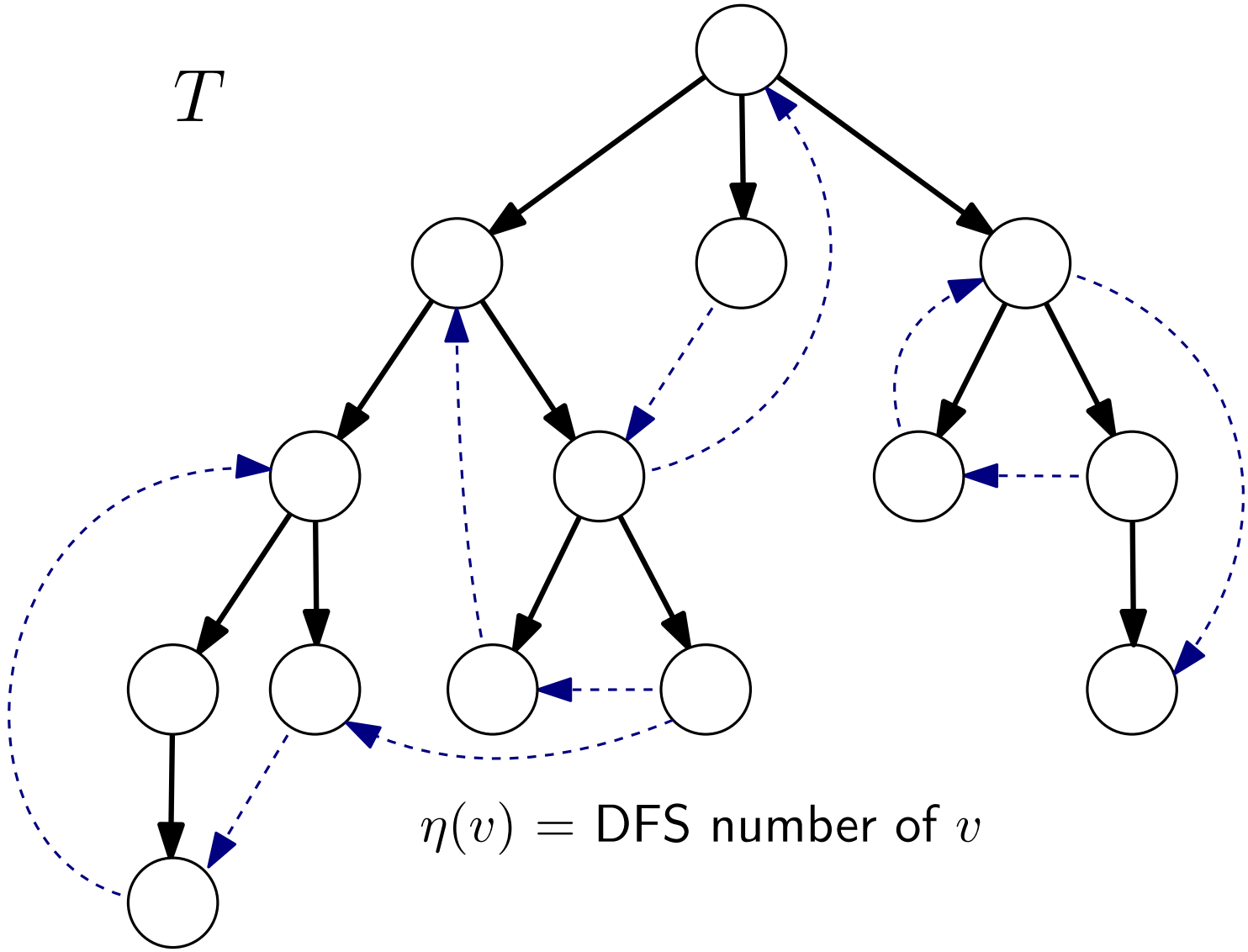
Strongly Connected Components

Tarjan's algorithm



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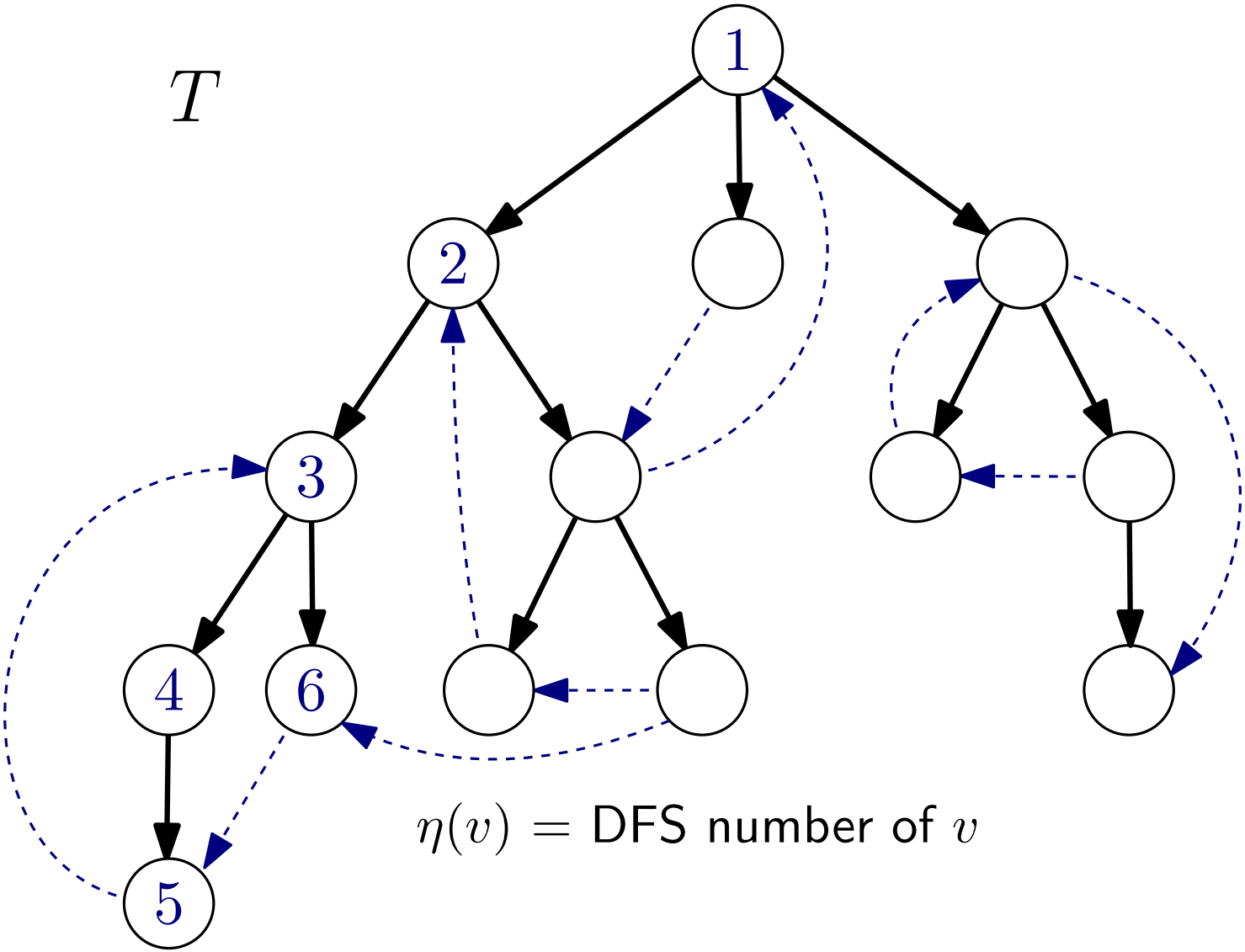

