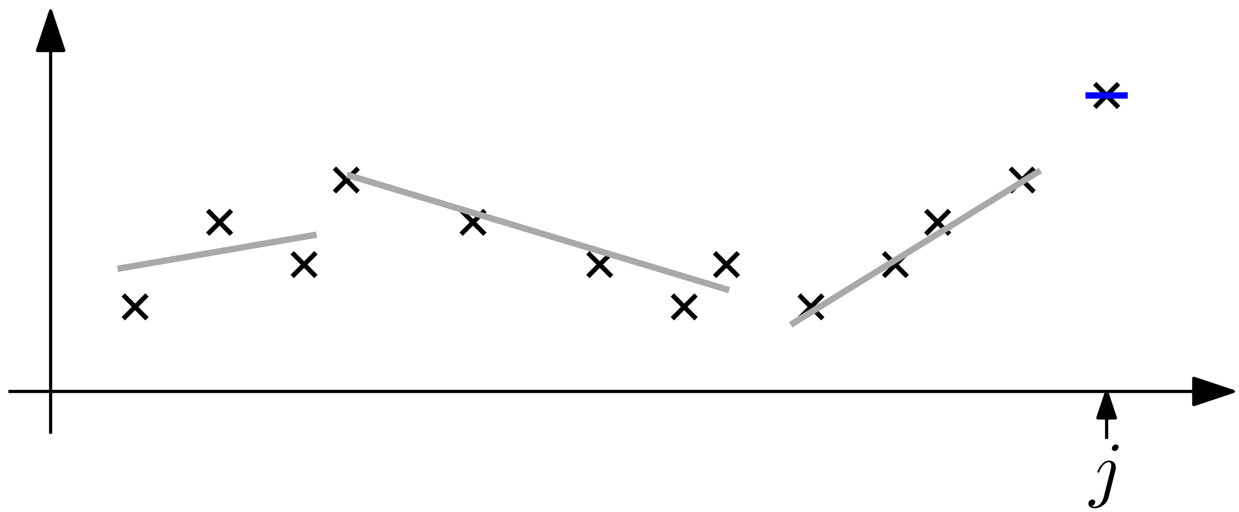

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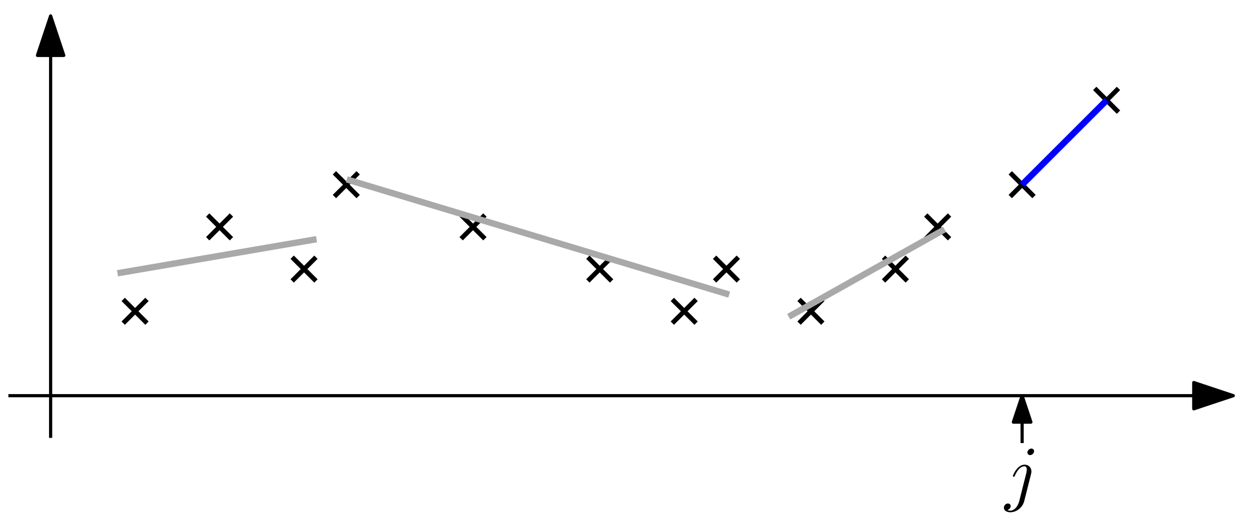

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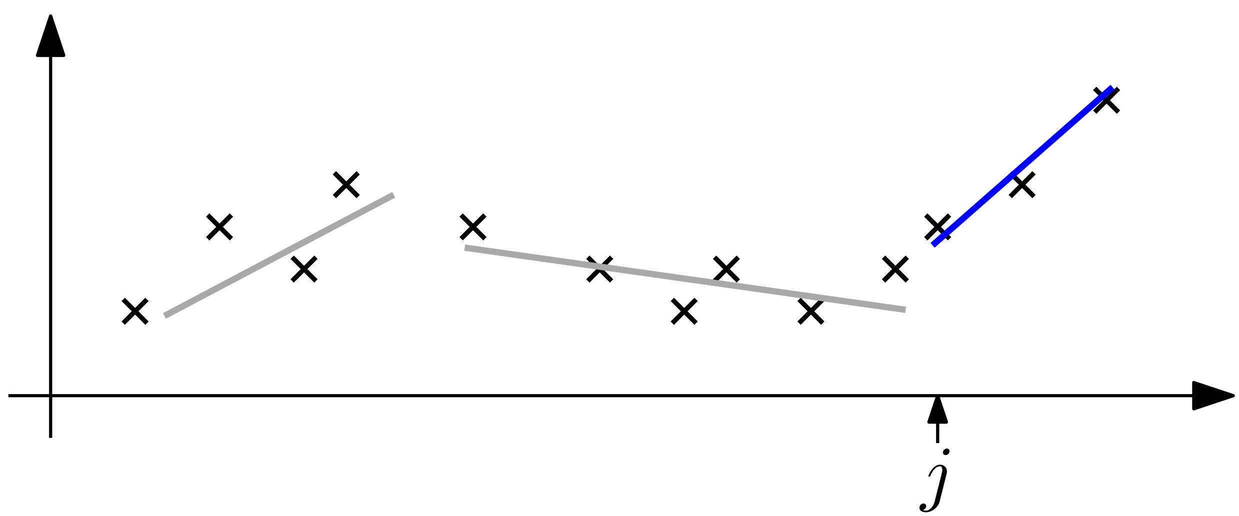

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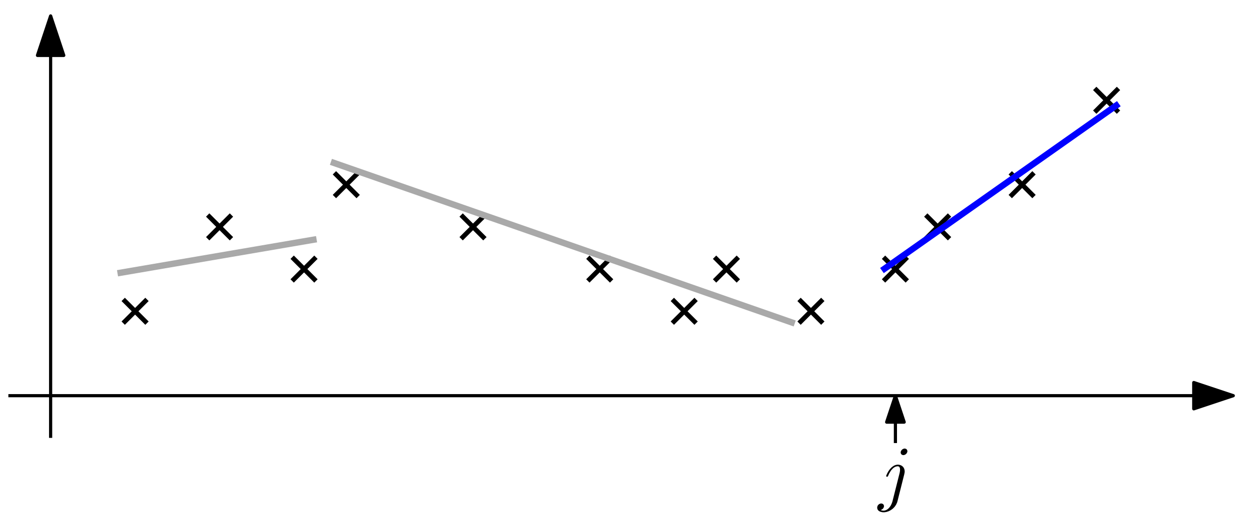