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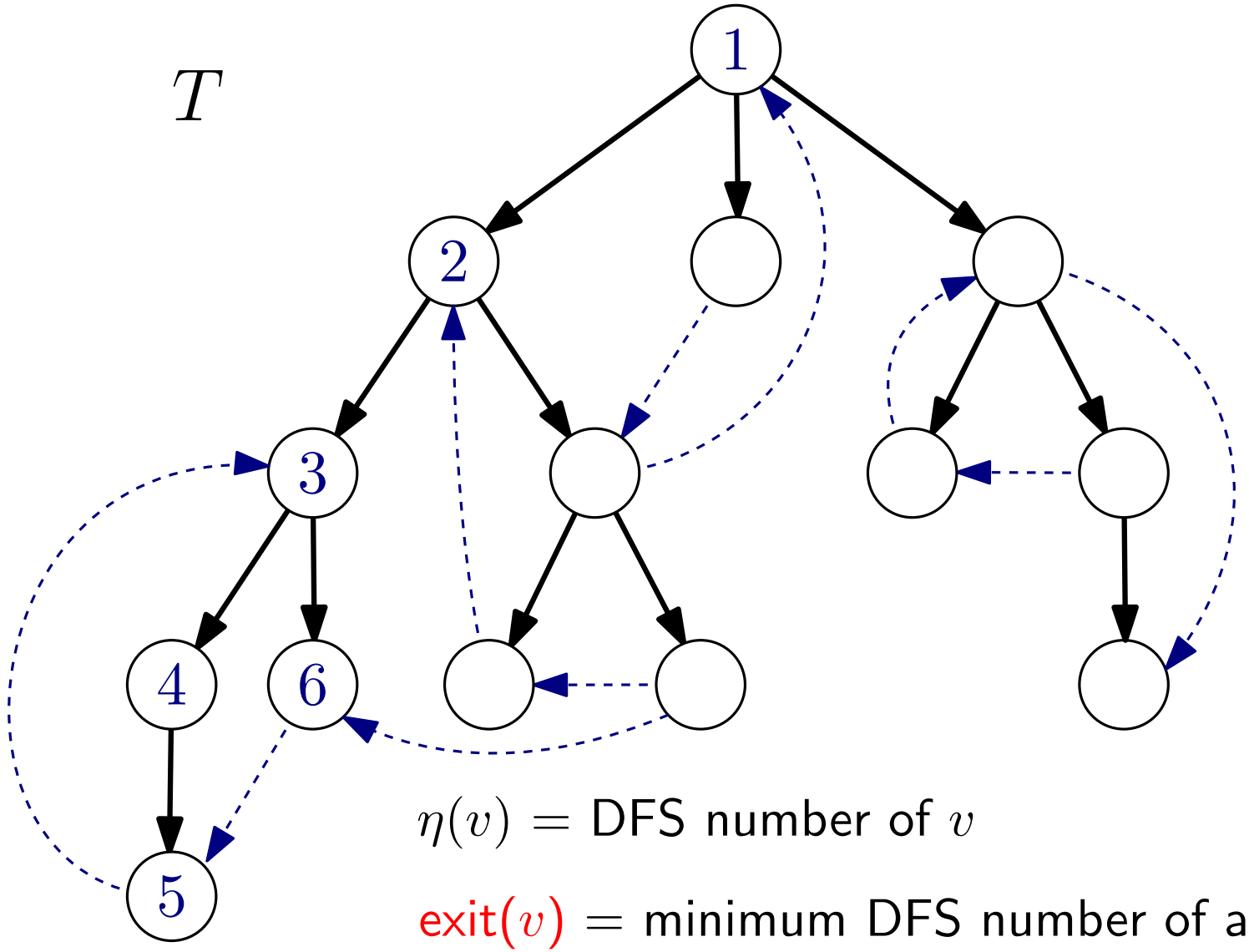

$\eta(v)$ = DFS number of v

exit(v) = minimum DFS number of a vertex u in a yet undiscovered SCC such that u is reachable from v via a path in T followed by at most one final non-tree edge.

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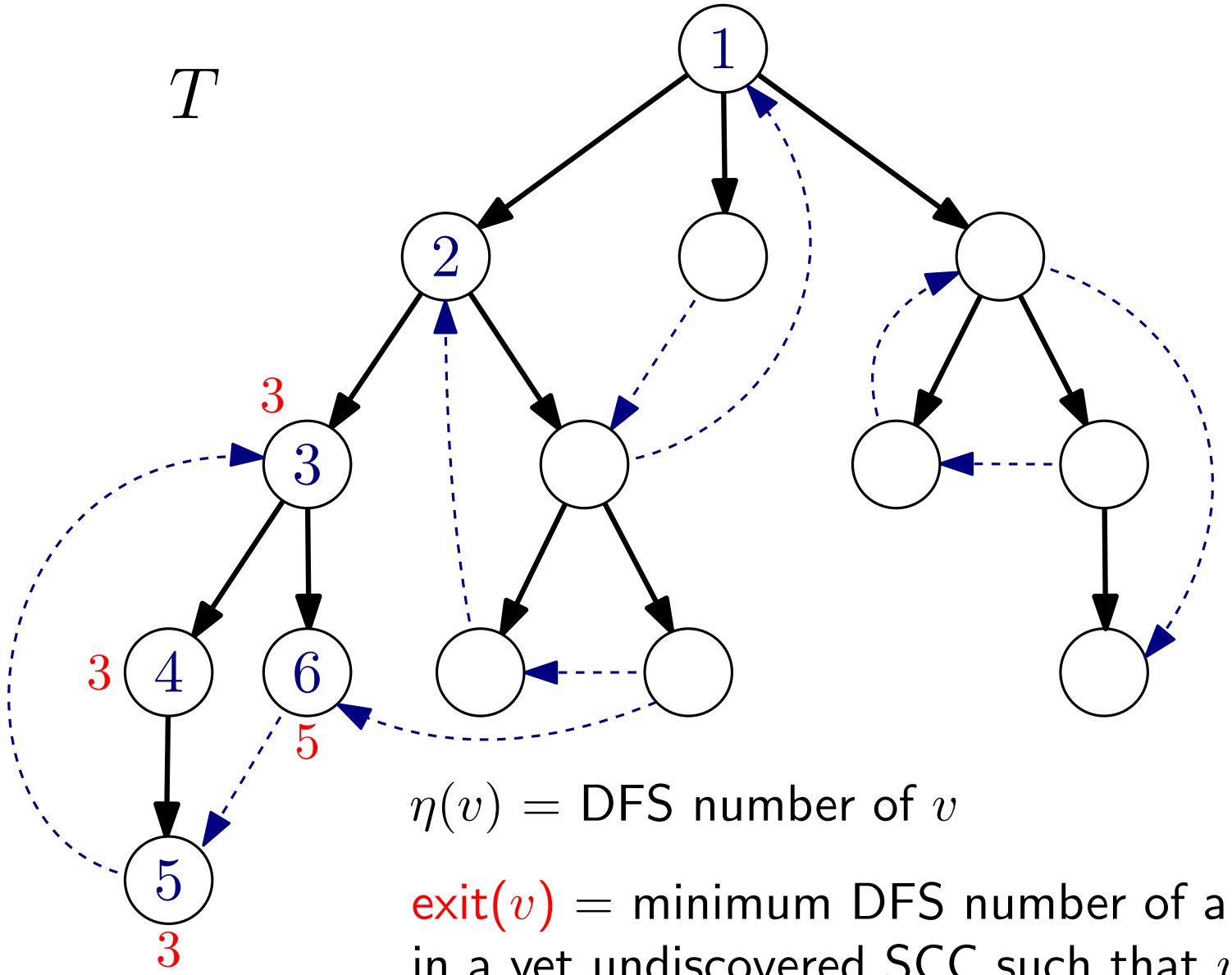