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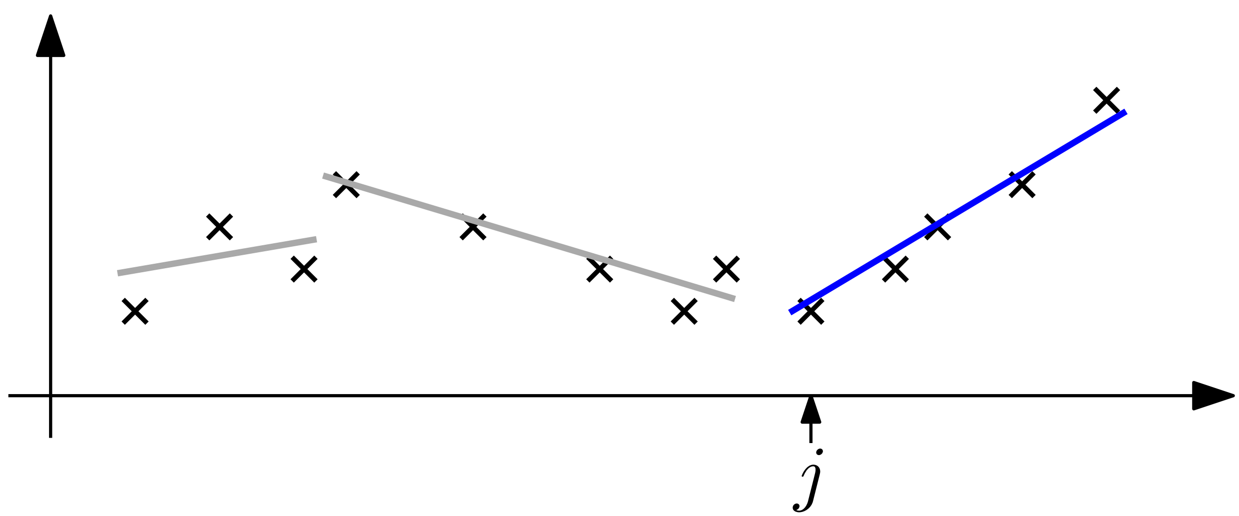

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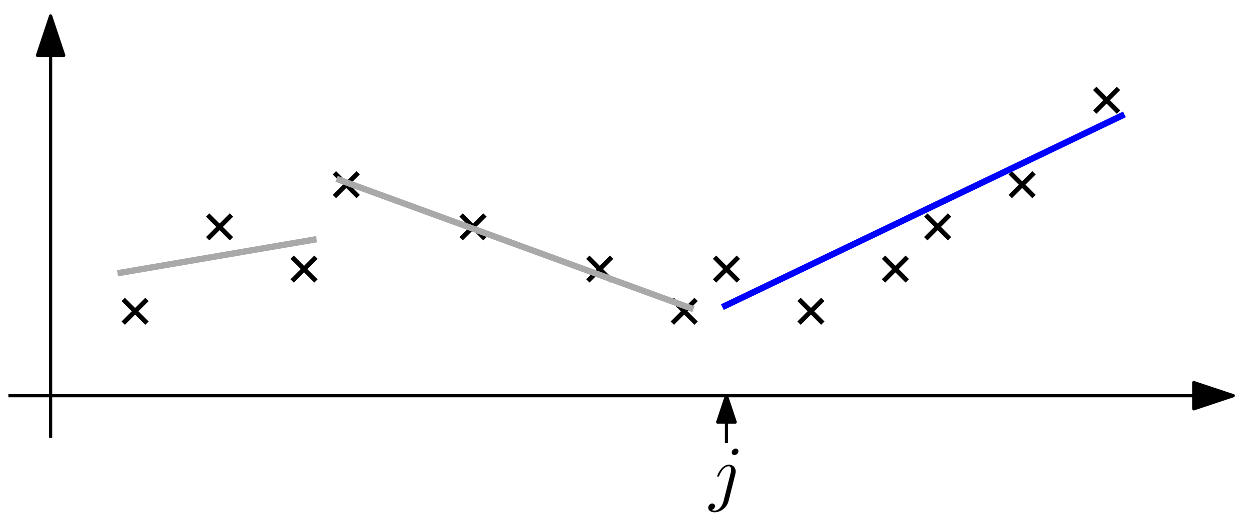