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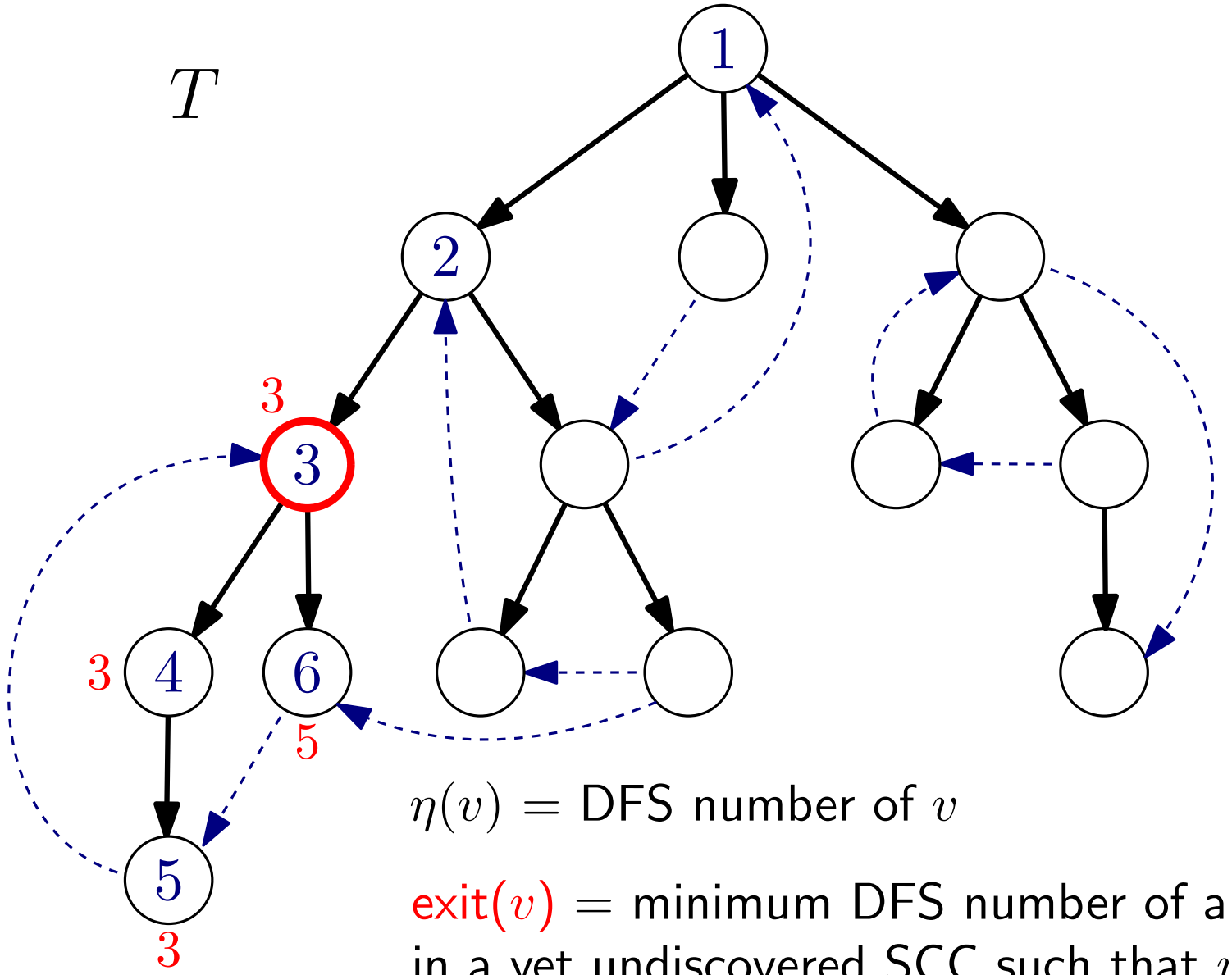

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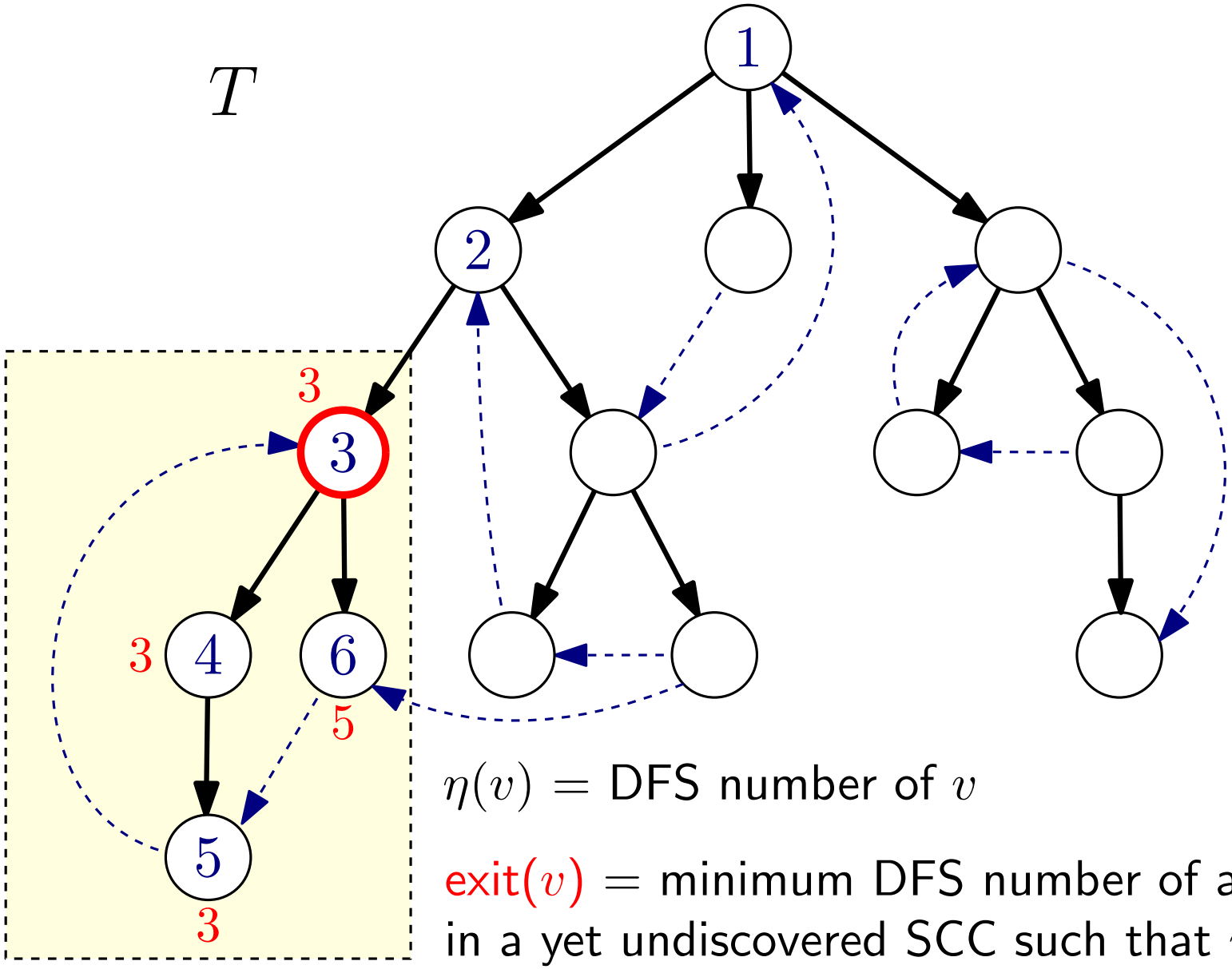

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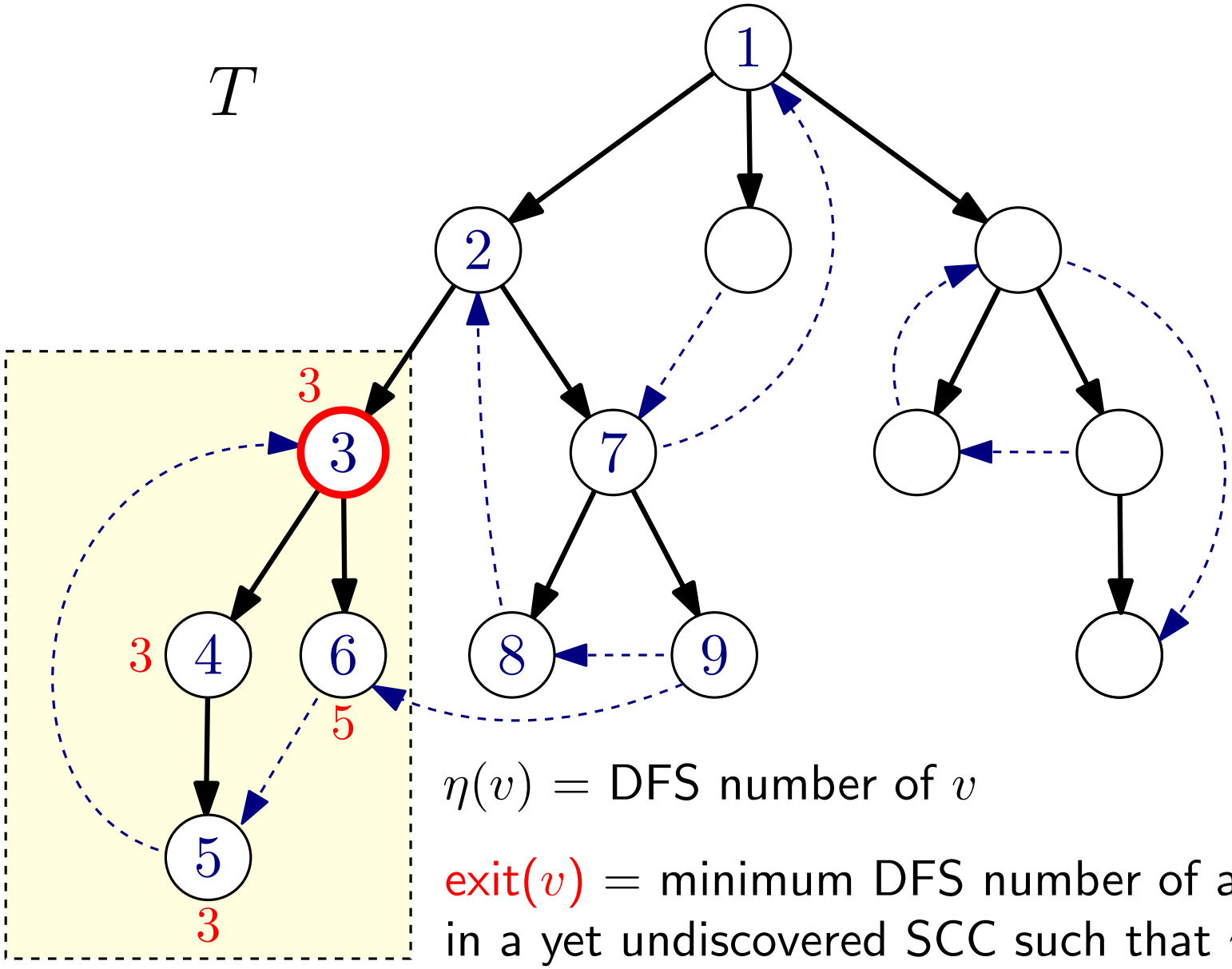

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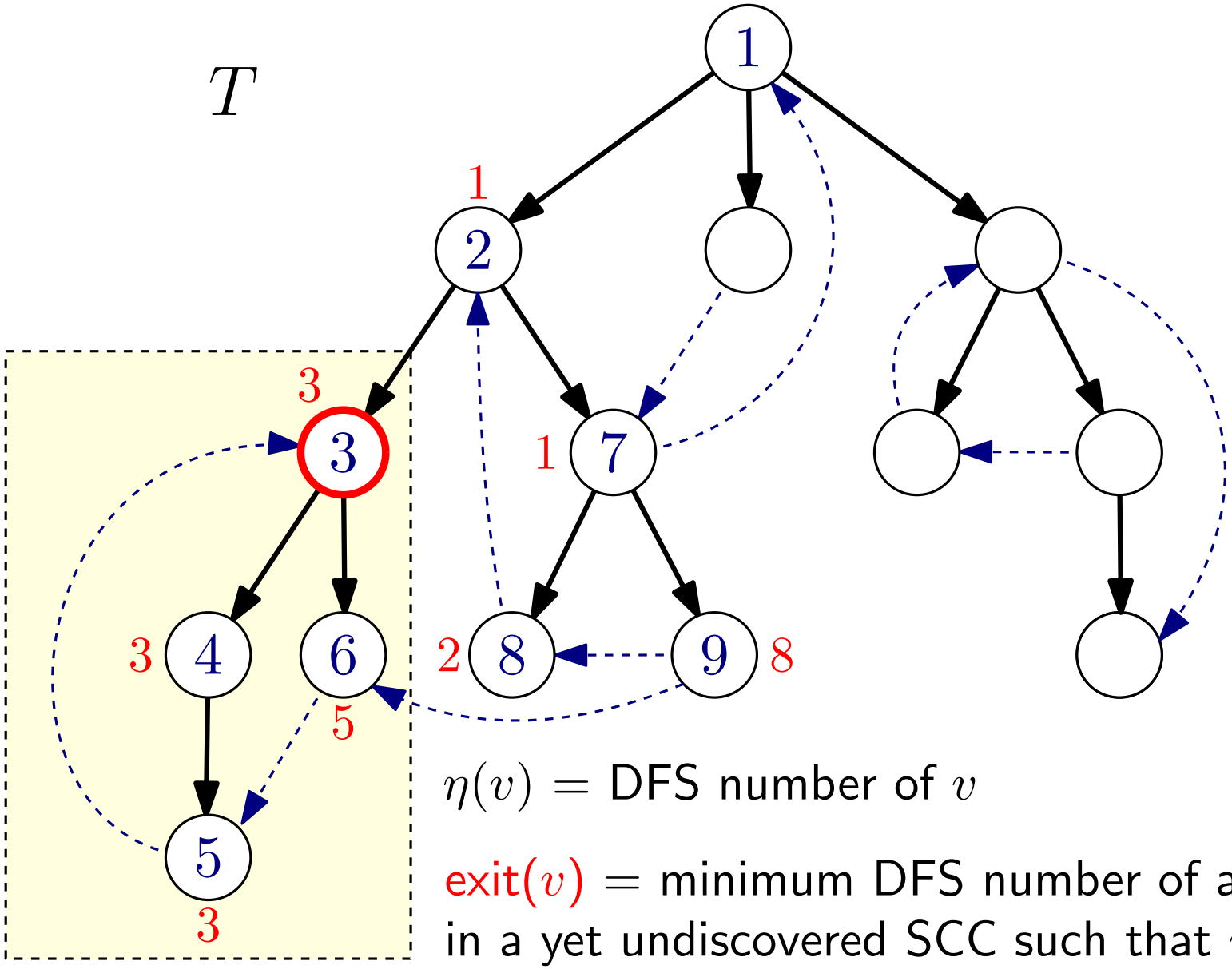