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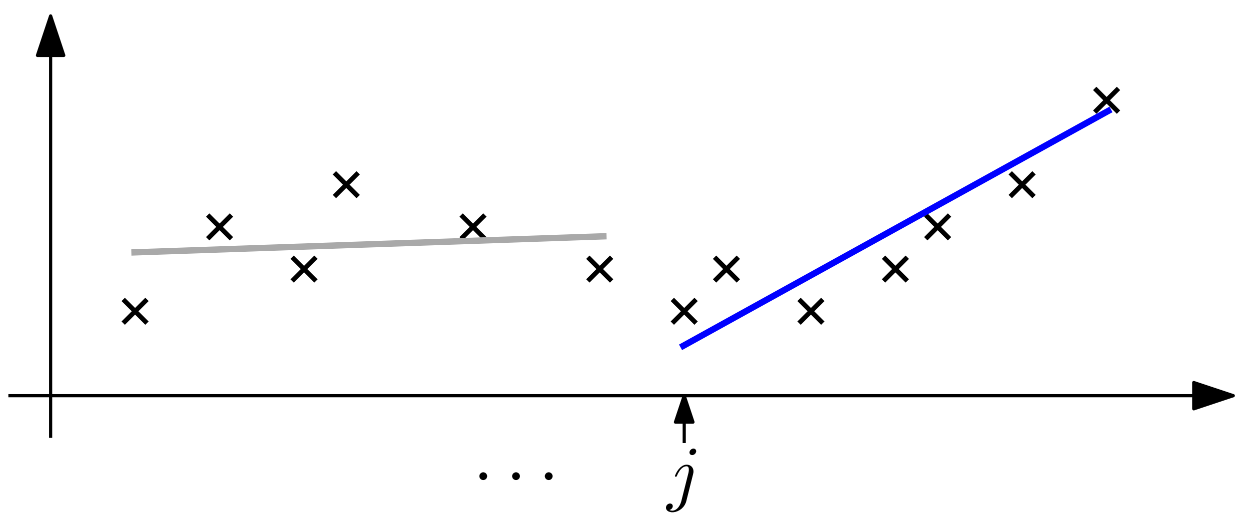

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A Dynamic Programming Algorithm

Subproblem definition:

OPT(i) = penalty incurred by an optimal solution for the instance that only considers the points in $\{p_1, p_2, \ldots, p_i\}$

Base case: OPT$(0) = 0$

Recursive formula: (for $i \geq 1$)

$$\text{OPT}(i) = \min_{j=1,\ldots,i} \left\{ C + \text{Err}(\{p_j, \ldots, p_n\}) + \text{OPT}(j-1) \right\}.$$

Order of subproblems: OPT(1), OPT(2), . . . , OPT(n)

Solution: OPT(n)

Time Complexity

How many subproblems?

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  — Need to find the best fit line ℓ for $\{p_j, \ldots, p_i\}$ $O(n)$
  — Need to find the error $\mathsf{Err}(\ell, \{p_j, \ldots, p_i\})$ $O(n)$

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Overall time: $O(n^3)$

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  — Need to find the error Err$(\ell, \{p_j, \ldots, p_i\})$ $O(n)$

Overall time: $O(n^3)$

Can we do better?

Improving the Time Complexity

Goal: find the best fit line $\ell(x) = a \cdot x + b$, and the error $\text{Err}(\ell, \{p_j, \ldots, p_i\})$ **quickly**

$$a = \frac{n \sum_{h=j}^{i} x_h y_h - \left(\sum_{h=j}^{i} x_h\right)\left(\sum_{h=j}^{i} y_h\right)}{n \sum_{h=j}^{i} x_h^2 - \left(\sum_{h=j}^{i} x_h\right)^2}$$

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$$\text{Err}(\ell, \{p_j, \ldots, p_i\}) = \sum_{h=j}^{i} \left(\ell(x_h) - y_h\right)^2$$

Finding ℓ quickly

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All the marked quantities can be found in **constant time** after a linear-time preprocessing using **prefix sums**.

Finding $\mathsf{Err}(\cdot, \cdot)$ quickly

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Time Complexity

How many subproblems? $O(n)$

How much time per subproblem? $O(n^2)$

- We need to test $O(n)$ choices of j.
- How much time per choice of j?

— Need to find the best fit line ℓ for $\{p_j, \ldots, p_i\}$ $O(n)$

— Need to find the error $\mathrm{Err}(\ell, \{p_j, \ldots, p_i\})$ $O(n)$

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Overall time: $O(n^2)$