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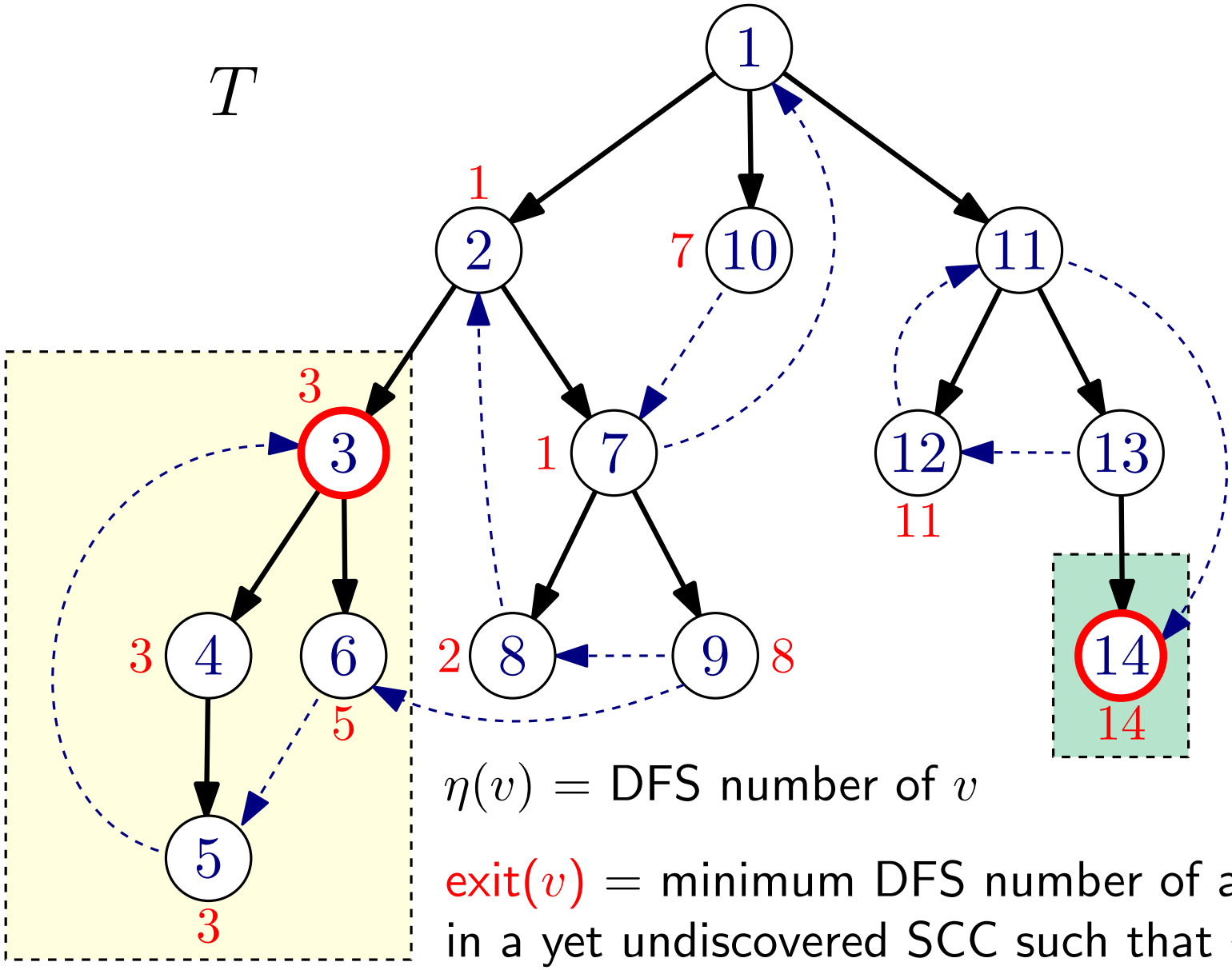

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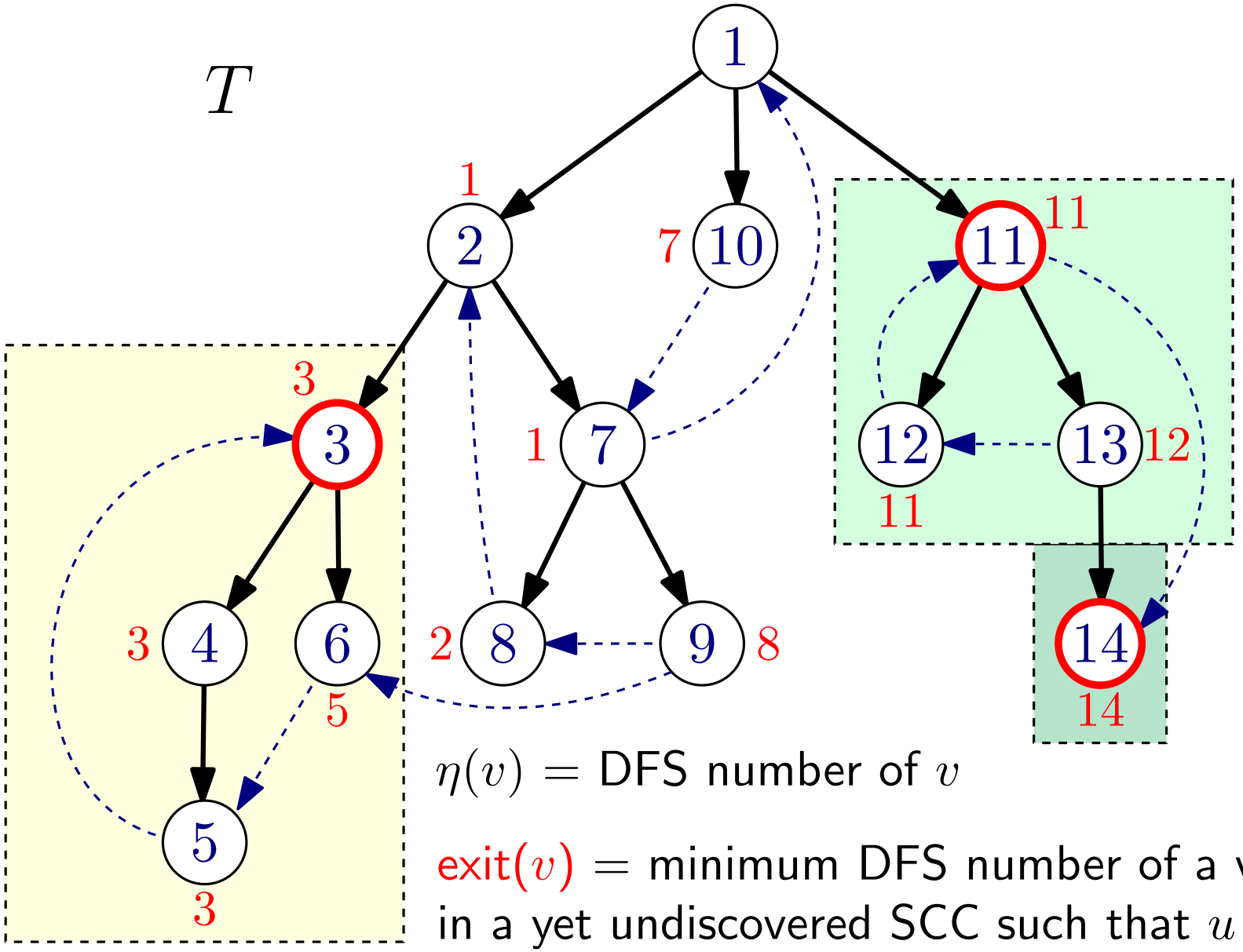

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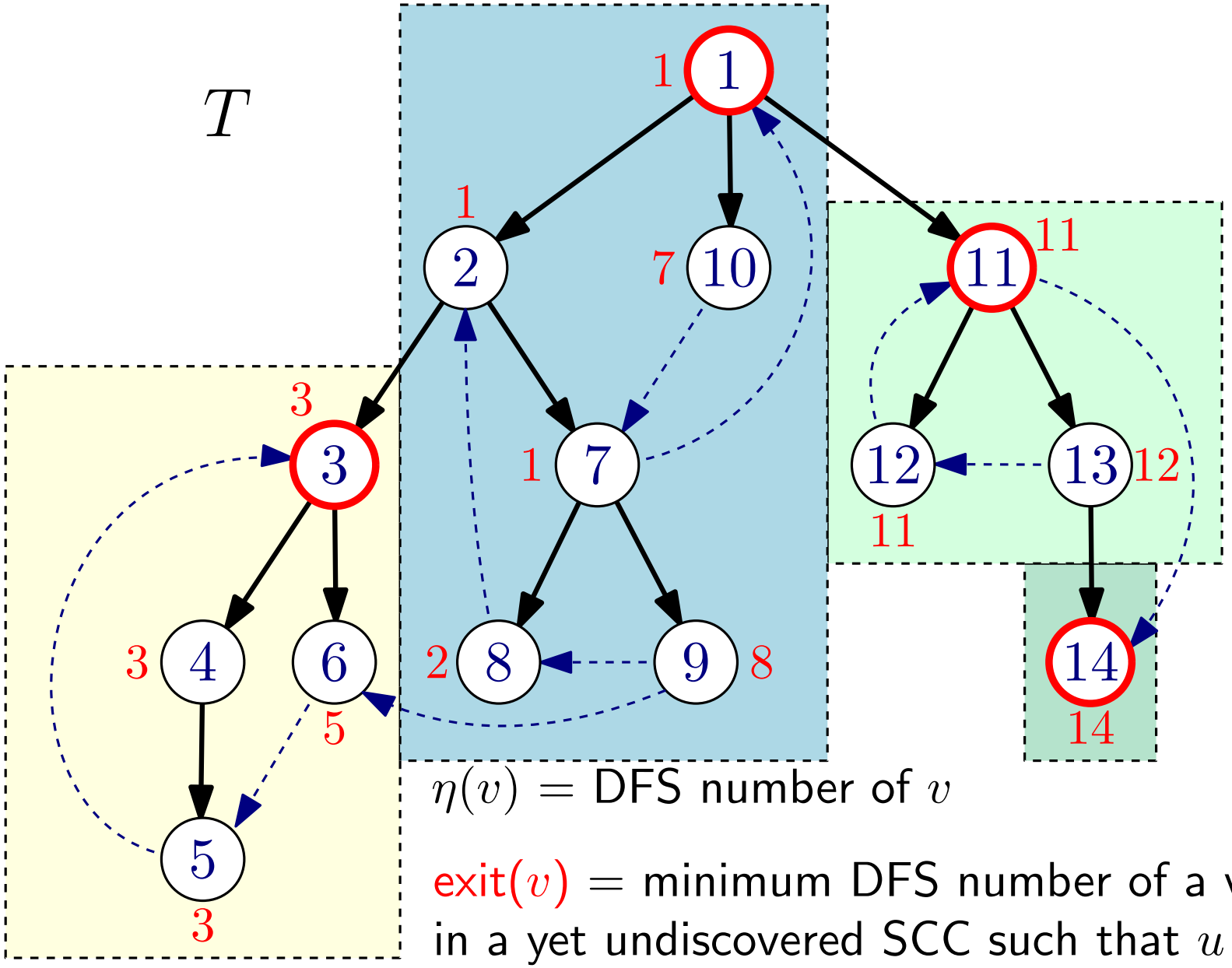

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Proof of correctness

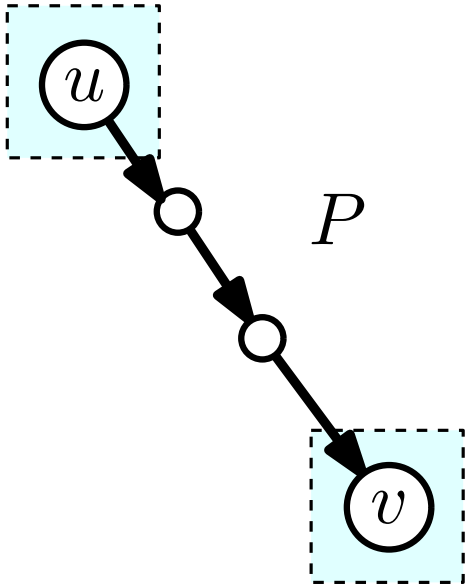
Claim: Let C be a SCC. The subgraph $T[C]$ of T induced by C is connected.

Proof:

Let u be the first vertex of C that is visited by the algorithm.

Let $v \in C$, with $v \neq u$.

- u must be an ancestor of v in T (by the properties of DFS).



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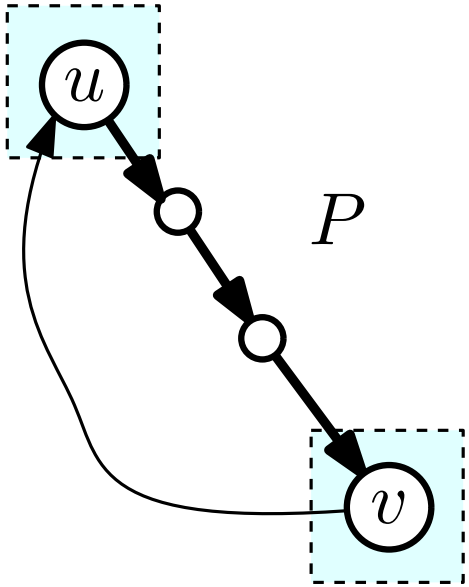

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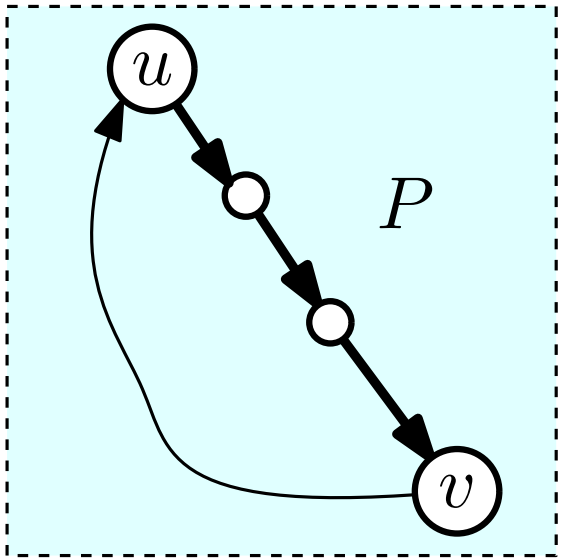

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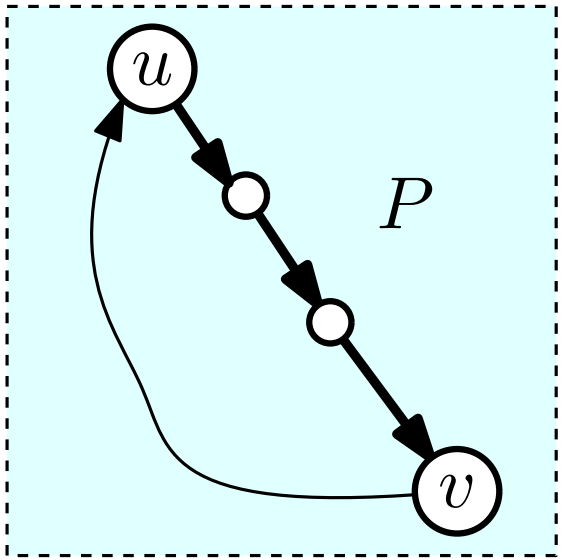

- There is a path from u to v in $G \implies$ the vertices in P are in $C \implies u$ and v must also be connected in $T[C]$.

□

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Definition: the *head* u of a SCC C is the (unique!) vertex of C having minimum depth in T.

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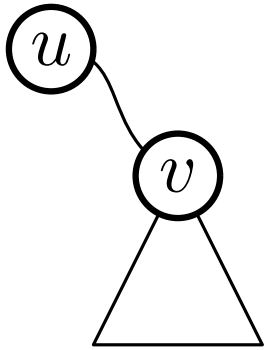

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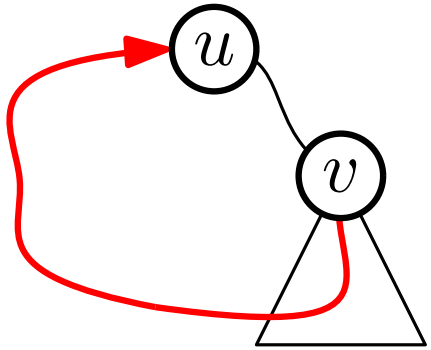

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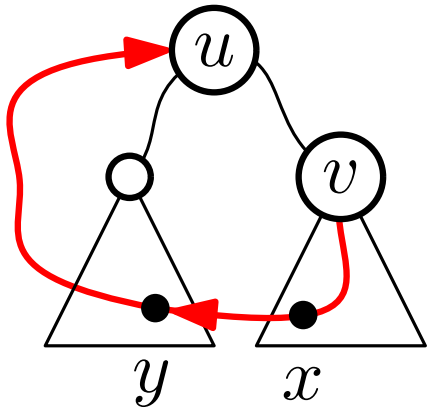
- Consider the first edge (x, y) of P such that $y \notin T_v$.



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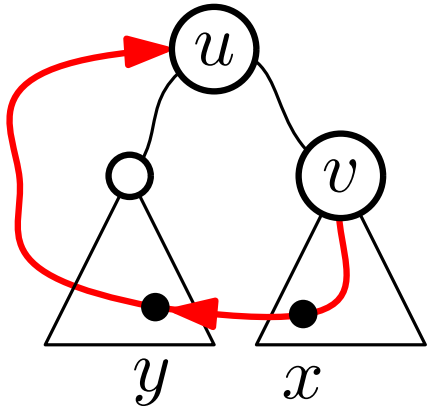

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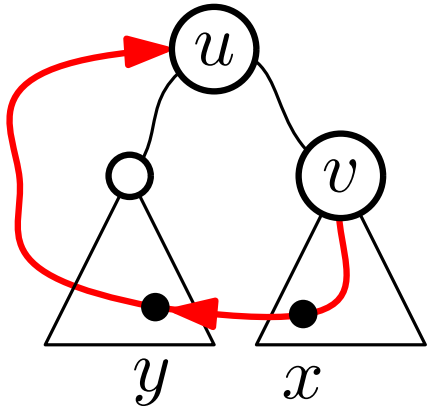

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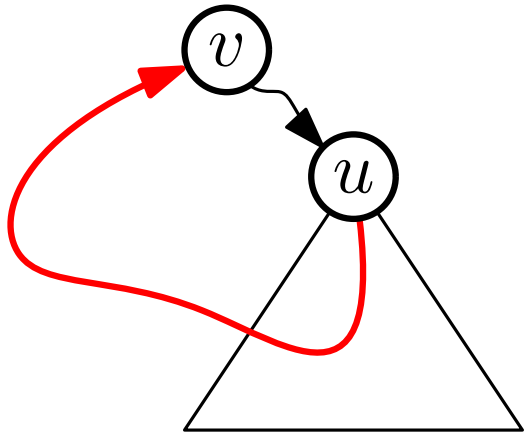

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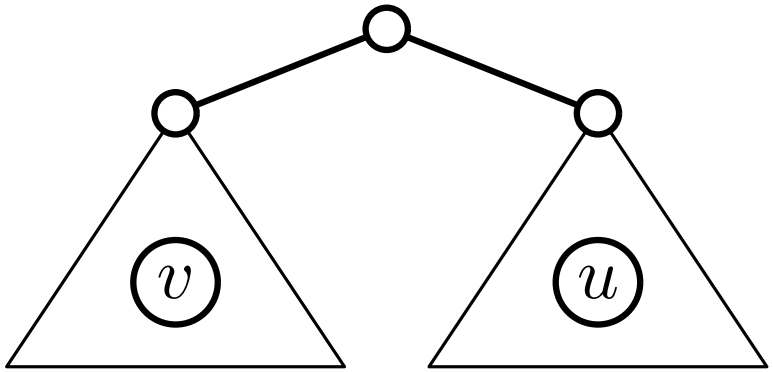

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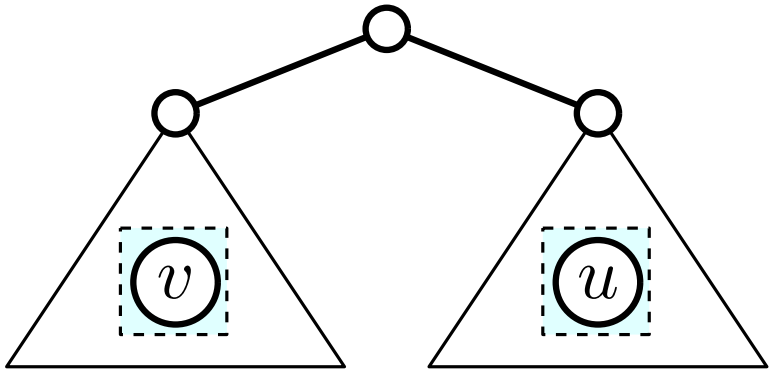

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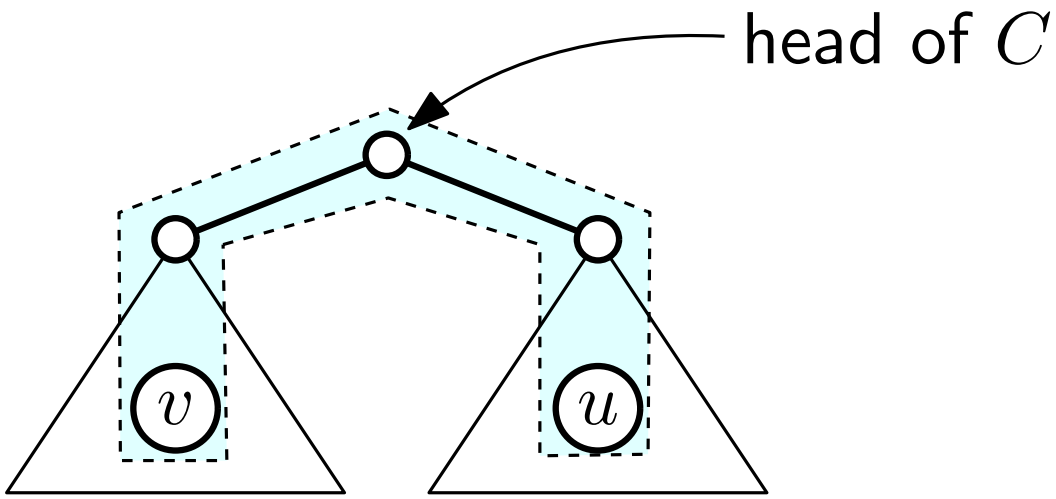

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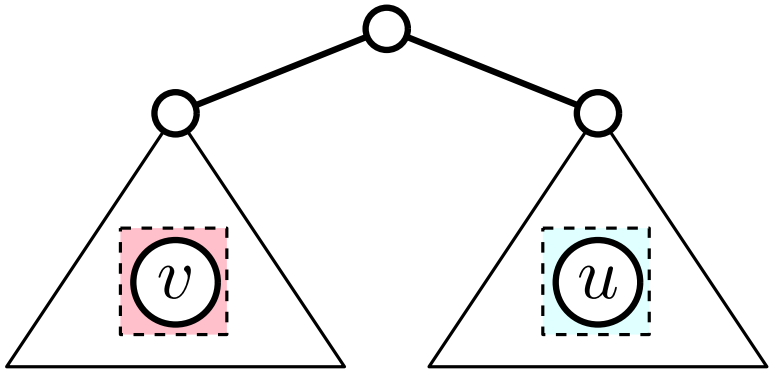

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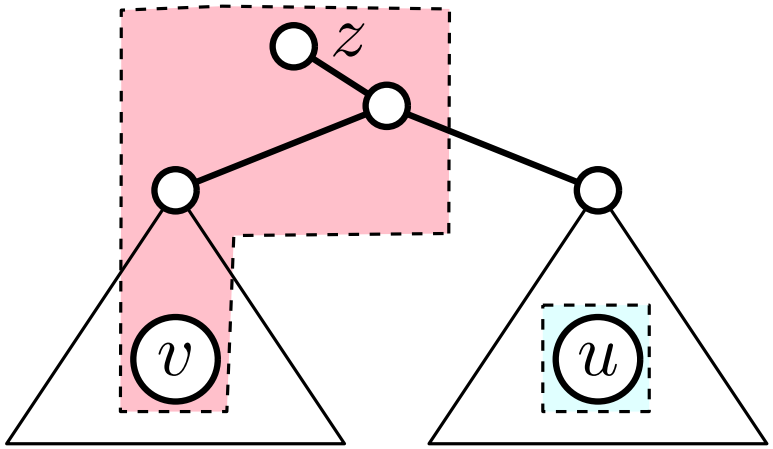

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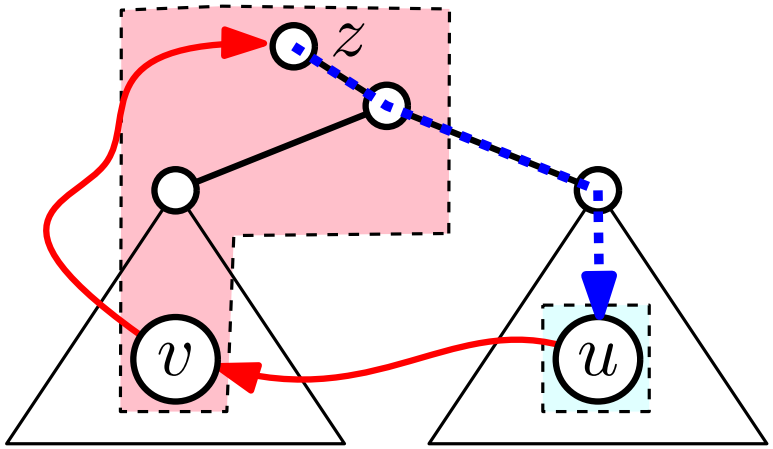

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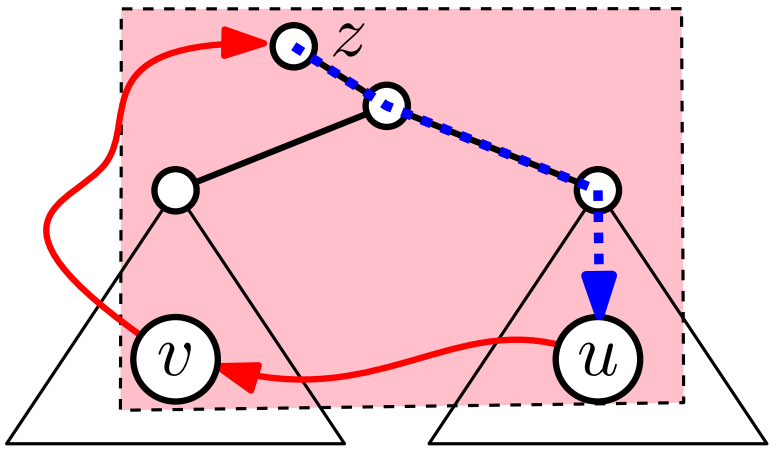

The Algorithm

While $\exists$ vertex $u \in G$ (that has not been deleted):

- cnt $\leftarrow 0$; $T \leftarrow (\{u\}, \emptyset)$
- SCC(u)

SCC(u):

- $\eta(u) \leftarrow$ cnt; cnt $\leftarrow$ cnt $+1$; $exit(u) \leftarrow \eta(u)$
- For each $(u, v) \in E$:
  - If v has not yet been visited:
    - Add (u, v) to T
    - SCC(v)
    - $exit(u) \leftarrow \min\{exit(u), exit(v)\}$
  - Else:
    - $exit(u) \leftarrow \min\{exit(u), \eta(v)\}$
- If $exit(u) = \eta(u)$:
  - Report a new SCC C containing all the descendants of u in T
  - Delete the vertices in C from G and T
    (vertices can be "deleted" in constant time by marking them)

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The Algorithm

While $\exists$ vertex $u \in G$ (that has not been deleted):

- cnt $\leftarrow 0$; $T \leftarrow (\{u\}, \emptyset)$ $S \leftarrow$ Empty stack
- SCC(u)

SCC(u):

- $\eta(u) \leftarrow$ cnt; cnt $\leftarrow$ cnt $+1$; $exit(u) \leftarrow \eta(u)$ Push u into S
- For each $(u, v) \in E$:
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    - SCC(v)
    - $exit(u) \leftarrow \min\{exit(u), exit(v)\}$
  - Else:
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- If $exit(u) = \eta(u)$:
  - $C = \emptyset$; do $z \leftarrow$ Pop from S; $C \leftarrow C \cup \{z\}$ while $z \neq u$;
  - Delete the vertices in C from G and T

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